CHAPTER-III

Estimation Problems Related to Exponential Distribution

and their Sequential Solutions

3.1 Introduction and Setup of the Estimation Problems

3.1.1 Introduction

Exponential distribution is applied in a variety of statistical procedures. Among the most prominent applications are those in the field of life testing. The life testing (or life characteristics as it is often called) can often be usefully represented by an exponential random variable with (usually) a simple associated theory. Sometimes the representation is not adequate; in such cases a modification of the exponential distribution (very often a Weibull distribution) is used. Another application is producing usable approximate solutions to difficult distributional problems.

Exponential distribution is clearly related to many important distributions. If a variable $X(\theta)$ is such that $Y = (X - \theta)^{p}$ has the exponential distribution, then $X$ is said to have Weibull distribution with shape parameter $p (>-1)$. Another modification of exponential distribution gives Laplace’s First Law of Error or Double (or bilateral) exponential distribution. Mixture of exponential distribution give general gamma or general Erlang distribution, which is used in queing theory, reliability and psychology.

Historically, the exponential distribution was the first life time model for which statistical methods were extensively developed. Early work by Sukhatme (1937) and later work by Epstein and Sobel (1953,1954,1955) and Epstein (1954,1960) gave numerous results and popularized the exponential as a lifetime distribution, especially in the area of industrial life
testing. Many authors have contributed to statistical methodology of the distribution. The lengthy bibliographies of Mendenhall (1958), Govindarajuulu (1964) and Johnson and Kotz (1970, chapter 18) give some idea of very large number of papers in this area.

Woodroofe (1977) considered the problem of sequential minimum risk point estimation for the mean of a gamma distribution under squared-error loss function. For the proposed sequential procedure, he obtained the second-order approximations. For the same estimation problem, Isogai and Uno (1995) proposed an ‘improved’ estimator and showed its dominance (in terms of risk) over the usual uniformly minimum variance unbiased estimator considered by Woodroofe (1977). Similar results for the other distributions have been obtained by Isogai and Uno (1993, 1994). Chaturvedi and Rani (1997) considered the problem of interval estimation of the mean survival time and reliability function. For the mean survival time, they developed sequential procedure in order to construct a confidence interval of ‘pre assigned width and coverage probability.’ For the reliability function, sequential procedure was adopted to construct a confidence interval of ‘fixed ratio-width.’ Both the sequential procedures were proved to be ‘asymptotically efficient and consistent’ in Chow-Robbins (1965) sense.

3.1.2 The Set-up of the Estimation Problems

Suppose \( X \) is the random variable representing the failure time is exponentially distributed with probability density function

\[
g(x; \sigma) = \frac{1}{\sigma} \exp \left(-\frac{x}{\sigma}\right); \quad (x, \sigma > 0). \tag{3.1.1}
\]

We consider three estimation problems related to the exponential distribution (3.1.1) which are the problem of minimum risk point estimation of \( \sigma \) under the squared-error loss function, the problem of construction of fixed width confidence interval for \( \sigma \) and the problem
of constructing fixed-ratio width confidence interval for the reliability function. The failure of the fixed sample size procedures to deal with these estimation problems is established.

For the model (3.1.1), it is assumed that $\sigma$ is unknown. We know that [see, Patel, Kapadia and Owen (1976, p. 186)] the UMVUE, as well as, MLE of $\sigma$ is

$$\hat{\sigma}_n = \frac{1}{n} \sum_{i=1}^{n} x_i, \text{ with } E(\hat{\sigma}_n) = \sigma, \text{ Var}(\hat{\sigma}_n) = \left(\frac{\sigma^2}{n}\right). \quad (3.1.2)$$

The reliability function at a specified mission time $t (> 0)$ is

$$R(t) = \exp\left(-\frac{t}{\sigma}\right). \quad (3.1.3)$$

Our first estimation problem is the minimum risk point estimation of $\sigma$ by $\hat{\sigma}_n$. Let the loss incurred in estimating $\sigma$ by $\hat{\sigma}_n$ be

$$L(\sigma, \hat{\sigma}_n) = A(\hat{\sigma}_n - \sigma)^2 + Cn, \quad (3.1.4)$$

where $A (> 0)$ is the known weight and $C (> 0)$ is the known cost per unit sample observation. The risk corresponding to the loss function (3.1.4) is

$$R_n(C) = \frac{A\sigma^2}{n} + Cn. \quad (3.1.5)$$

The value $n^*$ of $n$, which minimizes the risk (3.1.5) is

$$n^* = \left(\frac{A}{C}\right)^{\frac{1}{2}} \sigma, \quad (3.1.6)$$

and substituting $n = n^*$ in (3.1.5), the corresponding minimum risk is

$$R_{n^*}(C) = 2Cn^*. \quad (3.1.7)$$

However, in the absence of any knowledge about $\sigma$, no fixed sample size procedure uniformly minimizes the risk.
Our second estimation problem is the construction of fixed-width confidence interval for \(\sigma\). Suppose one wishes to construct a confidence interval for \(\sigma\), having width 2\(d\) and coverage probability 1-\(\alpha\). We propose \(I_n = [\hat{\sigma}_n - d, \hat{\sigma}_n + d,]\) for \(\sigma\).

