ABSTRACT

A line dominating set \( D \subseteq V(L(G)) \) is a split line dominating set, if the subgraph \( V(L(G)) - D \) is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number in \( L(G) \) and is denoted by \( \gamma_{sl}(G) \). In this chapter, we introduce the new concept in domination theory. Also, we study the graph theoretic properties of \( \gamma_{sl}(G) \) and many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters were found.
INTRODUCTION

In this chapter, we follow the notations of [1]. The concept of domination in graphs with its many variations is well studied in graph theory (see [2] and [3]).

A line dominating set $D \subseteq V(L(G))$ is a split line dominating set, if the subgraph $\langle V(L(G)) - D \rangle$ is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number of $G$ and is denoted by $\gamma_{sl}(G)$. In this chapter, we introduce the new concept in domination theory. Also we study the graph theoretic properties of $\gamma_{sl}(G)$ and many bounds were obtained in terms of elements of $G$ and its relationships with other domination parameters were found. Throughout this chapter, we consider the graphs with $p \geq 4$ vertices.

RESULTS

Initially, we give the split line domination number for some standard graphs, which are straightforward in the following Theorem.

Theorem 5.1:

a. For any cycle $C_p$ with $p \geq 4$ vertices,

$$\gamma_{sl}(C_p) = \frac{p}{3} \text{ for } p \equiv 0(\text{mod}\, 3).$$

$$= \left\lceil \frac{p}{3} \right\rceil \text{ otherwise.}$$
**b.** For any path $P_p$ with $p \geq 4$ vertices,

$$\gamma_d(P_p) = n,$$ for $p = 3n + 1$, $n = 1, 2, 3, \ldots,$

$$= \frac{p}{3} \text{ for } p \equiv 0 \pmod{3}.$$

$$= \left\lceil \frac{p}{3} \right\rceil \text{ otherwise.}$$

**Theorem 5.2:** A split line dominating set $D \subseteq V(L(G))$ is minimal if and only if for each vertex $x \in D$, one of the following condition holds:

a. There exists a vertex $y \in V(L(G)) - D$ such that $N(y) \cap D = \{x\}$.

b. $x$ is an isolated vertex in $\langle D \rangle$.

c. $\langle (V(L(G)) - D) \cup \{x\} \rangle$ is connected.

**Proof:** Suppose $D$ is a minimal split line dominating set of $G$ and there exists a vertex $x \in D$ such that $x$ does not hold any of the above conditions. Then for some vertex $v$, the set $D_1 = D - \{v\}$ forms a split line dominating set of $G$ by the conditions (a) and (b). Also by (c), $\langle V(L(G)) - D \rangle$ is disconnected. This implies that $D_1$ is a split line dominating set of $G$, a contradiction.

Conversely, suppose for every vertex $x \in D$, one of the above statements hold. Further, if $D$ is not minimal, then there exists a vertex $x \in D$ such that $D - \{x\}$ is a split line dominating set of $G$ and there exists a vertex $y \in D - \{x\}$ such that $y$ dominates $x$. That is $y \in N(x)$. Therefore, $x$ does not satisfy (a) and (b), hence it must satisfy (c). Then there exists a vertex $y \in V(L(G)) - D$ such that $N(y) \cap D = \{x\}$.
Since $D - \{x\}$ is a split line dominating set of $G$, then there exists a vertex $z \in D - \{x\}$ such that $z \in N(y)$. Therefore $w \in N(y) \cap D$, where $w \neq x$, a contradiction to the fact that $N(y) \cap D = \{x\}$. Clearly, $D$ is a minimal split line dominating set of $G$.

The following Theorem characterizes the split line domination and line domination number of graphs.

**Theorem 5.3:** For any connected graph $G$, $\gamma_{sl}(G) = \gamma_l(G)$ if $L(G)$ contains the set of end vertices.

**Proof:** Let $v \in V(L(G))$ be an end vertex and there exists a support vertex $u \in N(v)$. Further, let $D$ be a split line dominating set of $G$. Suppose $u \in D$, then $D$ is a $\gamma_{sl}$-set of $G$. Suppose $u \notin D$, then $v \in D$ and hence $(D - \{v\}) \cup \{u\}$ forms a minimal $\gamma_{sl}$-set of $G$. Repeating this process for all end vertices in $L(G)$, we obtain a $\gamma_{sl}$-set of $G$ containing all the end vertices and $\gamma_{sl}(G) = \gamma_l(G)$.

The following Theorem relates the split line domination and domination number in terms of vertices of a graph.

