1.1 Introduction

This thesis entitled SOME LINE DOMINATION RESULTS IN THE FIELD OF GRAPH THEORY is in the area of Graph Theory which is one of the important and ever growing branches of Mathematics. Graph Theory is intimately related to many branches of Mathematics including Group Theory, Matrix Theory, Numerical Analysis, Probability, Topology, Operational Research and Combinatorics and many more. Graph Theory, in its essence, can be described as the study of relations of finite sets, which are visualized with vertices and edges in a two dimensional plane. This thesis aims at introducing certain new parameters in Graph Theory and to study their relationship with other domination parameters.

1.2 A brief history of Graph Theory and its relevance

The basic ideas of Graph Theory were introduced in eighteenth century by the great mathematician Leonard Euler. Since then it has been the source of interest for many researchers and it has achieved remarkable development leading to fruitful generalizations and extensions, yielding interesting and beautiful combinatorial results.

The past fifty years have been a spectacular growth in both pure and applied graph theory, due to its range of applications in many fields like Engineering, Physical, Social and Biological Sciences, Linguistics, Discrete, Optimization problems, Combinatorial problems and Classical Algebraic problems. There is virtually no end to list the problems that can be solved using graph theory.
Graph theory is a very fascinating subject. Its origin is as diverse as its applications. One simple way of representing the structure of any system is a graph. A graph is a simple diagram consisting of points (vertices) and lines (edges). Graphs are useful in enhancing the understanding of the organization and behavioral characteristics of complex systems.

The study of domination in graphs originated around 1850 with the problem of placing minimum number of queens on a $n \times n$ chess board so as to cover or to dominate every square by at least one queen. The solutions of these problems are nothing but dominating sets in the graph, whose vertices are the squares of the chess board and edges are the possible moves of a queen on the chess board. The theory of domination in graphs was introduced by Ore [28] and Berge [4], [5] is an extremely rich area of research in graph theory.

Among the various concepts of graph theory, the concept of domination, independence, etc., have existed for a long time. The concept of DOMINATION IN GRAPHS originated in 1850 with the problem of placing minimum number of queens on a $n \times n$ chess board so as to cover or to dominate every square. The problem of dominating the squares of chess board can be stated more generally as a problem of dominating the vertices of a graph.

For a comprehensive survey of the chess board problems, see Cockayne [9], Kraitchik [22], Spencer and Cockayne [30] and Wagner and Giest [32].
The next appearance of domination in the literature was also associated with game applications. In their book on game theory in 1953, Von Neumann and Morgenstern [27] considered domination in digraphs to find solutions (kernels) for a co-operative $n$-person game.

In 1958, domination was formalized as a theoretical area in graph theory by Berge [4] and [5]. He referred to the domination number as the coefficient of external stability. In 1962, Ore [28] was the first to use the term domination for undirected graphs and also he introduced the concept of minimal and minimum dominating sets of vertices in graphs.

Until 1975, there had not been any considerable progress in this topic. In 1975, Cockayne and Hedetniemi [13], the pioneers of the theory of domination in graphs, published the first paper entitled OPTIMAL DOMINATION IN GRAPHS. Later in 1977, another fundamental paper by the same authors [14] entitled TOWARDS A THEORY OF DOMINATION IN GRAPHS appeared in which they surveyed all the existing results up to 1977 and considered many problems related to concepts viz. colorability, clique size and Nordhaus–Gaddum type results etc. In 1990, Hedetniemi and Laskar [20] published their bibliography on DOMINATION IN GRAPHS AND SOME BASIC DEFINITIONS OF DOMINATION PARAMETERS, containing at that time about 400 entries. This bibliography has grown to over 1200 entries at the end of 1997. In 1998, the publication of the first large two volume text books on domination, FUNDAMENTALS OF DOMINATION IN GRAPHS and DOMINATION IN GRAPHS: ADVANCED TOPICS edited by
Haynes, Hedetniemi and Slater [18] and [19]. Also Cockayne, Gamble and Shepherd [12] have established domination parameters for the Bishop graphs.

Finally, Chartand and Lesniak [8] have included a chapter on domination in their revised book, GRAPHS AND DIGRAPHS. During the past 28 years of dominating sets in graphs has emerged as a significant area of research not only in graph theory but also in Combinatorial Optimization and Analysis of Algorithms.