It follows from (3.1.2) and the Central Limit Theorem that

\[
\sqrt{n}/\sigma(\hat{\sigma}_n - \sigma) \xrightarrow{\text{L}} Z, \quad \text{as} \quad n \to \infty
\]

where \(Z\) denote a Standard Normal Variate. Utilizing (3.1.4), we get

\[
p(\sigma \in I_n) = p(|\hat{\sigma}_n - \sigma| \leq d)
\]

\[
\approx p\left|Z\right| \leq d\sqrt{n}/\sigma
\]

\[
= 2\Phi\left(d\sqrt{n}/\sigma\right) - 1,
\]

where \(\Phi(\bullet)\) denotes the cumulative distribution function of standard normal variate.

Let ‘\(a\)’, be the constant determined by the equation

\[
2\Phi(a) - 1 = 1 - \alpha.
\]

It follows from (3.1.9) and (3.1.10) that, in order to achieve \(p(\sigma \in I_n) \geq 1 - \alpha\), the sample size required is the smallest positive integer \(n \geq n_0\).

\[
\text{where} \quad n_0 = \left(\frac{a^2\sigma^2}{d^2}\right).
\]

But in the ignorance of \(\sigma\), no fixed sample size procedure can achieve the goals of pre assigned width and coverage probability simultaneously for all values of \(\sigma\).
Our final estimation problem is the construction of ‘fixed-ratio width’ confidence interval for the reliability function. It follows from (3.1.3) that the MLE of \( R(t) \) is

\[
\hat{R}_n(t) = \exp\left(-\frac{t}{\hat{\sigma}_n}\right) \tag{3.1.12}
\]

For pre-fixed \( \delta \in (0,1) \) and \( \alpha \in (0,1) \), suppose we have to construct a confidence interval \( I_n \) for \( R(t) \) such that the ratio between the upper and lower confidence limits of \( I_n \) is \( \delta^{-1} \) and \( P[R(t) \in I_n] \geq 1 - \alpha \).

We call \( I_n \) as ‘fixed-ratio width’ confidence interval. We define

\[
I_n = [\hat{R}_n(t), \delta^{-1}\hat{R}_n(t)]
\]

Since \( R(t) \) is a probability, it can be kept within the interval \([0,1]\) by a proper choice of \( \delta \).

Moreover, the precision of \( I_n \) is fixed in the sense that \( \left\{\frac{\hat{R}_n(t)}{R(t)}\right\} \) lies in the fixed interval \([\delta,1]\).

If we take \( \delta \to 1 \), then \( I_n \) is an ‘optimal’ confidence interval because \( \left\{\frac{\hat{R}_n(t)}{R(t)}\right\} \to 1 \) with probability at least \( 1 - \alpha \).

Now we prove a lemma and its corollary, which will be needed to calculate the coverage probability corresponding to \( I_n \).

**Lemma 1:** As \( n \to \infty \),

\[
E(\hat{\sigma}_n^{-1}) = \frac{1}{\sigma} \left(1 + \frac{1}{n}\right) + o(n^{-1}) \tag{3.1.13}
\]

and

\[
\text{Var}(\hat{\sigma}_n^{-1}) = \frac{1}{n\sigma^2} + o(n^{-1}) \tag{3.1.14}
\]
Proof: Using the fact that \( \frac{2n\hat{\sigma}_n}{\sigma} \sim \chi^2_{(n)} \), we have for \( k > 0 \),

\[
E(\hat{\sigma}_n^{-k}) = \left( \frac{2n}{\sigma} \right)^k E(\chi^2_{(2n)})^{-k} = \left( \frac{n}{k} \right)^k \left( \frac{\sqrt{n-k}}{\sqrt{n}} \right)^k. \tag{3.1.15}
\]

From (3.1.15), we get

\[
E(\hat{\sigma}_n^{-1}) = \frac{1}{\sigma} \left( 1 + \frac{1}{n} \right) + o(n^{-1}), \text{ and (3.1.13) holds.}
\]

Furthermore, from (3.1.15),

\[
E(\hat{\sigma}_n^{-2}) = \frac{1}{\sigma^2} \left( 1 + \frac{3}{n} \right) + o(n^{-1}). \tag{3.1.16}
\]

Result (3.1.14) follows from (3.1.13) and (3.1.16).

