2.1. INTRODUCTION

In this chapter we consider only trees and \( T \) always denotes the underlying tree in all our discussion.

The study of pseudosimilar vertices began because of its significance in any proof of the Reconstruction Conjecture. Motivated by the possible influence of the study of these vertices in the complexity of the isomorphism problems, Corneil, Klawe and Kirkpatrick [7,8] initiated their study of these vertices. They characterized the structure of a tree with two pseudosimilar vertices and this led to the interesting result that a vertex of a tree cannot have two pseudosimilar mates, or, some similar mates and one pseudosimilar mate. They generalized the notion of pseudosimilarity into \( k \)-pseudosimilarity, by deleting all the vertices upto a distance \( k \) from the given vertex. Their study of the concepts was facilitated by their notation \( \langle X_1, X_2, \ldots, X_n \rangle \) representing a tree in terms of its subtrees.

In [7], they have shown that, in a tree if \( u \not\sim^0 v \) and \( u \not\sim^1 v \) then \( u \) and \( v \) are similar and have raised the following questions.
(1) What is the feasible set of values for \( k \) such that \( u \prec v \) but \( u \prec^k v \) for all values of \( k \) in such a set?

(2) Is the set \( \{1, 2, \ldots, n\} \), for any given \( n \geq 2 \), such a feasible set? In particular is \( \{1, 2\} \) a feasible set?

Here we show that, in a tree

(a) \( u \prec^0 v \) and \( u \prec^2 v \) imply \( u \prec v \),
(b) \( u \prec^1 v \) and \( u \prec^2 v \) imply \( u \prec v \), if \( d(u, v) \neq 1 \) or \( 3 \) and
(c) for no other combination of \( i \) and \( j \), \( u \prec^1 v \) and \( u \prec^j v \) will imply \( u \prec v \).

Thus regarding the first question in [7], all the double combinations for feasible values of \( k \) are determined in this chapter. The triple combinations for feasible values of \( k \) will be considered in the next chapter.

Regarding the second question, an affirmative answer, in the form of examples forming an infinite family is given here. This and many other constructions depend on an infinite family \( \{ S_i \} \) of rooted trees. We begin with the construction of this family in the next section.

We also prove some edge versions of the above results.
2.2 TREES WITH PRESCRIBED PROPERTIES

Recall that the forest $S_1 - F^k$ will be denoted by $S^k_1$, where $(S^*_1, s^*)$ is a rooted tree.

Theorem 2.2.1: Given $n \geq 2$, there exist rooted trees $(S_i, s_i)$, $1 \leq i \leq n$, such that

(i) $S_i \not\cong S_j$ for $i \neq j$ and
(ii) $S_i^k \not\cong S_j^k$ for all $k \geq 0$ and for $1 \leq i, j \leq n$.

Proof: Consider the table of rooted trees given in Figure 2.1. In this table, the triangle and the diamond represent any two rooted trees.

The trees $(S_i, s_i)$, $1 \leq i \leq n$, are formed as follows.

Choose the first $(n+1)$ rooted trees in the $i$th row of the table. Join the roots of these $(n+1)$ trees to a new vertex called $s_i$, which is going to be the root of $S_i$. For example in the case $n = 3$, the trees are given in Figure 2.2.

Note that if the triangle and diamond are isomorphic as rooted trees then $S_1 \not\cong S_2$.

If the two trees of Figure 2.3 are isomorphic as
rooted trees then $S_1 \not\cong S_2 \not\cong S_3$. So, we may avoid these situations, since we need $S_i \not\cong S_j$ for $i \neq j$.

Since all the trees of any particular column are isomorphic (but not as rooted trees) we have $S_i^O \not\cong S_j^O$, for all $i, j \geq 1$.

Each tree of the table can be thought of as a combination of two rooted trees, say $A$ and $B$ and the tree can be denoted by $A \cdot B$; that is, the new graph is obtained by identifying the roots of $A$ and $B$. Consider the only three places in which the rows $i$ and $i+1$ ($i \geq 1$) differ as rooted trees. If these trees of the $i^{th}$ row are of the form $A \cdot B$, $C \cdot D$ and $E \cdot F$, then note that the corresponding three trees of the $(i+1)^{th}$ row are of the form $E \cdot D$, $A \cdot F$ and $C \cdot B$. This ensures that $S_i^k \cong S_{i+1}^k$ for all $k$, $i \geq 1$, which in turn indicates $S_i^k \cong S_j^k$ for all $i, j \geq 1$, and for all $k \geq 0$.

We now show that $S_i \not\cong S_j$ if $i \neq j$. Suppose that they are isomorphic. Let $\alpha$ be an isomorphism from $S_i$ to $S_j$. In $S_j$, $s_j$ and $\alpha(s_i)$ are $k$-removal similar for all $k \geq 0$. Hence by Note 1.2.6, we get $s_j \sim \alpha(s_i)$ in $S_j$ and hence

$$(S_j, s_j) \not\cong (S_j, \alpha(s_i)) \not\cong (S_i, s_i).$$
but that $S_i$ and $S_j$ are not isomorphic as rooted trees is easy to see. Hence $S_i \not\cong S_j$ for $i \neq j$, which completes the proof.

**Note 2.2.2.** The table used in the formation of $S_i$'s is slightly off symmetric, in the sense that to form the $n$ number of $S_i$'s, $n$ rows and $(n+1)$ columns are to be taken. It could be made symmetric if two consecutive rows differ in exactly two positions instead of three positions as in our table. If that is the case, the two differing positions should be of the form $A \cdot B$, $C \cdot D$ and $A \cdot C$, $B \cdot D$. The necessity of $A \not\cong D$ is obvious, as otherwise, the two rows do not differ, except for a change in the positions of two elements, which does not matter. Also $A \cdot B \not\cong A \cdot C$, $B \cdot D \not\cong C \cdot D$ and $A \cdot B \not\cong A \cdot C$, $B \cdot D \not\cong C \cdot D$. We have proved the impossibility of this in the sixth chapter (Theorem 6.4.4).

**Note 2.2.3.** We see that the smallest trees for $S_i$ are got by taking the triangle as $K_2$ and the diamond as $K_1$. The smallest trees $S_1$ and $S_2$ are shown in Figure 2.4.

2.3 THREE LEMMAS

Three lemmas are proved in this section in their general form which will be used in the later sections.
Lemma 2.3.1.: Let \( u, v \in V(T) \). Let

(i) \( u \neq v \) and

(ii) for some \( k \geq 1 \), the collection

\[
\{ n_1 R_u^1 , n_2 R_v^2 , n_3 R_u^3 , n_4 R_v^4 , \ldots , n_k R_k^k \}
\]

and

\[
\{ n_1 R_v^1 , n_2 R_u^2 , n_3 R_v^3 , n_4 R_u^4 , \ldots , n_k R_k^k \}
\]

for some \( n_1 > 0, n_1 > 0 \) and let \( x \) and \( y \) take the value \( v \) or \( u \) depending on whether \( k \) is even or odd.