The theory of domination in graphs has a wide range of applications. Among these applications, the most often discussed is a communication network. This network consists of communication links between a fixed set of sites. The problem is to select a smallest set of sites at which the transmitters are placed so that every other site in the network is joined by a direct communication link to the site, which has a transmitter. In other words, the problem is to find a minimum dominating set in the graph corresponding to this network.

### 1.3 Definitions and Terminologies

Here we recall some of the basic definitions and notations, which are needed for the subsequent chapters. However for any undefined terms, may be found in Harary [17]. The graphs considered here are finite, undirected, without loops or multiple edges unless otherwise specified.

A graph $G$ consisting of a finite non-empty set $V = V(G)$ of $p$ vertices (or points) together with prescribed set $E$ of $q$ unordered pairs
of distinct vertices of $V$. Each pair $e = \{u, v\}$ of vertices in $E$ is called an edge (or line) and $e$ is said to join $u$ and $v$. We write $e = uv$ and say that $u$ and $v$ are adjacent vertices and we say that $u$ and $v$ are incident with an edge $e$. If two distinct edges $x$ and $y$ are incident with a common vertex, then they are adjacent edges.

The cardinality of the vertex set of a graph $G$ is called the order of $G$ and is denoted by $p$. The cardinality of its edge set is called the size of $G$ and is denoted by $q$. A graph with $p$ vertices and $q$ edges is called a $(p, q)$-graph. A $(0, 1)$-graph is called a trivial graph.

A graph $G$ is isomorphic to a graph $H$, if there exists a bijection $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$. If $G$ is isomorphic to a graph $H$, we write $G \cong H$.

A subgraph of $G$ is the graph whose vertex set and edge set are the subsets of the vertex set and the edge set of $G$ respectively. A spanning subgraph is a subgraph containing all the vertices of a graph $G$, while its edge set is a subset of edge set of $G$. The most important subgraph which we shall encounter is the induced subgraph. For any set $S$ of vertices of a graph $G$, the induced subgraph denoted by $\langle S \rangle$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of set $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$.

The removal of a vertex $v$ from a graph $G$ results in the subgraph $G - v$ of $G$ consisting of all vertices of $G$ except $v$ all edges not incident with $v$. The removal of an edge $e$ from $G$ yields the spanning subgraph $G - e$ containing all the edges of $G$ except $e$. 
The degree of a vertex \( v \) in a graph \( G \) denoted by \( \deg v \) is the number of edges adjacent to \( v \). If every edge is incident with two vertices, then it contributes two degrees to the sum of the degrees of the vertices. The minimum degree among the vertices of \( G \) is denoted by \( \min \deg(G) \) or \( \delta(G) \), while \( \Delta(G) \) denoted as \( \max \deg(G) \) is the largest such number. If \( \delta(G) = \Delta(G) = r \), then all the vertices of the graph \( G \) have the same degree \( r \) and \( G \) is called \( r \)-regular. In particular, for \( r = 2 \), \( G \) is called a 2-regular. The vertex \( v \) is isolated if \( \deg(v) = 0 \) and it is called an end vertex if \( \deg(v) = 1 \). The edge incident to an end vertex is a pendant edge (or end edge).

The degree of an edge \( e = uv \) of \( G \) is defined by \( \deg(e) = \deg(u) + \deg(v) - 2 \). The minimum and maximum degree of an edge in \( G \) is denoted by \( \delta'(G) \) and \( \Delta'(G) \).

A walk of a graph \( G \) is an alternating sequence of vertices and edges, beginning and ending with vertices immediately preceding and following it. The walk \( v_0e_1v_1e_2v_2...e_nv_n \) is a closed walk if \( v_0 = v_n \), otherwise it is an open walk. It is called a trivial if all the edges are distinct and it is called a path if all vertices are distinct. A closed walk is called cycle if its \( n \) vertices are distinct and \( n \geq 3 \). The length of a path or a cycle is the number of edges in it. A graph of order \( n \) which is a path or a cycle is denoted by \( P_n \) or \( C_n \) respectively.

The distance \( d(u,v) \) between two vertices \( u \) and \( v \) of a graph \( G \) is the length of the shortest path joining them if any. Otherwise, \( d(u,v) = \infty \). A shortest \( u-v \) path is often called a geodesic. The
diameter $\text{diam}(G)$ of a connected graph $G$ is the length of any longest geodesic.

