ANNEXURE II
COVERING PROBLEM

II A.1 Introduction

As an illustration of the meaning of the term "Covering", consider the customers to be residences in a community, the facilities to be fire stations and let residence $i$ be covered if there is a fire station located within a 10 minutes drive of the residence. As another example, let the facilities be plants and let customer $i$ be covered or served by a plant if it is assigned to either sites 1, 2, 3, ..., $n$ where $n$ is the total number of sites.

The general formulation of the above type of problems is as follows:

$$\text{Minimize } Z = \sum_{j=1}^{n} e_j \cdot y_j$$

Subject to:

$$\sum_{j=1}^{n} d_{ij} y_j \geq 1, \quad i = 1, \ldots, m \quad \ldots \ldots \quad \ldots \ldots \quad \text{P.1}$$

$$y_j = (0,1), \quad j = 1, \ldots, n$$

Where $d_{ij}$ is the value referred to as covering coefficient and take on the value of one if customer $i$ is covered by site $j$; otherwise, $d_{ij}$ equals zero. The variable $y_j$ is set equal to one if facility is assigned to site $j$ and
to zero, otherwise. By the constraint of the problem, it is required that each of the customers be covered by at least one of the facilities. The variable $e_j$ is the cost of assigning facility to the site $j$ and the variables $m$ and $n$ are the number of customers and the number of potential sites respectively. Then the objective is to cover the customers at minimum cost of assigning a facility to site $j$.

If the covering coefficient is either 0 or 1 as defined earlier and if there is no restriction on the total number of facilities available, then the objective is to serve all the customers with the minimum number of facilities assigned or with minimum total cost of assigning facilities. This type of problem is called as total cover problem.

For the same definition of covering coefficient, if there is restriction on the total number of facilities available, then always it will not be possible to serve all the customers with the available number of facilities. Then the objective is to serve as many customers as possible by assigning at most number of available facilities with minimum total cost of assigning facilities. This type of problem is called as partial cover problem.

If the covering coefficient is not either 0 or 1 but it is either the distance involved in serving the customer from the given site or the cost of serving the customer from the given site then the objective is to serve all the customers with the at most available number of facilities with minimum total distance or with minimum total cost respectively. This

Generally speaking there are four broad approaches reported in the literature for solving covering problems. The first of these is an implicit enumeration approach (LAWLER 1966, LEMKE et al 1971, PIERCE 1968), such as branch and bound. A second approach is to use cutting plane methods and solve iteratively a number of linear programming problems (BELLMORE et al 1971). Third approach is to employ reduction techniques (BALINSKI 1965, LAWLER 1966, ROTH 1969). The fourth approach is the use of heuristic methods (IGNIZIO 1971).

II A.2 Generalized Covering Problem

Consider the minimization version of the following form of covering problem.
Minimize $Z = \sum_{i=1}^{m} \min_{j \in q(x)} a_{ij}$

Subject to:
\[ \sum_{j=1}^{n} x_j \leq k \]
\[ x_j = (0,1), \text{ for all } j \]

Where $q(x) = \{ j: x_j = 1 \}$; $q$ must be non-empty.

Let $a_{ij}$ be the distance between customer $i$ and site $j$. Thus the objective is to locate at most $k$ facilities such that the total distance travelled between customer locations and facility locations is minimized. The term $\min_{j \in q(x)} a_{ij}$ indicates that if a customer can be served by more than one assigned facility he will choose that facility which is closest. The set $q(x)$ contains the indices for the sites that are assigned facilities.

As an illustration of a generalized covering problem, consider a facility location problem (FRANCIS, 1974), involving five customers and four potential sites for locating at most three facilities. The distance between customer locations and potential sites are given as follows:

\[ D = (d_{ij}) = \begin{pmatrix}
1 & 9 & 17 & 24 \\
10 & 2 & 8 & 15 \\
16 & 8 & 2 & 11 \\
20 & 12 & 4 & 5 \\
24 & 16 & 10 & 1
\end{pmatrix} \]
The number of trips made per month between a facility and customer \( i \), \( b_i \) equals 75, 171, 153, 137, and 805 for \( i = 1, \ldots, 5 \), respectively. The objective is to allocate facilities to sites and assign customers to sites in such a way that the total distance travelled per month is minimum. A maximum of three facilities is available for allocation to sites.