Then \( u \sim v \).

Proof: Suppose the conclusion is not true. Let \( T = \langle X_1, X_2, \ldots, X_n \rangle \), \( x_1 = u, x_n = v \) be a counterexample to the lemma, with the minimum number of vertices. For the sake of definiteness let us fix \( k \) to be even. The case when \( k \) is odd can be treated similarly.

Case 1: Let \( d = d(u, v) \geq 2 \).

Here \( n \geq 3 \). Since \( u \neq v \) and \( u \neq v \), by Note 1.2.3, \( | X_1 | = | X_n | > | X_i | \) for \( 1 \leq i < n \).

Assumption (ii) of the lemma gives
Using Equation (2.1) and considering the components with the maximum number of vertices in the trees of Equation (2.2), we get

\[ \langle x_3, x_4, \ldots, x_n \rangle \cong \langle x_1, x_2, \ldots, x_{n-2} \rangle \]

Hence \( u \not\equiv v \). (2.3)

Assumption (i) and the Equation (2.3) imply \( u \sim v \), by Note 1.2.5, a contradiction to our assumption that the lemma is not true for \( T \).

**Case 2**: Let \( d = 1 \).

Hence \( n = 2 \) and we are through if we show that \( x_1 \not\equiv x_2 \). Now \( u \not\equiv v \) implies \( \{ x_1^0, x_2 \} \not\cong \{ x_2^0, x_1 \} \)

Hence \( x_1 \not\equiv x_2 \) and \( x_1^0 \not\equiv x_2^0 \) (2.4)
Condition (ii) implies
\[ \{ n_1(x_1^1, x_2^0), n_2(x_2^2, x_1^1), \ldots, n_k(x_k^k, x_1^k) \} \]
\[ \cong \{ n_1(x_2^1, x_1^0), n_2(x_2^2, x_1^1), \ldots, n_k(x_k^k, x_2^k) \}. \]

Since \( x_1^0 \cong x_2^0 \), we get
\[ \{ (n_1+n_2)x_1^1, (n_2+n_3)x_2^2, \ldots, (n_{k-1}+n_k)x_1^{k-1}, n_kx_1^k \} \]
\[ \cong \{ (n_1+n_2)x_2^1, (n_2+n_3)x_1^2, \ldots, (n_{k-1}+n_k)x_2^{k-1}, n_kx_2^k \}. \]
\[ \ldots \text{(2.5)} \]

Let \( f : X_2 \rightarrow X_1 \) be an isomorphism.
Consider the vertices \( x_1 \) and \( f(x_2) \) in \( X_1 \). These vertices satisfy the assumptions of the lemma because of Equations (2.4) and (2.5). The minimality of \( T \) implies that \( x_1 \sim f(x_2) \) in \( X_1 \).

Let \( g : X_1 \rightarrow X_1 \) be an automorphism of \( X_1 \) such that \( g(f(x_2)) = x_1 \). Now \( g \cdot f \) is an isomorphism from \( X_2 \) to \( X_1 \) taking \( x_2 \) to \( x_1 \). That is, \( X_1 \cong X_2 \) and the proof is complete.

The proofs for the next two lemmas are similar to that of the previous lemma and are hence omitted.
Lemma 2.3.2: Let $u, v \in V(T)$. Let
\[
\{ n_0^R u, n_1^R v, n_2^R u, \ldots, n_k^R x \} \equiv \{ n_0^R v, n_1^R u, n_2^R v, \ldots, n_k^R y \}
\]
for some $n_0, n_1 \geq 1, k \geq 1, n_i \geq 0$ and $x$ and $y$ take the value $u$ or $v$ depending upon $k$ being odd or even. Then $u \sim v$. \(\square\)

Lemma 2.3.3: Let $u, v \in V(T)$. Let
\begin{align*}
\text{(i) } & u \not\sim^1 v \quad \text{and} \\
\text{(ii) } & \{ n_0^R u, n_2^R v, n_3^R u, \ldots, n_k^R x \} \equiv \\
& \{ n_0^R v, n_2^R u, n_3^R v, \ldots, n_k^R y \}
\end{align*}
for some $n_0 \geq 1, n_1 \geq 0, k \geq 0$ and $x$ and $y$ take the value $u$ or $v$ depending upon $k$ being odd or even. Then $u \sim v$. \(\square\)

The following corollaries follow directly from the lemmas.

Corollary 2.3.4: $u \not\sim^0 v$ and $\{ m R_1^u, R_2^v \} \equiv \{ m R_v^1, R_u^2 \}$, for some $m \geq 1$ imply $u \sim v$. \(\square\)

Corollary 2.3.5: $\{ m R_0^u, R_1^v \} \equiv \{ m R_v^0, R_u^1 \}$, for some $m \geq 1$, implies $u \sim v$. \(\square\)
Corollary 2.3.6: \( u \phi^1 v \) and \( \{ R^0_u, R^2_v \} \) imply \( u \sim v \).

2.4 ZERO AND ONE

First we prove a stronger version of Theorem 5.1 of [8] which states that \( u \phi^0 v \) and \( u \phi^1 v \) imply \( u \sim v \).

Theorem 2.4.1: If \( u \phi^0 v \) and \( u \phi^1 v \) then \( u \sim v \).

Proof: If \( d = d(u,v) \geq 2 \), then by Note 1.2.5, we get \( u \sim v \). Hence let \( d = 1 \).

Suppose \( u \not\sim v \).

By Note 1.2.2, we get

\[ T = \langle Y_1, Y_2, \ldots, Y_t, Y_{t+1}, \ldots, Y_{2t} \rangle \text{ where } t > 1, \]

\( Y_1 \leq Y_i \) for \( 1 \leq i \leq t \), \( u = Y_t \) and \( v = Y_{t+1} \).

We consider two cases depending upon the value of \( t \).

Case 1: Let \( t = 2 \).

Here \( T = \langle Y_1, Y_2, Y_3, Y_4 \rangle \), \( Y_1 \leq Y_3 \), \( Y_2 \leq Y_4 \), \( u = Y_2 \) and \( v = Y_3 \).