A graph $G$ is connected if every pair of vertices are joined by a path. A maximal connected subgraph of $G$ is called a component of $G$. Therefore a disconnected graph has at least two components. A graph $G$ is totally disconnected if it has no edges.

A vertex $v$ of $G$ is called a cut vertex if its removal results in a disconnected graph that is, $G-v$ has at least two components. Similarly, a bridge is an edge of a graph $G$ whose removal disconnects $G$. A graph $G$ is said to be separable if it has at least one cut vertex or $k(G)=1$. Otherwise $G$ is nonseparable. Thus, a nonseparable graph is a connected, non trivial graph without cut vertices.

A block of a graph $G$ is a maximal nonseparable subgraph. A block is called an end block of $G$ if it contains exactly one cut vertex of $G$.

A graph is complete if every pair of its vertices are adjacent. A complete $(p, q)$ - graph is therefore a regular graph of degree $p-1$ having $q = \frac{p(p-1)}{2}$ edges and is denoted by $K_p$.

The complement $\overline{G}$ of a graph $G$ has $V(G)$ as its vertex set but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A graph $G$ is called a self complimentary graph if it is isomorphic to its compliment $\overline{G}$.
A bigraph (or bipartite graph) $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and other end in $V_2$. If every vertex of $V_1$ joins every vertex of $V_2$ then $G$ is called a complete bipartite graph. If $V_1$ and $V_2$ have $m$ and $n$ vertices respectively, then we write $G = K_{m,n}$. A star is a complete bipartite graph with $m=1$ and it is denoted by $K_{1,n}$.

The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. The edge connectivity $\lambda(G)$ of a graph $G$ is the minimum number of edges whose removal results in a disconnected graph or a trivial graph.

A graph is acyclic if it has no cycles. A tree is a connected acyclic graph. Any graph without cycles is a forest. An acyclic spanning subgraph of $G$ is called a spanning tree of $G$.

A wheel $W_p$ is a graph with $p$ vertices such that one of its vertices has degree $p-1$, while the degree of the other vertices are three and all such vertices lie on a cycle of length $p-1$. A spider is a tree with the property that the removal of all end paths of length two of $T$ results in an isolated vertex, called the head of a spider.

The concept of the line graph is so natural that it has been independently discovered by many authors giving different names. The line graph of a graph $G$, denoted by $L(G)$, is the graph whose vertices
are the edges of $G$, with two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ are adjacent.

A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph is called a cover for $G$. The smallest number of vertices in any cover of $G$ is called its covering number and is denoted by $\alpha_0$. A set of edges which cover all the vertices of $G$ is called an edge cover of $G$. The smallest number of edges in any edge covers of graph $G$ is called edge covering number of $G$ and is denoted by $\alpha_1$.

A subset $S$ of a vertex set in a graph $G$ is said to be independent if no two vertices in $S$ are adjacent in $G$. The maximum number of vertices in an independent set is called the independence number of $G$ and is denoted by $\beta_0$.

A subset $S$ of edges in a graph $G$ is said to be independent if no two edges in $S$ are adjacent in $G$. The set $S$ is called a maximal independent set provided it is not a proper subset of some other independent set. The maximum cardinality of an edge independent set of $G$ is called the edge independence number or matching number of $G$ and is denoted by $\beta_1$.

A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An $n$ – coloring of a graph $G$ uses $n$ colors; it there by partitions $V$
into \( n \) color classes. The \textit{chromatic number} \( \chi(G) \) is defined as the minimum \( n \) for which \( G \) has an \( n \)–coloring.

The \textit{open neighborhood} \( N(v) \) of a vertex \( v \) in a graph \( G \) is the set of all vertices adjacent to \( v \) in \( G \) and \( N[v] = N(v) \cup \{v\} \) is called the \textit{closed neighborhood} of \( v \).

For any real number \( x \), \( \lfloor x \rfloor \) denotes the \textit{largest integer less than or equal to} \( x \) and \( \lceil x \rceil \) denotes the \textit{smallest integer greater than or equal to} \( x \).

The \textit{open neighborhood} \( N(e) \) of an edge \( e \) in a graph \( G \) is the set of all edges adjacent to \( e \) in \( G \) and the \textit{closed neighborhood} is denoted by \( N[e] = N(e) \cup \{e\} \).