**Corollary 1:** As \( n \to \infty \),

\[
\sqrt{n} \sigma (\hat{\sigma}_n^{-1} - \sigma^{-1}) \xrightarrow{L} N(0,1).
\]

**Proof:** The result follows from lemma 1 and the Central Limit Theorem.

Now, from (3.1.3), (3.1.12) and Corollary 1,

\[
P[R(t) \in I_n] = P \left[ \exp \left( -\frac{t}{\hat{\sigma}_n} \right) \leq \exp \left( -\frac{t}{\sigma} \right) \leq \delta^{-1} \exp \left( -\frac{t}{\hat{\sigma}_n} \right) \right] = P \left[ 0 \leq t^\theta (\hat{\sigma}_n^{-1} - \sigma^{-1}) \leq \ln \delta^{-1} \right] = \Phi \left( \sigma \sqrt{nt^{-1} \ln \delta^{-1}} \right) - \frac{1}{2}. \tag{3.1.16}
\]

Let \( 'b' \) be the constant defined by

\[
\Phi(b) - \frac{1}{2} = 1 - \alpha. \tag{3.1.17}
\]
From (3.1.16) and (3.1.17), in order to achieve \( P[R(t) \in I_n] \geq 1 - \alpha \), the sample size required is the smallest positive integer \( n \geq n_1 \),

where

\[
n_1 = \frac{b^2 t^2}{\sigma^2 \left( \ln \delta^{-1} \right)^2}.
\] (3.1.18)

Since \( \sigma \) is unknown, the fixed sample size procedure cannot achieve the goal of fixed-ratio width for all values of \( \sigma \).

In section 3.2, a sequential procedure is proposed in order to estimate \( \sigma \) point wise and obtained are the associated second-order approximations. Also proposed is an ‘improved’ estimator and have shown its dominance over the Uniformly Minimum Variance Unbiased Estimator (UMVUE).

In section 3.3, is developed a sequential procedure fixed-width confidence interval for \( \sigma \) and have proved its asymptotic efficiency and consistency.

In section 3.4, a sequential procedure is adopted in order to construct a fixed–ratio width confidence interval for the reliability function and have shown it to be asymptotically efficient and consistent.

### 3.2 Sequential Procedure for the Point Estimation of the Scale Parameter

Let us start with a sample of size \( m \geq 1 \). Then, the stopping time \( N = N(C) \) is defined by

\[
N = \inf \left\{ n \geq m : n \geq \left( \frac{A}{C} \right)^{\frac{1}{2}} \hat{\sigma}_n \right\}.
\] (3.2.1)

After stopping, we estimate \( \sigma \) by \( \hat{\sigma}_N \), incurring the risk
\[ R_N(C) = AE(\hat{\sigma}_N - \sigma)^2 + CE(N). \]  \hspace{1cm} (3.2.2)

In what follows, we obtain second-order approximations for the risk (3.2.2). Before proving the main result, we state some lemmas.

**Lemma 2:** For all \( m \geq 2 \), as \( c \to 0 \),

\[ E(N) = n^* + v - 1 + o(1), \]

where \( v \) is specified.

**Proof:** The result follows by using equation (1.1) and the Theorem 2.4 of Woodroofe (1977).

**Lemma 3:** For \( \eta \in (0,1) \), as \( C \to 0 \),

\[ P(N \leq \eta n^*) = O\left(C^{m^*}\right). \]

**Proof:** The result follows from lemma 2.3 of Woodroofe (1977).

**Lemma 4:** Let the random variable ‘U’ be defined by \( |U - 1| \leq \left| \frac{N}{n} - 1 \right| \), then

\[ U \xrightarrow{a.s.} 1 \quad \text{as} \quad C \to 0, \]  \hspace{1cm} (3.2.3)

and

\[ U^4 \text{ is uniformly integrable for all } m \geq 5. \]  \hspace{1cm} (3.2.4)

**Proof:** From (3.2.1), we notice the inequality,

\[ \left( \frac{A}{C} \right)^{1/2} \hat{\sigma}_N \leq N \leq \left( \frac{A}{C} \right)^{1/2} \hat{\sigma}_N + (m - 1). \]