\( u \phi^1 v \Rightarrow \{ Y^1_2, Y^O_1, Y^O_3, Y_4 \} \leq \{ Y^O_3, Y^O_4, Y^O_2, Y_1 \} \)

\( \Rightarrow \{ Y^1_2, Y^O_1, Y^O_3, Y_2 \} \leq \{ Y^O_1, Y^O_2, Y^O_2, Y_1 \} \)
\[ Y_1 \leq Y_2 \text{ and } \{ Y_1^1, 2Y_1^0 \} \leq \{ Y_2^1, 2Y_2^0 \} \]
\[ Y_1 \not\leq Y_2 \text{ by Corollary 2.3.5} \]
\[ u \sim v, \text{ a contradiction.} \]

**Case 2:** Let \( t > 3 \). \( u \not\leq v \) implies
\[ \{ Y_t^1, Y_{t-1}^0, Y_{t-2}^0, \ldots, Y_t^1 \} \]
\[ \cong \{ Y_t^1, Y_t^0, Y_3^0, Y_4, \ldots, Y_t^0, Y_{t-1}^1 \} \]

Considering the largest and the next largest components on both the sides, we get
\[ <Y_2, Y_3, \ldots, Y_t> \cong <Y_1, Y_2, \ldots, Y_{t-1}> \]
and \[ <Y_3, Y_4, \ldots, Y_t> \cong <Y_1, Y_2, \ldots, Y_{t-2}> \quad (2.6) \]

Let \( T_1 = <Y_1, Y_2, \ldots, Y_t> \). Then the Equation (2.6) implies \( Y_1 \not\sim_M Y_t \) and \( Y_1 \not\leq_M Y_t \) in \( T_1 \). Hence \( Y_1 \sim Y_t \)
in \( T_1 \), by Note 1.2.5, since \( d(Y_1, Y_t) \geq 2 \). This implies \( u \sim v \) in \( T \), a contradiction. \( \square \)

Consider the rooted trees \((W_i, w_i), i = 1,2\) defined in Chapter 1.

**Note 2.4.2:** The tree
\[ T = <W_2, (W_1, w_1 = u), (W_2, w_2 = v), W_1> \]
shows that \( u \not\leq_M v \) and \( u \not\leq_M v \) need not imply \( u \sim v \).
2.5 ONE AND TWO

Theorem 2.5.1: Let $d = d(u, v) \neq 1$ or $3$. Then $u \not\sim v$ and $u \not\sim v$ imply $u \sim v$ ($u$ and $v$ are vertices of the underlying tree $T$ and $\sim$ is the same as $\sim_1$).

Proof:

Case 1: Let $d = 2$.

Let $T = \langle X_1, X_2, X_3 \rangle$, $x_1 = u$ and $x_3 = v$.

$u \not\sim v \implies \{ x_1, x_2, x_3 \} \not\cong \{ x_1, x_2, x_1 \}$

$\implies x_1 \not\cong x_3$ and $x_1 \not\cong x_3$ \hspace{1cm} (2.7)

$u \not\sim v \implies \{ x_1, x_2, x_3 \} \not\cong \{ x_2, x_2, x_1 \}$

$\implies \{ x_2, x_3 \} \not\cong \{ x_2, x_1 \}$ \hspace{1cm} (2.8)

Equations (2.7) and (2.8) imply $x_1 \not\cong x_3$, by Corollary 2.3.6. Hence $u \sim v$.

Case 2: Let $d \geq 4$.

Let $T = \langle X_1, X_2, \ldots, X_n \rangle$ where $x_1 = u$, $x_n = v$ and $n \geq 5$.

$u \not\sim v \implies \{ x_1, x_2, X_3, X_4, \ldots, X_n \} \not\cong \{ x_1, x_2, X_3, X_4, \ldots, X_n \}$

$\implies \{ x_n, x_{n-1}, X_1, X_2, \ldots, X_{n-2} \}$
Since \( n \geq 5 \), we have \( d(x_2, x_{n-1}) \geq 2 \). Now Equations (2.9), (2.10) and Note 1.2.5, imply \( x_2 \sim x_{n-1} \). Hence by Note 1.2.1,
\[
(\langle x_1, x_2 \rangle, x_2) \models (\langle x_{n-1}, x_n \rangle, x_{n-1}).
\]

If \( x_1 \not\models x_n \) then \( x_2 \not\models x_{n-1} \) and hence \( u \sim v \). Suppose that \( x_1 \not\models x_n \). This implies the existence of a rooted tree \((H, h)\) such that
\[
x_2 \models (\langle H, x_n \rangle, h) \text{ and } x_{n-1} \not\models (\langle H, x_1 \rangle, h).
\]
(See Figure 2.5).

\[
x_2 \sim x_{n-1} \Rightarrow x_3 \models x_{n-2}
\]
(2.11)

\[
u \not\models \quad \Rightarrow \quad \{x_1^1, H^0, x_n, \langle x_3, x_4, \ldots, x_n \rangle\} \models \{x_1^1, H^0, x_1, \langle x_1, x_2, \ldots, x_{n-2} \rangle\}.
\]

\[
\Rightarrow \quad x_1 \models x_n \quad \text{and} \quad x_1^1 \models x_n^1
\]
(2.12)
Now Equations (2.12) and (2.13) imply $X_1 \not\subseteq X_n$, by Corollary 2.3.6. This contradiction completes the proof.

The examples in Figure 2.6 show that when $d = d(u, v) = 1$ or $3$, $u \not\varphi^1 v$ and $u \not\varphi^2 v$ need not imply $u \sim v$.

That theorem 2.5.1 cannot be strengthened further is shown by the following two examples.

Let $P_{20}$ denote the path of length 20 rooted at one of the end vertices.

**Example 2.5.2:** The trees in Figure 2.7 show that $u \not\varphi^1 v$ and $u \not\varphi^2 v$ do not imply $u \sim v$. In the first case $d = 2$ and in the second $d \geq 4$.

**Example 2.5.3:** The vertices $u$ and $v$ of the tree in Figure 2.8 satisfy $u \varphi^1 v$ and $u \varphi^2 v$ but $u \not\varphi v$.
(The trees $M_1$ and $M_2$ are as defined in Chapter 1).
2.6 ZERO AND TWO

Let \( T = \langle X_1, X_2, \ldots, X_n \rangle \).

Lemma 2.6.1: If \( n \) is an even integer greater than two, \( X_1 \not\sim X_{i+2} \) for \( 1 \leq i \leq n-2 \) and \( x_1 \not\sim x_n \)
then \( x_1 \not\sim x_2 \).

Proof: If \( x_1 \not\approx x_2 \) then \( |X_3| < |X_1| \) by Note 1.2.3. This implies \( x_1 \sim x_n \) and hence \( x_1 \not\sim x_2 \).

Lemma 2.6.2: Let \( n \geq 6 \), \( x_2 \not\sim x_{n-1}, x_2 \not\sim x_{n-1} \), \( X_1 \not\sim X_m \) and \( X_2 \not\sim X_{m+1} \) for some \( m \), \( 3 \leq m \leq n-3 \). Then \( X_1 \not\sim X_{i+2} \) for \( 1 \leq i \leq n-4 \).