The \textit{open neighborhood} \( N(S) \) of a set \( S \) of vertices in a graph \( G \) is the set of all vertices adjacent to the vertices in \( S \) and \( N[S] = N(S) \cup \{S\} \) is called the \textit{closed neighborhood} of \( S \).

A set \( D \subseteq V(G) \) is said to be a dominating set of \( G \), if every vertex in \( V \setminus D \) is adjacent to some vertex in \( D \). The minimum cardinality of vertices in such a set is called the \textit{domination number} of \( G \) and is denoted by \( \gamma(G) \). For more details on \( \gamma(G) \) see [2], [25] and [33]. A dominating set \( D \) is called \textit{perfect dominating set}, if every vertex in \( V \setminus D \) is adjacent to exactly one vertex in \( D \).
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The concept of *edge domination number* was introduced by Gupta [16] and Mitchell and Hedetniemi [26]. A set \( S \subseteq E(G) \) in a graph \( G \) is an edge dominating set if every edge in \( E - S \) is adjacent to at least one edge in \( S \). The minimum cardinality of edges in such a set is called an *edge domination number* of \( G \) and is denoted by \( \gamma'(G) \).

Related to this parameter we refer [3], [21] and [24]. Cockayne, Dawes and Hedetniemi [10] introduced the concept of *total domination* in graphs. A set \( D \subseteq V(G) \) is said to be total dominating set of \( G \), if the subgraph \( \langle D \rangle \) has no isolated vertices. The total domination number of \( G \), denoted by \( \gamma_t(G) \), and is the minimum cardinality of a total dominating set of \( G \). Similarly, a set \( S \subseteq E(G) \) is said to be total edge dominating set of \( G \), if the subgraph \( \langle S \rangle \) has no isolated edges. The *total edge domination number* of \( G \), denoted by \( \gamma'_t(G) \), and is the minimum cardinality of a total edge dominating set of \( G \). The total edge domination in graphs was introduced by S.Velammal and S.Arumugam [31].

Sampathkumar and Walikar [29] established the new concept of domination called the *connected domination number* of a graph \( G \). A dominating set \( D \subseteq V(G) \) is said to be connected dominating set of \( G \), if the subgraph \( \langle D \rangle \) is connected. The connected domination number of \( G \), denoted by \( \gamma_c(G) \), is the minimum cardinality of a connected dominating set of \( G \). Similarly, an edge dominating set \( S \subseteq E(G) \) is said to be connected edge dominating set of \( G \), if the subgraph \( \langle S \rangle \) is
connected. The connected edge domination number of $G$, denoted by $\gamma'_c(G)$, is the minimum cardinality of a connected edge dominating set of $G$. The connected edge domination in graphs was introduced by S.Velammal and S.Arumugam [31].

The concept of independent domination in graphs was defined by Allan and Laskar [1]. A dominating set $D$ is called an independent dominating set of $G$, if $D$ is also independent. The independent domination number of a graph $G$, denoted by $i(G)$, equals $\min\{|D|; D$ is an independent dominating set of $G\}$. Related to this parameter, refer [6], [7] and [11]. Similarly, an edge dominating set $S$ is called an independent edge dominating set of $G$, if $S$ is also independent. The independent edge domination number of a graph $G$, denoted by $\gamma'_i(G)$, equals $\min\{|S|; S$ is an independent edge dominating set of $G\}$.

Let $D \subseteq V(G)$ and $v \in D$, we say that $v$ has a private neighbor (with respect to $D$) if there is a vertex in $V - D$ whose only neighbor in $D$ is $v$. Let $PN(v,D)$ denote the private neighbors set of $v$ with respect to $D$.

Kulli [23] introduced the following new concept in domination theory. A dominating set $D \subseteq V(G)$ is said to be split dominating set of $G$, if the subgraph $\langle V - D\rangle$ is disconnected. The split domination number of $G$, denoted by $\gamma_s(G)$, and is the minimum cardinality of a split dominating set of $G$. 
A dominating set $D \subseteq V(G)$ is said to be a cototal dominating set of $G$, if the subgraph $\langle V - D \rangle$ has no isolated vertices. The cototal domination number of $G$, denoted by $\gamma_{ct}(G)$, and is the minimum cardinality of a cototal dominating set of $G$. This concept was introduced by Kulli [23].