Or

\[ \left( \frac{\hat{\sigma}_N}{\sigma} \right) \leq \frac{N}{n} \leq \left( \frac{\hat{\sigma}_N}{\sigma} \right) + \frac{(m - 1)}{n^*}. \]  \hspace{1cm} (3.2.5)
Using the facts that \( \lim_{C \to 0} N = \infty \) and \( \hat{\sigma}_N \xrightarrow{a.s.} \sigma \) as \( N \to \infty \), we obtain from (3.2.5) that

\[
1 \leq \liminf_{C \to 0} \left( \frac{N}{n^*} \right) \leq \limsup_{C \to 0} \left( \frac{N}{n^*} \right) \leq 1
\]

or

\[
\lim_{C \to 0} \left( \frac{N}{n^*} \right) = 1 \text{ a.s}
\]

(3.2.6)

Result (3.2.3) now follows from the definition of \( U \).

We note that on the event \( \{ N \leq \eta n^* \} \),

\[
|U - \bar{I}| \leq 1 - \left( \frac{m}{n^*} \right), \text{ i.e. } U^{-1} \leq \left( \frac{n^*}{m} \right).
\]

Denoting by \( I(\cdot) \), the usual indicator function and \( k \) any positive generic constant independent of \( C \), we have

\[
E[U^{-4} I(N \leq \eta n^*)] \leq k n^* P(N \leq \eta n^*)
\]

which on applying Lemma 3 gives that

\[
E[U^{-4} I(N \leq \eta n^*)] = O \left( C^{m/2-2} \right)
\]

\[
= o(1), \text{ for all } m \geq 5
\]

(3.2.7)

Furthermore, on the event \( \{ N > \eta n^* \}, |U - \bar{I}| \leq 1 - \eta, \text{ i.e., } U^{-1} \leq \eta^{-1} \).

Thus,

\[
E \left[ U^{-4} I(N > \eta n^*) \right] \leq k P(N > \eta n^*)
\]

\[
= o(1),
\]

(3.2.8)

since \( N \) terminates with probability one.

Result (3.2.4) now follows on combining (3.2.7) and (3.2.8).
Lemma 5: \( \left( \frac{N}{n^*} \right)^4 \) is uniformly integrable, \( \text{ (3.2.9) } \)

\[
\frac{(N - n^*)^2}{(n^*)^2} \text{ is uniformly integrable for all } m \geq 2, \quad \text{ (3.2.10) }
\]

and

\[
\frac{(N - n^*)^4}{(n^*)^2} \text{ is uniformly integrable for all } m \geq 3, \quad \text{ (3.2.11) }
\]

**Proof:** Result (3.2.9) follows from lemma 2.1 of Woodroofe (1977) and results (3.2.10) and (3.2.11) follow from theorem 2.3 of Woodroofe (1977).

Lemma 6: \( \left\{ \frac{(N - n^*)}{(n^*)^{1/2}} \right\} \) and \( \left\{ \frac{(S_n - N)}{(n^*)^{1/2}} \right\} \) are asymptotically uncorrelated for all

\[
m \geq 3, \quad \text{ (3.2.12) }
\]

and

\[
\left\{ \frac{(N - n^*)}{(n^*)^{1/2}} \right\} \text{ and } \left\{ \frac{(S_n - N)}{(n^*)^{1/2}} \right\} \text{ are asymptotically uncorrelated for all}
\]

\[
m \geq 3; \quad \text{ (3.2.13) }
\]

where \( S_n = \sum_{j=1}^{n} \left( \frac{Z_j}{\sqrt{2}} \right) \), with \( Z_j \sim \chi^2(2) \).

**Proof:** Using \( S_n \), we can rewrite the stopping rule (3.2.1) as

\[
N = \inf \left\{ n \geq m : S_n \leq \left( \frac{n^2}{n_0} \right) \right\}, \text{ and hence it follows that }
\]
\[ S_N - N \leq N \left( \frac{N}{n} - 1 \right) \] \hspace{1cm} (3.2.14)

Using (3.2.14), Cauchy-Schwartz inequality and lemma 5, gives us (3.2.12).

A similar proof holds for (3.2.13).

**Lemma 7**: As \( C \to 0 \), \( \frac{(N - n^*)}{(n^*)^{1/2}} \xrightarrow{L} N(0,1) \).

**Proof**: The lemma follows from a result of Bhattacharya and Mallik (1973).

Now we prove the main theorem, which provides the second-order approximations for the risk.