Proof: \( x_2 \not\sim x_{n-1} \) and \( x_2 \not\sim x_{n-1} \) imply

\[
T = \langle Y_1, Y_2, \ldots, Y_{(k+1)t+1} \rangle \text{ where } t > 1, \ 1 \leq i \leq t,
\]

\[
k \geq 1, \ x_2 = Y_t, \ x_{n-1} = Y_{kt+1} \text{ and } Y_j \not\sim Y_{rt+j}, \ 1 \leq j \leq t,
\]

\[
r \geq 1, \text{ by Note 1.2.2.}
\]

Hence the trees \( X_m \) and \( X_{m+1} \) are isomorphic to some \( Y_s \) and \( Y_{s+1} \) respectively where \( 1 \leq s \leq t \). Hence

\[
\langle Y_s, Y_{s+1} \rangle \cong \langle X_m, X_{m+1} \rangle \cong \langle X_1, X_2 \rangle \cong \langle Y_1, Y_2, \ldots, Y_t \rangle.
\]

This implies \( t = 2, \ x_1 \not\sim Y_1 \) and \( x_2 \not\sim Y_2 \) or \( x_1 \not\sim x_2 \not\sim Y_1 \not\sim Y_2 \). Hence the result.
Lemma 2.6.3: Let $T = \langle Y_1, Y_2, \ldots, Y_{(k+1)t+1-1} \rangle$ with $k \geq 1$, $t > 1$, $1 \leq i \leq t$, $Y_j \neq Y_{r+t+j}$, $1 \leq j \leq t$, $r \geq 1$ and $T_1 = \langle Y_1, Y_2, \ldots, Y_{kt+i-1} \rangle$. If $Y_2 \sim Y_{kt+i-2}$ in $T_1$, then $Y_t \sim Y_{kt+i}$ in $T$.

Proof: Since $Y_2 \sim Y_{kt+i-2}$ in $T_1$, there is an automorphism of $T_1$ interchanging $Y_2$ and $Y_{kt+i-2}$. Using this automorphism it is easy to show that $Y_j \neq Y_{t+i-j}$ for $j < 1$ and $Y_j \neq Y_{t+i-j}$ for $1 \leq j \leq t$. This implies that $Y_t \sim Y_{kt+i}$ in $T$.

Theorem 2.6.4: If $d = d(u, v)$ is an even integer greater than two, then $u \overset{0}{\sim} v$ and $u \overset{2}{\sim} v$ imply $u \sim v$.

Proof: The proof is the same as that of Case 3 and Case 4 of the next theorem. The only difference is that in Case 3, when $t$ is even we have to use the assumption that $d$ is even, which will imply that the $Y_i$'s at $u$ and $v$ are both equal to $Y_2$.

The proof of the following theorem involves the detailed study of the subtrees involved in forming the tree $T$, and the structures of the subtrees are different in different cases that arise. Hence consideration of many cases and subcases become unavoidable in the proof. The same applies to the proof of two more theorems.
Theorem 2.6.5: If \( u \not\equiv v \) and \( u \not\equiv v \) then \( u \sim v \).

Proof: Suppose \( u \not\equiv v \). Then by Note 1.2.4,
\[
T = \langle Y_1, Y_2, \ldots, Y_{(k+1)t+i-1} \rangle \quad \text{where} \quad t > 1, \quad Y_t = u,
\]
\[
y_{kt+1} = v, \quad k \geq 1, \quad 1 \leq i \leq t \quad \text{and} \quad Y_j \not\equiv Y_{rt+j},
\]
\[
1 \leq j \leq t \quad \text{and} \quad r \geq 1.
\]

Suppose \( t = 2 \). We see that \( i = 2 \) implies \( u \sim v \) and hence let \( i = 1 \). Here
\[
T = \langle Y_1, (Y_2, Y_2 = u), Y_1, Y_2, \ldots, Y_2, (Y_1, Y_1 = v), Y_2 \rangle.
\]

Now \( u \not\equiv v \) implies \( \{ Y_2, Y_1 \} \not\equiv \{ Y_1, Y_2 \} \) and hence \( Y_1 \not\equiv Y_2 \) and \( Y_1^0 \not\equiv Y_2^0 \). Also \( u \not\equiv v \) implies
\[
\{ Y_1^0, Y_2^0, Y_1^1, Y_2^1 \} \not\equiv \{ Y_1^1, Y_2^1, Y_1^0, Y_2^0 \} \quad \text{and} \quad \text{This in turn implies} \quad \{ Y_2^1, 2Y_1^0 \} \not\equiv \{ Y_2^0, 2Y_1^1 \} \quad \text{since} \quad Y_1^0 \not\equiv Y_2^0.
\]

Now Corollary 2.3.4 implies \( Y_1 \not\equiv Y_2 \) and \( u \sim v \), a contradiction. Hence let \( t \geq 3 \).

We consider five cases depending upon \( d = d(u, v) \).

Case 1: Let \( d = 1 \).

Let \( T = \langle X_1, X_2 \rangle \) where \( x_1 = u \) and \( x_2 = v \).

Now \( u \not\equiv v \) and \( u \not\equiv v \) imply \( X_1 \not\equiv X_2 \), \( X_1^0 \not\equiv X_2^0 \) and \( \{ x_1^0, x_2^1 \} \not\equiv \{ x_2^0, x_1^1 \} \). Using Corollary 2.3.4, we get...
\( x_1 \not\cong x_2 \) and hence \( u \sim v \), a contradiction.

**Case 2:** Let \( d = 3 \).

Here \( v = y_{t+3} \) and \( T = \langle y_1, y_2, \ldots, y_{2t+2} \rangle \)

\[ u \phi^0 v = \langle y_1, y_2, \ldots, y_{t-1} \rangle \cong \langle y_{t+4}, y_{t+5}, \ldots, y_{2t+2} \rangle \]

\[ \cong \langle y_4, y_5, \ldots, y_{t+2} \rangle \text{ and } \]

\[ u \phi^2 v = \langle y_1, y_2, \ldots, y_t \rangle \cong \langle y_{t+3}, y_{t+4}, \ldots, y_{2t+2} \rangle \]

\[ \cong \langle y_3, y_4, \ldots, y_{t+2} \rangle \].

Hence in \( T_1 = \langle y_1, y_2, \ldots, y_{t+2} \rangle \), \( y_2 \) and \( y_{t+1} \) satisfy \( \phi^0_M \) and \( \phi^1_M \). Since \( t \geq 3 \), we get \( d(y_2, y_{t+1}) \geq 2 \) and this implies \( y_2 \sim y_{t+1} \) in \( T_1 \) by Note 1.2.5. Now Lemma 2.6.3 implies \( u \sim v \) in \( T \), a contradiction.

**Case 3:** Let \( d \geq 5 \).

Since \( u \phi^2 v \) implies \( u \phi^2 v \), we get

\[ \langle y_1, y_2, \ldots, y_{k+1-3} \rangle \cong \langle y_{t+3}, y_{t+4}, \ldots, y_{(k+1)t+1-1} \rangle \]

\[ \cong \langle y_3, y_4, \ldots, y_{k+1-1} \rangle \].