A dominating set $D \subseteq V(G)$ is a restrained dominating set of $G$, if every vertex not in $D$ is adjacent to a vertex in $D$ and to a vertex in $V(G) - D$. The restrained domination number of a graph $G$, is denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set in $G$. The concept of restrained domination in graphs was introduced by Domke et. al., [15].

A set $D \subseteq V(L(G))$ be a line dominating set in $L(G)$, if every vertex not in $D$ is adjacent to a vertex in $D$. The minimum cardinality of vertices in such a set is called a line domination number in $L(G)$ and is denoted by $\gamma_1(G)$.

A Roman dominating function on a line graph $L(G) = (V', E')$ is a function $f : V' \to \{0, 1, 2\}$ satisfying the condition that every vertex $u'$ for which $f(u') = 0$ is adjacent to at least one vertex $v'$ for which $f(v') = 2$ in $L(G)$. The weight of a Roman dominating function is the value $f(V') = \sum_{u \in V'} f(u')$. The minimum weight of a Roman dominating
function on a line graph $L(G)$ is called the Roman domination number of $L(G)$ and is denoted by $\gamma_r(L(G))$.

Additional definitions will be introduced as and when required.

1.4 An outline of the present investigation

The First chapter is introductory in nature and it gives some basic definitions and terminologies in graph theory.

In the Second chapter the concept COTOTAL DOMINATION IN LINE GRAPHS is introduced. A dominating set $D$ of a line graph $L(G)$ is a cototal dominating set if the induced subgraph $\langle V(L(G)) - D \rangle$ has no isolated vertices. The cototal domination number $\gamma_c(L(G))$ is the minimum cardinality of a cototal dominating set of $L(G)$. In this chapter, we study the graph theoretic properties of $\gamma_c(L(G))$ and many bounds were obtained in terms of elements of $G$. Also its relationship with other domination parameters were found.

Chapter Three deals with REGULAR TOTAL DOMINATION IN LINE GRAPHS. In this chapter, we introduce a new concept in domination theory. A dominating set $D$ of $L(G)$ is a regular total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertices and $\deg(v) = 1, \forall v \in D$. The regular total domination number $\gamma_{rt}(L(G))$ of $L(G)$ is the minimum cardinality of a regular total
dominating set. Further, we study the graph theoretic properties of $\gamma_n(L(G))$ and many bounds were obtained in terms of elements of $G$ and its relationship with other domination parameters were also found. Also we show that the decision problem for regular total dominating set is an NP-complete even for bipartite graphs.

Chapter Four deals with the concept of INVERSE LINE DOMINATION IN GRAPHS. Let $D$ be a line dominating set in $L(G)$, if $V(L(G)) - D$ contains another dominating set $D^{-1}$, then $D^{-1}$ is called an inverse line dominating set with respect to $D$. The minimum cardinality of vertices in such a set is called an inverse line domination number of $L(G)$ and is denoted by $\gamma^{-1}_l(G)$. In this chapter, we study the graph theoretic properties of $\gamma^{-1}_l(G)$ and many bounds were obtained in terms of elements of $G$ and its relationship with other domination parameters were found. Throughout this chapter, the graphs with $p \geq 3$ vertices are considered.

In the Fifth Chapter, the concept of DEGREE EQUITABLE LINE DOMINATION IN GRAPHS is introduced. A line dominating set $D \subseteq V(L(G))$ is called a degree equitable line dominating set, if for every $v \in V(L(G)) - D$ there exists a vertex $u \in D$ such that $uv \in E$ in $L(G)$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of vertices in such a set is called a degree equitable line dominating set in $L(G)$ and is denoted by $\gamma_d(G)$. In this chapter, the graph theoretic properties of
\( \gamma_{el}(G) \) were studied and many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters also found.

In chapter Six the concept of SPLIT LINE DOMINATION IN GRAPHS is introduced. A line dominating set \( D \subseteq V(L(G)) \) is a split line dominating set, if the subgraph \( (V(L(G)) - D) \) is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number of \( G \) and is denoted by \( \gamma_{sl}(G) \). In this chapter, the new concept in domination theory being introduced. Also the graph theoretic properties of \( \gamma_{sl}(G) \) were studied. And many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters were found. Throughout this chapter, we consider the graphs with \( p \geq 4 \) vertices.
REFERENCES