**Theorem 1**: For all \( m \geq 3 \), as \( C \to 0 \),

\[ R_N(C) = R_{n^*}(C) + 3C + o(C) \]

**Proof**: We can write (3.2.2) as

\[ R_N(C) = A\sigma^2 \mathbb{E} \left[ \frac{1}{N^2} \left( \sum_{j=1}^{N} Z_j - N \right)^2 \right] + CE(N) \]

\[ = C \mathbb{E} \left[ f \left( \frac{N}{n^*} \right) \left( S_N - N \right)^2 \right] + CE(N), \]

(3.2.15)

where \( f(x) = x^{-2} \).

Expanding \( f(x) \) by Taylor’s series, we obtain for \( |U - 1| \leq \left| \frac{N}{n} - 1 \right| \),

\[ R_N(C) = CE \left[ \left( 1 - 2 \left( \frac{N}{n} - 1 \right) + 3 \left( \frac{N}{n} - 1 \right)^2 \right) \left( S_N - N \right)^2 \right] + CE(N). \]

(3.2.16)

Applying lemmas 2, 4, 5, 6, 7 and Wald’s lemma, we obtain from (3.2.16) that for all \( m \geq 3 \), as \( C \to 0 \),
\[ R_N(C) = CE(N) - \frac{2C}{n} E(N-n^*)E(N) + \frac{3C}{n} E(N) + CE(N) \]

\[ = 2C(n^* + \nu - 1 + o(1)) - \frac{2C}{n} (\nu - 1 + o(1))(n^* + \nu - 1 + o(1)) + \frac{3C}{n} (n^* + \nu - 1 + o(1)) \]

\[ = 2Cn^* + 3C + o(C), \]

and the theorem follows.

In what follows, we obtain an ‘improved estimator of \( \sigma \).

Let us propose the estimator

\[ \hat{\delta}_N = \hat{\sigma}_N + K C^{1/2} A^{-1/2} \quad \text{for } \sigma, \]

where \( N \) is determined by the rule (3.2.1) and \( k \) is to be obtained. The risk corresponding to the estimator \( \hat{\delta}_N \) is

\[ R_N(C) = AE\left(\hat{\delta}_N - \sigma\right)^2 + CE(N) \]

\[ = R_N(C) + 2kA^{1/2}C^{1/2}E(\hat{\sigma}_N - \sigma) + k^2 C. \quad (3.2.17) \]

Denoting by \( \psi_c = \frac{2N^2}{n} - S_N \), we have

\[ AE(\hat{\sigma}_N - \sigma) = AE\left[ \frac{\sigma}{N} (S_N - 2N) \right] \]

\[ = A^{1/2} C^{1/2} E \left( N - n^* - \left( \frac{n^*}{N} \right) \psi_c \right), \]

giving

\[ A^{1/2} E(\hat{\sigma}_N - \sigma) = C^{1/2} E \left[ (N - n^*) - \left\{ 1 - \frac{(N - n^*)}{n^*} + \frac{(N - n^*)^2}{n^* U^3} \right\} \psi_c \right] \quad (3.2.18) \]
It follows from theorem 2.1 of Woodroofe (1977) that $\psi_c$ and $\frac{(N-n^*)}{n^*}$ are asymptotically independent and $E(\psi_c) \rightarrow \nu$ as $C \rightarrow 0$. Utilizing these results and arguments similar to these given in the proof of theorem 1, we obtain from (3.2.18) that for all $m \geq 3$, as $C \rightarrow 0$,

$$A^{\frac{1}{2}}E(\hat{\sigma}_N - \sigma) = -C^{\frac{1}{2}} + o(C). \quad (3.2.19)$$

Utilizing (3.2.19), we obtain from (3.2.17) that

$$R_N'(C) = R_N(C) - 2kC + k^2C + o(C) \quad (3.2.20)$$

It is easy to see that the value $k_0$ of $k$ minimizing the function $g(k) = k^2C - 2kC$ is $k_0 = 1$ and substituting this value in (3.2.20), we get

$$R_N'(C) - R_N(C) = -C + o(C),$$

thereby concluding that $\hat{\delta}_N$ has smaller risk than $\hat{\sigma}_N$.

3.3 Sequential procedure to construct fixed width confidence interval for the scale parameter

Let us start with a sample of size $m \geq 1$. Then, motivated by (3.1.11), the stopping time $N = N(d)$ is the smallest positive integer $n \geq m$ such that

$$n \geq \left(\frac{a^2}{d^2}\right)\hat{\sigma}_n^2. \quad (3.3.1)$$

The properties of the sequential procedures (3.3.1) are mentioned in the following theorem.