Hence in \( T_1 = \langle y_1, y_2, \ldots, y_{k+1-1} \rangle \) we have \( y_2 \not\phi^0_M y_{k+1-2} \).

By Lemma 2.6.3, \( y_2 \not\sim y_{k+1-2} \) in \( T_1 \), since otherwise \( u \sim v \) in \( T \).
Since \( d \geq 5 \), the vertices \( Y_{t+1} \) and \( Y_{t+2} \)
are different from \( Y_{kt+1-2} \). We know \( Y_{t+1} \neq Y_1 \) and
\( Y_{t+2} \neq Y_2 \). By Lemma 2.6.2, \( Y_1 \) and \( Y_2 \) occur alternately
in the path \( (Y_1, Y_2, \ldots, Y_{kt+1-3}) \).

If \( t \) is even, \( Y_t \neq Y_2 \). Since \( u \not\sim v \) in \( T \),
\( Y_{kt+1} \) should be the same as \( Y_1 \). Now \( u \not\sim v \) implies
\[<Y_1, Y_2, \ldots, Y_{t-1}> \neq <Y_{kt+1}, Y_{kt+1+2}, \ldots, Y_{(k+1)t+1-1}>\]
\[\neq <Y_2, Y_3, \ldots, Y_{t-1}>.\]

Now Lemma 2.6.1 implies \( Y_1 \neq Y_2 \). Hence \( u \sim v \), and a
contradiction.

If \( t \) is odd then \( Y_t \neq Y_1 \) and since
\( Y_2 \neq Y_{t+1} \neq Y_1 \) we have \( Y_1 \neq Y_2 \). Thus \( Y_a \neq Y_b \) for
\( 1 \leq a, b \leq t \) and this gives \( u \sim v \) in \( T \), a contradiction.

**Case 4:** Let \( d = 4 \).

Here \( v = Y_{t+4} \).

As in Case 3, we get \( Y_2 \not\sim Y_{t+2} \) in \( T_1 \) and
\( Y_2 \not\sim Y_{t+2} \) in \( T_1 \) where \( T_1 = <Y_1, Y_2, \ldots, Y_{t+3}> \).
By Note 1.2.2, we can write $T_1$ as

$$\langle Z_1, Z_2, \ldots, Z_n, Z_{n+1}, \ldots, Z_{(s+1)n+1-j} \rangle$$
where $n > 1$,

$1 \leq j \leq n$, $Y_2 = Z_n, Y_{t+2} = Z_{sn+j}$, $s \geq 1$ and

$Z_p \not\leq Z_{rn+p}$ for $r \geq 1$, $1 \leq p \leq n$. Clearly

$$(\langle Z_1, Z_2, \ldots, Z_n \rangle, z_n) \not\leq (\langle Y_1, Y_2 \rangle, y_2)$$
and

$$(\langle Z_j, Z_{j+1}, \ldots, Z_n, Z_1, Z_2, \ldots, Z_{j-1} \rangle, z_j) \not\leq (\langle Y_{t+2}, Y_{t+3} \rangle, Y_{t+2}).$$

Hence $|\langle Y_1, Y_2 \rangle| = |\langle Y_{t+2}, Y_{t+3} \rangle|$.

Suppose $(\langle Z_1, Z_2, \ldots, Z_{n-1} \rangle, z_{n-1}) \not\leq Y_1$.

This implies $Z_n \not\leq Y_2 \not\leq Y_{t+2}$ (See Figure 2.9).

We know $Y_{t+3} \not\leq Y_3 \not\leq Z_{n+1} \not\leq Z_1$. Hence

$$\sum_{i=1}^{n} |Z_i| = |\langle Y_1, Y_2 \rangle| = |\langle Y_{t+1}, Y_{t+2} \rangle| = |\langle Z_n, Z_1 \rangle|.$$ 

This implies $n = 2$. We observe $Y_1 \not\leq Z_1$, $Y_2 \not\leq Z_2$,

$Y_{t+2} \not\leq Z_2$, $Y_{t+3} \not\leq Z_1$. Hence

$$(\langle Y_1, Y_2 \rangle, Y_2) \not\leq (\langle Y_{t+2}, Y_{t+3} \rangle, Y_{t+3}).$$

By Note 1.2.1, we get $Y_2 \sim Y_{t+2}$ in $T_1$, a contradiction.

Hence $(\langle Z_1, Z_2, \ldots, Z_{n-1} \rangle, z_{n-1}) \not\leq Y_1$. 

This gives $<Z_1, Z_2, ..., Z_{n-1}> \subseteq Y_2$, and the existence of a tree $(H, h)$ such that

$$Y_2 \trianglerighteq <Z_1, Z_2, ..., Z_{n-1}, H>, \quad h=Z_n$$

and

$$Z_n \trianglerighteq (<Y, H>, h) \quad (\text{See Figure 2.10}). \quad (2.14)$$

Clearly

$$Y_{t+3} \trianglerighteq Y_3 \trianglerighteq Z_{n+1} \trianglerighteq Z_1 \quad (2.15)$$

Also

$$<Y_{t+2}, Y_{t+3}, Y_{t+2}> \trianglerighteq \quad (Z_{sn+j}, Z_{sn+j+d}, ..., Z_{(s+1)n+j-1}, Z_{sn+j}).$$

Hence

$$<Z_j, Z_{j+1}, ..., Z_n, Z_1, Z_2, ..., Z_{j-1}, Z_j> \trianglerighteq \quad (2.16)$$

We consider three subcases depending upon $j$.

**Subcase 4.1:** Let $j = 1$.

Here Equation (2.16) becomes

$$<Z_1, Z_2, ..., Z_n, z_1> \trianglerighteq <Z_1, Z_2, ..., Z_{n-1}, H, Z_1, h>.$$

This isomorphism is impossible (since the branch at $h$ of the RHS (Right hand side) containing only $Z_1$ cannot be mapped inside the $Z_1$ at $z_1$ of the LHS, we get $Z_1 \trianglerighteq <Z_1, Z_2, ..., Z_{n-1}, H>, h)$, a contradiction).
Hence \( j = 1 \) is not possible

**Subcase 4.2:** Let \( 2 \leq j \leq n-1 \).

Since \( Z_j \subseteq \langle Z_1, Z_2, \ldots, Z_{n-1} \rangle \), from Equation (2.16) we get

\[
Z_j \cong \langle H, Z_1 \rangle, \text{ and } \langle Z_{j+1}, Z_{j+2}, \ldots, Z_n, Z_1, Z_2, \ldots, Z_{j-1}, Z_{j+1} \rangle \cong \langle Z_{n-1}, Z_{n-2}, \ldots, Z_1, z_{n-1} \rangle. \tag{2.17}
\]

If \( j = n-1 \), Equation (2.18) can be written as

\[
\langle Z_n, Z_1, \ldots, Z_{j-1}, z_n \rangle \cong \langle Z_j, Z_{j-1}, \ldots, Z_1, z_j \rangle. \tag{2.19}
\]

If \( j = n-2 \), Equation (2.18) becomes

\[
\langle Z_{n-1}, Z_n, Z_1, \ldots, Z_{j-1}, z_{n-1} \rangle \cong \langle Z_{n-1}, Z_{n-2}, \ldots, Z_1, z_{n-1} \rangle.
\]

Hence

\[
\langle Z_n, Z_1, Z_2, \ldots, Z_{j-1}, z_n \rangle \cong \langle Z_j, Z_{j-1}, \ldots, Z_1, z_j \rangle,
\]

which is the same as Equation (2.19).