Based on the data for the illustration, the facility location problem can be formulated using (P.2). The matrix \( a_{ij} = b_i \cdot d_{ij} \).

\[
\begin{array}{cccc}
75 & 675 & 1275 & 1800 \\
1710 & 342 & 1368 & 2565 \\
2448 & 1224 & 306 & 1683 \\
2740 & 1644 & 548 & 685 \\
19320 & 12880 & 8050 & 805 \\
\end{array}
\]

Subsequently, it will be found that the optimum solution to the generalized cover problem is to assign facilities to sites 2, 3, and 4, assign customers 1 and 2 to the facility at site 2, assign customers 3 and 4 to the facility at site 3, and assign customer 5 to the facility at site 4.

Notice that in the generalized covering problem formulation, an individual customer interacts with only one facility, whereas a given facility might interact with a number of customers. Furthermore, a customer interacts with the closest facility.
A number of solution procedures are available for solving this type of problems. Specifically, either dynamic programming, branch and bound or heuristic methods can be used to solve the stated problem.

This problem is of combinatorial in nature. Any technique which aims to get optimal solution of this type of problem will not be practically feasible. Hence in practice heuristic method is used to resort this kind of difficulties.

To solve the generalized covering problem which is stated, IGNIZIO (1971) has suggested a heuristic procedure. The logic of the heuristic is given in Figure II A.1.
INPUT:
NUMBER OF CUSTOMERS : m
NUMBER OF POTENTIAL SITES : n
AT MOST NUMBER OF FACILITIES AVAILABLE : k
DISTANCE MATRIX : \( (a_{ij}) \)

FIND THE SUM FOR THE COLUMNS IN THE DISTANCE MATRIX. SELECT THE POTENTIAL SITE FOR ASSIGNING FACILITY WITH MINIMUM COLUMN SUM. STORE THE ASSIGNED POTENTIAL SITE NUMBER (COLUMN NUMBER) IN THE ARRAY \( q \). STORE THE ELEMENTS IN THE SELECTED COLUMN OF THE DISTANCE MATRIX IN THE ARRAY \( f \). SET ACTUAL NUMBER OF POTENTIAL SITES SELECTED \( (kk) \) to 1.

IS \( k=1 ? \)

YES

B

NO

FOR EACH OF THE UNSELECTED POTENTIAL SITE, FIND THE SAVING IN THE TOTAL DISTANCE \( (STC_j) \) IF THAT POTENTIAL SITE IS SELECTED IN ADDITION TO THE CURRENT SELECTED POTENTIAL SITES IN ARRAY \( q \).

\[
STC_j = \sum_{i=1}^{m} \text{Max} \{ f_i - a_{ij}, 0 \} \text{ for } j \text{ not in array } q.
\]

FIND THE MAXIMUM OF \( STC_j \) FOR \( j \) NOT IN ARRAY \( q \)

A
SELECT THE POTENTIAL SITE WITH MAX.STC,, PLACE
THE SELECTED POTENTIAL SITE NUMBER IN ARRAY q.
UPDATE THE ACTUAL NUMBER OF POTENTIAL SITES
SELECTED (kk), kk=kk+1. UPDATE THE ARRAY f
(MINIMUM DISTANCE FOR THE CUSTOMERS WHEN THEY
ARE SERVED FROM THE POTENTIAL SITES IN ARRAY q).

IF k=2 AND kk=2
YES
IF k>2 AND kk=2
NO

FIND INCREASE IN TOTAL COST ITC_j FOR EACH
SELECTED POTENTIAL SITE WHEN IT IS REMOVED FROM
ARRAY q.

$$ITC_j = \sum_{i=1}^{m} \min_{p \in q \; p \neq j} [a_{ip}] - f_i$$

NO
IS
YES

j FOR MIN.ITC = LAST ELEMENT IN q ?

REMOVE j FROM ARRAY q
AND UPDATE ARRAY f.
DECREASE kk by 1.

END RESULTS REACHED. ASSIGN EACH
CUSTOMER TO THE POTENTIAL SITE WITH
MINIMUM DISTANCE

FIGURE II A.1 LOGIC OF THE GENERALIZED COVERING TECHNIQUE.