**Theorem 2:** $N$ terminates with probability one, \( \lim_{d \to 0} N = \infty \) a.s., \( (3.3.2) \)
\[
\lim_{d \to 0} \left( \frac{N}{n_0} \right) = 1 \text{ a.s} \quad \text{(3.3.4)}
\]

\[
\lim \mathbb{E} \left( \frac{N}{n_0} \right) = 1 \text{ a.s} \quad \text{(asymptotic efficiency)} \quad \text{(3.3.5)}
\]

and

\[
\lim_{d \to 0} p(\sigma \in I_n) = 1 - \alpha. \quad \text{(asymptotic consistency)} \quad \text{(3.3.6)}
\]

**Proof:**

Denoting by \( Y_n = \left( 2n \hat{\sigma}_n / \sigma \right) \) and using the fact that \( Y_n \sim 2n / 2\sqrt{n} \xrightarrow{d} Z \), where \( Z \) is a standard normal variate, it follows from the definition of \( N \) that

\[
p(N > n) = p(n < (a/d)^2 \hat{\sigma}_n^2)
\]

\[
= p\left( 2n \hat{\sigma}_n / \sigma \geq \left( 2nd\sqrt{n} / a\sigma \right) \right)
\]

\[
= p \left( Y_n \geq \sqrt{n} \left\{ (n/n_0)^{3/2} - 1 \right\} \right)
\]

\[
= 1 - \Phi \left\{ \sqrt{n} \left\{ (n/n_0)^{3/2} - 1 \right\} \right\}.
\]

(3.3.7)

Now applying a result of Zacks (1971, p.561) that, as \( x \to \infty \), \( 1 - \Phi(x) \approx x^{-1/2} \phi(x) \), where \( \phi(x) \) stands for the p.d.f. of a standard normal variate, we obtain from (3.3.7) that

\[
p(N > n) = O(n^{-3/2}) \quad \text{and (3.3.2) follows.}
\]

Result (3.3.3) follows from the definition of \( N \) at (3.3.1).

From (3.3.1), we notice the inequality
\( (a/d)^2 \hat{\sigma}_N^2 \leq N \leq (a/d)^2 \hat{\sigma}_N^2 + (m-1) \)

or,

\[ \left( \frac{\hat{\sigma}_N}{\sigma} \right)^2 \leq \frac{N}{n_o} \leq \left( \frac{\hat{\sigma}_N}{\sigma} \right)^2 + \frac{(m-1)}{n_o}. \]  \hspace{1cm} (3.3.8)

Applying (3.3.3) and the fact that \( \hat{\sigma}_N \xrightarrow{a.s.} \sigma \) as \( N \to \infty \), we obtain from (3.3.8) that

\[ 1 \leq \liminf_{d \to 0} \left( \frac{N}{n_o} \right) \leq \limsup_{d \to 0} \left( \frac{n_o}{N} \right) \leq 1, \]

and (3.3.4) holds.

From (3.3.4) and Fatou’s lemma,

\[ \liminf_{d \to 0} \left( \frac{N}{n_o} \right) \geq 1. \]  \hspace{1cm} (3.3.9)

Result (3.3.5) follows from (3.3.9) and

\[ \limsup_{d \to 0} E(N/n_o) \leq 1. \]  \hspace{1cm} (3.3.10)

In order to prove (3.3.10), let for \( \eta \in (0,1) \), \( \theta = (1+\eta)n_o \)

where \( \eta \) is arbitrary.

Now,

\[ E(N) = \sum_{n>\theta} n P(N = n) \]

\[ = \sum_{n=m} P(N = n) + \sum_{n>\theta} n P(N = n) \]

\[ \leq \theta + \sum_{n>\theta} n P(N = n), \]

or,

\[ E \left( \frac{N}{n_o} \right) \leq (1+\eta) + \frac{1}{n_o} \sum_{n>\theta} n P(N = n) \]  \hspace{1cm} (3.3.11)
Since \( \eta \) is arbitrary and \( n_0 \to \infty \) as \( d \to 0 \), result (3.3.10) follows from (3.3.11), if we can prove that, as \( d \to 0 \),

\[
\sum_{n>0} n P(N = n) < K, \tag{3.3.12}
\]

where \( K \) is free from \( d \). To this end we have

\[
\sum_{n>0} n P(N = n) \leq \sum_{n>0} (n+1) P \left[ n < \left( \frac{a}{d} \right)^2 \right]
\]

\[
= \sum_{n>0} (n+1) P \left[ \chi^2_{(2n)} > 2n \left( \frac{n}{n_0} \right)^{\frac{1}{2}} \right]
\]

\[
\leq \sum_{n>0} (n+1) \inf_{0<h<1/2} \left[ e^{-2hn} \left( \frac{n}{n_0} \right)^{\frac{1}{2}} e^{b(x^2_{(2n)})} \right]
\]

\[
\leq \sum_{n>0} (n+1) \inf_{0<h<1/2} \left[ e^{\left( -2hn(1+\eta)^{\frac{1}{2}} \right)(1-2h)^{-n}} \right] \tag{3.3.13}
\]