If \( j = n-3 \), Equation (2.18) becomes

\[
\langle Z_{n-2}, Z_{n-1}, Z_n, Z_1, Z_2, \ldots, Z_{j-1}, z_{n-2} \rangle \cong \langle Z_{n-1}, Z_{n-2}, \ldots, Z_1, z_{n-1} \rangle.
\]
Since \( Z_{n-2} \neq \langle Z_{n-2}, Z_{n-3}, \ldots, Z_1 \rangle \), we get
\[
Z_{n-2} \neq Z_{n-1}
\]
and hence
\[
(Z_n, Z_1, \ldots, Z_{j-1}, z_n) \neq (Z_j, Z_{j-1}, \ldots, Z_1, z_j),
\]
which is again the same as Equation (2.19).

Similar arguments hold for \( 2 \leq j \leq n-4 \) too and all the cases lead to Equation (2.19).

Using Equations (2.14) and (2.17) in Equation (2.19) we get
\[
(\langle Y_1, H, Z_1, Z_2, \ldots, Z_{j-1}, h \rangle) \neq (\langle Z_1, H, Z_{j-1}, Z_{j-2}, \ldots, Z_1, h \rangle).
\]
This gives
\[
Y_1 \neq Z_1.
\]

Using Equations (2.15) and (2.20) we get
\[
Y_1 \neq Y_3 \neq Y_{t+3}.
\]
Hence \( \langle Y_1, Y_2 \rangle, Y_2 \neq \langle Y_{t+2}, Y_{t+3} \rangle, Y_{t+2} \). By Note 1.2.1, \( y_2 \sim y_{t+2} \) in \( T_1 \), a contradiction.

**Subcase 4.3:** Let \( j = n \)
In this case, Equation (2.16) becomes

\[(<Z_n, Z_1, Z_2, \ldots, Z_{n-1}>, z_n) \not\isom (<Z_1, Z_2, \ldots, Z_{n-1}, H, Z_1>, h).\]

Using Equation (2.14) in this, we get

\[(<Y_1, H, Z_1, Z_2, \ldots, Z_{n-1}>, h) \not\isom (<Z_1, Z_2, \ldots, Z_{n-1}, H, Z_1>, h)\]

which implies \(Y_1 \not\isom Z_1\) and as in Subcase 4.2 we arrive at a contradiction.

**Case 5:** Let \(d = 2\).

Let \(T = \langle(X_1, x_1 = u), X_2, (X_3, x_3 = v)\rangle\) be a minimal tree where the theorem fails for \(u\) and \(v\). Now \(u \phi^0 v\) implies \(\{x_1^0, \langle X_2, X_3\rangle\} \not\isom \{x_3^0, \langle X_1, X_2\rangle\}\). This gives \(\langle X_1, X_2\rangle \not\isom \langle X_2, X_3\rangle\) and \(x_1^0 \not\isom x_3^0\) \hspace{1cm} (2.21)

Also \(u \phi^2 v\) and Equation (2.21) gives

\[x_1^2 \isom x_3^2.\] \hspace{1cm} (2.22)

Let \(A = \langle X_1, X_2\rangle\) and \(B = \langle X_2, X_3\rangle\).

Let \(f: A \longrightarrow B\) be an isomorphism as assured by Equation (2.21). Equations (2.21) and (2.22) imply that \(x_3\) and \(f(x_1)\) in \(B\) satisfy \(x_3 \phi^0 f(x_1)\) and \(x_3 \phi^1 f(x_1)\). If \(d_B(x_3, f(x_1)) = 2\), by the minimality assumption on \(T\), \(x_3 f(x_1)\) in \(B\); if \(d_B(x_3, f(x_1)) \neq 2\) then by the
previous cases considered we get \( x_3 \sim f(x_1) \) in \( B \).

Hence there is an isomorphism \( g : A \to B \) such that \( g(x_1) = x_3 \). This gives \( x_1 \not\sim x_3 \) and hence \( u \sim v \) in \( T \), a contradiction.

This completes the proof of the theorem.

We now show that Theorem 2.6.4 cannot be strengthened for odd numbers.

**Example 2.6.6:** That \( u \not\sim v \) and \( u \not\sim v \) do not imply \( u \sim v \) is shown by the tree

\[
T = \langle W_1, (W_2, w_2 = u), W_1, W_2, \ldots, W_2, (W_1, w_1 = v), W_2 \rangle.
\]

Here \( d > 1 \).

**Example 2.6.7:** That \( u \not\sim v \) and \( u \not\sim v \) do not imply \( u \sim v \) is shown by the tree,

\[
T = \langle S_1, (S_2, s_2 = u), S_1, S_2, S_1, \ldots, S_2, (S_1, s_1 = v), S_2 \rangle.
\]

Here \( d > 1 \).

It can be easily verified that in the tree

\[
T = \langle (S_1, s_1 = u), (S_2, s_2 = v) \rangle,
\]

we have \( u \not\sim v \) for all \( k > 1 \) and \( u \not\sim v \). This answers a question raised by Corneil et al. [8]. Hence \( u \not\sim v \) and \( u \not\sim v \) do not
imply $u \sim v$ if $i, j \geq 1$.

In [7], it has been shown that $u \varphi^0 v$ and $u \varphi^k v$ do not imply $u \sim v$, where $k$ is any integer $\geq 3$. Hence $\{(i,j) \mid u \varphi^i v$ and $u \varphi^j v$ imply $u \sim v\} = \{(0,1), (0,2)\}$.

The combination $(1,2)$ can be included in this provided $d = d(u, v) \neq 1$ or 3.

2.7 PSEUDOSIMILARITY AND EDGES

In this section we prove the edge versions of the results similar to those proved in the case of vertices. In [13], it is shown that many of the propositions concerning point deleted subgraph also hold for line deleted subgraph.

**Note 2.7.1** Let $T = \langle x_1, x_2, \ldots, x_n \rangle$, $n \geq 3$, $x = (x_1, x_2)$ and $y = (x_{n-1}, x_n)$. Then

1. $x_1 \sim x_n$ iff $x \sim y$, and,
2. $x_1 \varphi_k^k x_n$ iff $x \varphi_k^k y$, for $0 \leq k \leq n-2$.

**Proof**

1. If $x_1 \sim x_n$, then there is an automorphism of $T$ interchanging $x_1$ and $x_n$ [1]. This implies $x \sim y$.

   Conversely, if $x \sim y$, then there is an automorphism of $T$ sending $x$ to $y$ and hence $x_1 \sim x_n$.
(ii) $T - F^k_{x_1} = \{ x_1^k, x_2^{k-1}, \ldots, x_{k+1}^O, <x_{k+2}, x_{k+3}, \ldots, x_n> \}$, 

$T - F^k_{x_n} = \{ x_n^k, x_{n-1}^k, \ldots, x_{n-k}^O, <x_1, x_2, \ldots, x_{n-k}^1> \}$, 

$T - F^O_{x} = \{ x_1, <x_2, x_3, \ldots, x_n> \}$, 

$T - F^O_{y} = \{ x_n, <x_1, x_2, \ldots, x_{n-1}> \}$ and 

for $1 \leq k \leq n-2$, 

$T - F^k_{x} = \{ x_1^{k-1}, x_2^{k-1}, \ldots, x_{k+1}^O, <x_{k+2}, x_{k+3}, \ldots, x_n> \} \cup \{\text{some isolated vertices}\}$ 

$T - F^k_{y} = \{ x_n^{k-1}, x_{n-1}^{k-1}, \ldots, x_{n-k}^O, <x_1, x_2, \ldots, x_{n-k}^1> \} \cup \{\text{some isolated vertices}\}$.

Comparing $T - F^k_{x_1}$ and $T - F^k_{x_n}$, we see that the component with maximum number of vertices cannot be in $x_1^k$, since $|<x_1, x_2, \ldots, x_{n-k}^1>| > |x_1^k|$ (note that $n-k-1 \geq 1$). Hence it is in $T - F^k_{x_1} - x_1^k$. Similarly such a maximum component cannot be in $x_n^k$, and hence it is in $T - F^k_{x_n} - x_n^k$. Similar observations hold for the forests $x_1^{k-1}$ and $x_n^{k-1}$ in $T - F^k_{x}$ and $T - F^k_{y}$.

Since $T - F^k_{x_1} - x_1^k = T - F^k_{x} - x_1^{k-1} = \{\text{some isolated vertices}\}$
and $T - F^k_{x_n} - X^k_{x_n} \equiv T - F^k_y - X^{k-1}_{x_n}$ - [some isolated vertices] we get $x_1 \phi^k_M x_n$ iff $x \phi^k_M y$, as the required components with maximum number of vertices should be within these forests.

**Theorem 2.7.2:** Let $x, y \in E(T)$. Then $x \phi^O_M y$ and $x \phi^1_M y$ imply $x \sim_T y$.

**Proof:** Let $T = <X_1, X_2, \ldots, X_n>$, $(x_1, x_2) = x$, $(x_{n-1}, x_n) = y$ and $n \geq 3$. Then $x \phi^O_M y$ and $x \phi^1_M y$ imply $x_1 \phi^O_M x_n$ and $x_1 \phi^1_M x_n$ in $T$ by Note 2.7.1. Since $n \geq 3$, $d(x_1, x_n) \geq 2$. Hence by Note 1.2.5, $x_1 \sim x_n$. By Note 2.7.1, we have $x \sim y$.

**Theorem 2.7.3:** Let $x$ and $y$ be adjacent edges of a tree $T$. Then $T - x \leq T - y$ iff $x \sim y$.

**Proof:** One way is obvious.

Hence let us assume $T - x \leq T - y$.

Let $T = <X_1, X_2, X_3>$, $(x_1, x_2) = x$ and $(x_2, x_3) = y$.

Now $T - x \leq T - y$ implies $\{X_1, <X_2, X_3>\} \leq \{X_3, <X_1, X_2>\}$. This gives $X_1 \not\leq X_3$ and $<X_2, X_3> \leq \ldots$
\begin{align*}
\langle X_1, X_2 \rangle. \quad \text{This ensures } x_1 \not\sim_M^O x_3 \text{ and } x_1 \not\sim_M^1 x_3. \\
\text{Here } d(x_1, x_3) = 2. \quad \text{Hence by Note 1.2.5, } x_1 \sim x_3. \\
\text{By Note 2.7.1, } \sim y.
\end{align*}

\textbf{Theorem 2.7.4:} \quad \text{Let } x, y \in E(T) \text{ and } d(x, y) \text{ be odd. Then } x \not\sim^O y \text{ and } x \not\sim^2 y \implies x \sim y.

\textbf{Proof:} \quad \text{If } d(x, y) = 1 \text{ then Theorem 2.7.3 gives } x \sim y.

\text{Let } d(x, y) \text{ be odd and } \geq 3.

\text{Let } T = \langle X_1, X_2, \ldots, X_n \rangle, (x_1, x_2) = x \text{ and } (x_{n-1}, x_n) = y. \quad \text{Since } d(x, y) \text{ is odd and } \geq 3, \text{ we get that } d(x_1, x_n) \text{ is even and } \geq 4. \quad \text{Now } x \not\sim^O y \text{ and } x \not\sim^2 y \implies x_1 \not\sim_M^O x_n \text{ and } x_1 \not\sim_M^2 x_n. \quad \text{By Theorem 2.6.4, } x_1 \sim x_n. \quad \text{Hence } x \sim y, \text{ by Note 2.7.1.}

\textbf{Example 2.7.5:} \quad \text{There exists a tree } T \text{ with dissimilar edges } x \text{ and } y \text{ such that } x \not\sim^O y, x \not\sim^2 y \text{ and such that } d(x, y) = d \text{ is any given even integer.}

\text{Consider the trees shown in Figure 2.11.}

\textbf{Theorem 2.7.6:} \quad \text{Let } x, y \in E(T). \quad \text{Then } x \not\sim^1 y \text{ and } x \not\sim^2 y \implies x \sim y \text{ if } d(x, y) \neq 2.
Proof: Let \( T = \langle X_1, X_2, \ldots, X_n \rangle \), \( x = (x_1, x_2) \) and \( y = (x_{n-1}, x_n) \). Then \( d(x, y) \neq 2 \) implies \( n \neq 4 \).

Case 1: Let \( n = 3 \).

\[ x \not\in y \implies \{ x_1^O, x_2^O, x_3^O \} = \{ x_1, x_2, x_3^O \}. \]

Hence \( x_1 = x_3 \) and \( x_1^O = x_3^O \). Now \( x \not\in y \implies \{ x_1^1, x_2^1, x_3^O \} = \{ x_1^O, x_2^1, x_3^1 \} \). Since \( x_1^O = x_3^O \), we have \( x_1^1 = x_3^1 \).