The value \( h_0 \) of \( h \), which minimizes the function

\[
g(h) = e^{-2hn} \left( \frac{n}{n_0} \right)^{\frac{1}{2}} (1-2h)^{-n} \quad \text{is} \quad h_0 = \left( \frac{1}{2} \right)^{\frac{1}{2}} \left( 1 + \eta \right)^{-\frac{1}{2}}
\]

and substituting \( h_0 \) of \( h \) in (3.3.13), we get

\[
\sum_{n>0} n P(N = n) \leq \sum_{n>0} (n+1) \left[ (1 + \eta)^{\frac{1}{2}} e^{-(1+\eta)^{\frac{1}{2}}} \right]^n
\]

\[
= \sum_{n>0} C_n, \quad \text{say} \tag{3.3.14}
\]
Since \( \lim_{m \to \infty} c_{n}^{1/n} = (1 + \eta)^{1/2} e^{-\gamma/(1+\eta)^{1/2}} \) \( < 1 \), the series on the right side of (3.3.14) is convergent and (3.3.12) follows, which proves (3.3.5).

Finally, using (3.1.8), (3.3.2), (3.3.3), (3.3.4) and a result of Anscombe(1952),

\[
\frac{\sqrt{N}}{\sigma} (\hat{\sigma}_N - \sigma) \xrightarrow{d} Z \text{ as } d \to 0. \tag{3.3.15}
\]

Using (3.3.15), we get

\[
\lim_{d \to 0} p(\sigma \in I_N) = \lim_{d \to 0} p\left[ \frac{\sqrt{N}}{\sigma} |\hat{\sigma}_N - \sigma| \leq \frac{d\sqrt{N}}{\sigma} \right]
\]

\[
= \lim_{d \to 0} p\left[ |Z| \leq \frac{a\sqrt{N}}{n_0} \right]
\]

\[
= 2\Phi(a) - 1
\]

\[
= 1 - \alpha,
\]

and (3.3.6) holds.

### 3.4 Sequential Procedure to Construct Fixed-Ratio-Width Confidence Interval for the Reliability Function

In conformity with (3.1.18), we propose the following sequential procedure.

Let us take \( m \geq 1 \) to be the initial sample size. Then the stopping time \( N \equiv N(\delta) \) is the smallest positive integer \( n \geq m \) such that

\[
n \geq \frac{b^2\lambda^2}{(\ln \delta)^2} \left( \frac{\hat{\sigma}_n}{\hat{\sigma}_n} \right)^2. \tag{3.4.1}
\]

After stopping, we construct the confidence interval \( I_N = [\hat{R}_N(t), \delta^{-1}\hat{R}_N(t)] \) for \( R(t) \).

The following theorem establish the results that the sequential procedure (3.4.1) is ‘asymptotically efficient and consistent.’
**Theorem 3**: \( N \) terminates with probability one

\[
\lim_{\delta \to 1} N = \infty, \tag{3.4.2}
\]

\[
\lim_{\delta \to 1} \left( \frac{N}{n_1} \right) = 1 \text{ a.s.}, \tag{3.4.3}
\]

\[
\lim_{\delta \to 1} E \left( \frac{N}{n_1} \right) = 1, \text{ ‘asymptotic efficiency’} \tag{3.4.4}
\]

and

\[
\lim_{\delta \to 1} \mathbb{P} \left( R(t) \in I_N \right) = 1 - \alpha, \text{ ‘asymptotic consistency’} \tag{3.4.5}
\]

**Proof**: Let \( T_n \) be a random variable such that \( T_n \sim \chi^2_{(2n)} \). Obviously, \( E(T_n) = \{2(n-1)\}^{-1} \) and

\[
\text{Var}(T_n) = \left\{4(n-1)^2(n-2)\right\}^{-1}.
\]

Denoting by \( Y_n = \left\{2(n-1)(n-2)^{1/2} \{T_n - (2(n-1)^{-1}\}\} \right\} \),

it follows from the Central Limit Theorem that \( Y_n \xrightarrow{L} Z \) as \( n \to \infty \), where \( Z \) is a Standard Normal Variate.