By Note 1.2.6, \( x_1 \not\in x_3 \). Hence \( x \not\sim y \).

Case 2: Let \( n > 5 \).

\[ x \not\in y \implies x_2 \not\in_M x_{n-1} \], proceeding as in Note 2.7.1, we can see that the components with the maximum number of vertices are \( \langle X_3, X_4, \ldots, X_n \rangle \) \( \not\in \langle X_1, X_2, \ldots, X_{n-2} \rangle \) and similarly \( x \not\in y \implies x_2 \not\in_M x_{n-1} \). Since \( n > 5 \) we have \( d(x_2, x_{n-1}) > 2 \).

Hence by Note 1.2.5, \( x_2 \not\sim x_{n-1} \).

This gives that \( (\langle X_1, X_2 \rangle, x_2) \not\in (\langle X_{n-1}, X_n \rangle, x_{n-1}) \).

If \( x_1 \not\in x_n \), then \( x_2 \not\in x_{n-1} \) and hence \( x \not\sim y \). So let \( x_1 \not\in x_n \). Now \( x_2 \not\sim x_{n-1} \) implies the existence of a
rooted tree \((H, h)\) such that
\[ x_2 \cong (H, x_n), h \text{ and } x_{n-1} \cong (H, x_1), h \]
(the tree \(T\) is as in Figure 2.5).

\[ x \in \mathcal{P} \text{ implies } \{x^0, H^0, x_n, (x_3, x_4, \ldots, x_n)\} \cong \{x_n^0, H^0, x_1, (x_1, x_2, \ldots, x_{n-2})\}. \]

Hence \(x_1 \cong x_n\) and \(x_1^0 \cong x_n^0\) \hspace{1cm} (2.23)

since \(|x_1| < |x_{n-1}|\) and \(|x_n| < |x_2|\).

Now \(x_2 \sim x_{n-1}\) implies \(x_3 \cong x_{n-2}\).

Hence \(x_3^0 \cong x_{n-2}^0\) \hspace{1cm} (2.24)

Also \(x_2^1 \cong \{x_n^0, H^1\} \cong \{x_1^0, H^1\} \cong x_{n-1}^1\) \hspace{1cm} (2.25)

\[ x \in \mathcal{P} \text{ implies } \{x_1^1, x_2^1, x_3^0, (x_4, x_5, \ldots, x_n)\} \cong \{x_n^1, x_{n-1}^1, x_{n-2}^0, (x_1, x_2, \ldots, x_{n-2})\}. \]

Using Equations (2.24) and (2.25) in the above we get, \(x_1^1 \cong x_n^1\). \hspace{1cm} (2.26)

From Equations (2.23) and (2.26) we get \(x_1 \cong x_n\), a contradiction.
Example 2.7.7: There exists a tree $T$ with dissimilar edges $x$ and $y$ such that $x \not\sim y$, $x \not\phi^2 y$ and $d(x, y) = 2$.

The tree $T$ is shown in the Figure 2.12.

Theorem 2.7.8: Let $x, y \in E(T)$ and $d(x, y)$ be odd and not equal to 3. Then $x \not\phi^1 y$ and $x \not\phi^3 y$ imply $x \sim y$.

Proof: Let $T = \langle x_1, x_2, \ldots, x_n \rangle$, $x = (x_1, x_2)$ and $y = (x_{n-1}, x_n)$. Since $d(x, y)$ is odd and other than three, $n$ is odd and other than five.

Case 1: Let $d(x, y) = 1$.

Hence $n = 3$.

$x \not\phi^1 y$ implies $x_1 \not\preceq x_3$ and $x_1^0 \not\preceq x_3^0$. Now $x \not\phi^3 y$ implies $\{x_1^2, x_3^1\} \not\preceq \{x_3^2, x_1^1\}$. From Corollary 2.3.4, we get $x_1 \not\preceq x_3$ and hence $x \sim y$.

Case 2: Let $d(x, y)$ be odd and $\geq 5$.

Here $n$ is odd and $\geq 7$.

Clearly $x \not\phi^1 y$ and $x \not\phi^3 y$ imply $x_2 \not\phi^0_M x_{n-1}$ and $x_2 \not\phi^2_M x_{n-1}$. Since $d(x_2, x_{n-1})$ is even and $\geq 4$, by Theorem 2.6.4 we get $x_2 \sim x_{n-1}$. Hence
(\langle x_1, x_2 \rangle, x_2) \preceq (\langle x_{n-1}, x_n \rangle, x_{n-1}). If \( x_1 \not\preceq x_n \) then \( x \sim y \). Hence let \( x_1 \not\preceq x_n \). This ensures the existence of a rooted tree \((H, h)\) such that 
\( x_2 \preceq (\langle H, x_n \rangle, h) \) and \( x_{n-1} \preceq (\langle H, x_1 \rangle, h) \).

\( x \phi^1 y \) implies \( x_1 \preceq x_n \) and \( x_1^0 \preceq x_n^0 \). Using 
\( x_3 \preceq x_{n-2} \) (since \( x_2 \sim x_{n-1} \)), \( x_2^2 \preceq \{ x_n^1, H^2 \} \), and 
\( x_{n-1}^2 = \{ x_n^1, H^2 \} \) in \( x \phi^3 y \) (as in Theorem 2.7.6),
we get \( \{ x_1^2, x_n^1 \} \equiv \{ x_n^2, x_1^1 \} \). From Corollary 2.3.4, 
we get \( x_1 \not\preceq x_n \), a contradiction.

Example 2.7.9: There exists a tree \( T \) with dissimilar edges \( x \) and \( y \) such that \( x \phi^1 y, x \phi^3 y \) and \( d(x, y) = 3 \).

The tree \( T \) is shown in Figure 2.13.

Example 2.7.10: For any given even integer \( d \), there exist trees with dissimilar edges \( x \) and \( y \) such that \( x \phi^1 y, x \phi^3 y \) and \( d(x, y) = d \).

The trees are shown in Figure 2.14.
FIGURE 2.1 A matrix of trees
FIGURE 2.2 The trees $S_1$, $S_2$, and $S_3$

FIGURE 2.3 Non-isomorphic rooted trees
FIGURE 2.4 The smallest examples
FIGURE 2.6 AN EXAMPLE
Figure 2.7 An example

Figure 2.8 An example
FIGURE 2.9 An illustration for proof of Theorem 2.6.5

FIGURE 2.10 An illustration for proof of Theorem 2.6.5
FIGURE 2.11  k-pseudosimilar edges for $k=0,2$

FIGURE 2.12  k-pseudosimilar edges for $k=1,2$
FIGURE 2.13 k-pseudosimilar edges for k=1,3

FIGURE 2.14 Another example