Using this result and a result of Zacks (1971, p.561), it follows from the definition of \( N \) that

\[
\mathbb{P}(N > n) \leq \mathbb{P} \left[ T_n > \left( \frac{(nn_1)^{-1/2}}{2} \right) \right]
\]

\[
= \mathbb{P} \left[ Y_n > (n-1)(n-2)^{1/2} \left( (nn_1)^{-1/2} - (n-1)^{-1} \right) \right]
\]

\[
\approx 1 - \Phi \left( (n-1)(n-2)^{1/2} \left( (nn_1)^{-1/2} - (n-1)^{-1} \right) \right)
\]

\[
= O \left( n^{-1} \right) \text{ where } \Phi(.) \text{ stands for c.d.f. of a Standard Normal Variate},
\]

and (3.4.2) follows.

Result (3.4.3) follows from the definition of \( N \) at (3.4.1).
From (3.4.1), we notice the inequality

\[
\frac{b^2 t^2}{(\ln \delta^{-1})^2} (\hat{\sigma}_N)^2 \leq N \leq \frac{b^2 t^2}{(\ln \delta^{-1})^2} (\hat{\sigma}_N)^2 + (m - 1),
\]

or

\[
\left( \frac{\sigma}{\hat{\sigma}_N} \right)^2 \leq \frac{N}{n_1} \leq \left( \frac{\sigma}{\hat{\sigma}_N} \right)^2 + \frac{(m - 1)}{n_1}.
\] (3.4.7)

Taking the Limit of (3.4.7) as \( \delta \to 1 \) and using (3.4.3) we get

\[
1 \leq \liminf_{\delta \to 1} \left( \frac{N}{n_1} \right) \leq \limsup_{\delta \to 1} \left( \frac{N}{n_1} \right) \leq 1,
\]

and (3.4.4) holds.

From (3.4.4) and Fatou’s Lemma,

\[
\liminf_{\delta \to 1} E \left( \frac{N}{n_1} \right) \geq 1.
\] (3.4.8)

Let for arbitrary \( \eta \in (0,1), \theta = (1 - \eta) n_1 \).

Now,

\[
E(N) = \sum_{n=m}^{\infty} n P(N = n)
\]

\[
= \sum_{n=m}^{\theta} n P(N = n) + \sum_{n > \theta} n P(N = n)
\]

\[
\leq \theta + \sum_{n > \theta} n P(N = n),
\]

or

\[
E \left( \frac{N}{n_1} \right) \leq (1 + \eta) + \frac{1}{n_1} \sum_{n > \theta} n P(N = n)
\] (3.4.9)

Now,
\[
\sum_{n=0}^{N} p(N = n) \leq \sum_{n=0}^{N+1} (n+1)p(N = n+1)
\leq \sum_{n=0}^{N+1} (n+1)p \left[ n < \frac{b^2(1-2p)}{(\ln \delta^{-1})^2} \left( \sigma_n^2 \right)^{-1/2} \right]
\leq \sum_{n=0}^{N+1} (n+1)p \left[ \chi^2_{(2n)} < 2n \left( 1 + \eta^{-1/2} \right) \right]
\leq \sum_{n=0}^{N+1} (n+1) \inf_{\theta < \chi^2_{(2n)}} \left\{ \exp \left[ 2hn(1 + \eta^{-1/2}) \right] \right\} \left\{ \exp \left( -h\chi^2_{(2n)} \right) \right\}^{-n}.
\]

(3.4.10)

From Cauchy’s \( n \)th root test, the series on the right side of (3.4.10) is convergent. Thus we obtain from (3.4.9) that

\[
\limsup_{\delta \to 1} E \left( \frac{N}{n_1} \right) \leq 1.
\]

(3.4.11)

Result (3.4.5) now follow on combining (3.4.8) and (3.4.11).

Finally, from corollary 1, (3.4.2), (3.4.3), (3.4.4) and a result of Anscombe (1952),

\[
\sigma \sqrt{n_1} \left( \hat{\sigma}_n^{-1} - \sigma^{-1} \right) \xrightarrow{\text{L}} N(0,1) \text{ as } \delta \to 1.
\]

(3.4.12)

From (3.4.12) and dominated convergence theorem,

\[
\lim_{\delta \to 1} \mathbb{P}[R(t) \in I_N] = \lim_{\delta \to 1} \mathbb{P}\left[ 0 \leq \hat{\sigma}_n^{-1} - \sigma^{-1} \leq t \ln \delta^{-1} \right]
= \lim_{\delta \to 1} \mathbb{P}\left[ 0 \leq \sigma \sqrt{n_1} (\hat{\sigma}_n - \sigma) \leq b \right]
= \Phi(b) - \frac{1}{2}
= 1 - \alpha,
\]

and (3.4.6) hold.