**Theorem 5.4:** For any connected $(p,q)$-graph $G$, $\gamma_{sl}(G) + \gamma(G) \leq p$.

**Proof:** Let $C = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the set of all non end vertices in $G$. Further, let $S \subseteq C$ be the set of vertices with $\text{diam}(u_i, v_i) \geq 3$, $\forall u_i, v_i \in S$, $1 \leq i \leq k$. Clearly, $N[S] = V(G)$ and $S$ forms a $\gamma$-set of $G$. Suppose $\text{diam}(u, v_i) < 3$. Then there exists at least one vertex $x \in V(G) - S$
such that, either \( x \in N(v) \) or \( x \in N(v') \), where \( v \in S \) and \( v' \in S \cup \{x\} \). Then \( S \cup \{x\} \) forms a minimal dominating set of \( G \). Now in \( L(G) \), let \( F = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G)) \) be the set of vertices corresponding to the edges which are incident to the vertices of \( S \) in \( G \). Further, let \( D \subseteq F \) be the minimal set of vertices which covers all the vertices in \( L(G) \), also making the subgraph \( \langle V(L(G)) - D \rangle \) contains at least two components. Clearly, \( D \) forms a minimal split line dominating set of \( G \). Hence, it follows that \( |D| \cup |S \cup \{x\}| \leq |V(G)| \) and gives \( \gamma_{sl}(G) + \gamma(G) \leq p \).

The following Theorem relates the split line domination and total domination number of \( G \).

**Theorem 5.5:** For any connected graph \( G \), \( \gamma_{sl}(G) + \gamma_t(G) \leq \alpha_0(G) + \beta_0(G) + 1 \).

**Proof:** Let \( C = \{v_1, v_2, \ldots, v_n\} \subseteq V(G) \) be the minimal set of vertices with \( \text{dist}(u, v) \geq 2 \) for all \( u, v \in C \), covers all the edges in \( G \). Clearly, \( |C| = \alpha_0(G) \). Further, if for any vertex \( x \in C \), \( N(x) \subseteq V(G) - C \). Then \( C \) itself is an independent vertex set. Otherwise, \( C_1 \cup C_2 \) where \( C_1 \subseteq C \) and \( C_2 \subseteq V(G) - C \), forms a maximum independent set of vertices \( |C_1 \cup C_2| = \beta_0(G) \). Now, let \( S = C \cup C^c \), where \( C \subseteq C \) and \( C^c \subseteq V(G) - C \), be the minimal set of vertices with \( N[S] = V(G) \) and \( \text{deg}(x) \geq 1 \), \( \forall x \in S \) in the subgraph \( \langle S \rangle \). Clearly, \( S \) forms a minimal total
dominating set in $G$. Now by the definition of line graph, let $F = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G))$ be the set of vertices corresponding to the edges which are incident with the vertices of $S$ in $G$. Let there exists a set $D \subseteq F$ of vertices which are minimally independent and covers all the vertices in line graph. Clearly, $D$ itself is a $\gamma_S$ - set of $G$. Therefore, it follows that $|D| \cup |S| \leq |C| \cup |C_1 \cup C_2| \cup 1$ and hence $\gamma_S(G) + \gamma(G) \leq \alpha_0(G) + \beta_0(G) + 1$.

The following Theorem relates the split line domination, connected domination and domination number of a graph.

**Theorem 5.6:** For any connected graph $G$, $\gamma_S(G) + \gamma_c(G) \leq \text{diam}(G) + \gamma(G) + \alpha_0(G)$.

**Proof:** Let $C \subseteq V(G)$ be the minimal set of vertices which covers all the edges in $G$ with $|C| = \alpha_0(G)$. Further, there exists an edge set $J \subseteq J$, where $J$ is the set of edges which are incident with the vertices of $C^\prime$, constituting the longest path in $G$ such that $|J| = \text{diam}(G)$. Let $S = \{v_1, v_2, \ldots, v_k\} \subseteq C$ be the minimal set of vertices which covers all the vertices in $G$. Clearly, $S$ forms a minimal dominating set of $G$. Suppose the subgraph $\langle S \rangle$ is connected, then $S$ itself is a $\gamma_c$ - set. Otherwise, there exists at least one vertex $x \in V(G) - S$ such that $S_1 = S \cup \{x\}$ forms a minimal connected dominating set of $G$. Now, in $L(G)$, let $F = \{u_1, u_2, \ldots, u_k\} \subseteq V(L(G))$ be the set of vertices such that
\{u_j\} = \{e_j\} \in E(G), 1 \leq j \leq k, \text{ where } \{e_j\} \text{ are incident with the vertices of } S. \text{ Further, let } D \subseteq F \text{ be the set of vertices with } N[D] = V(L(G)) \text{ and if the subgraph } \langle V(L(G)) - D \rangle \text{ contains more than one component. Then } D \text{ forms a split line dominating set of } G. \text{ Otherwise, there exists at least one vertex } \{u\} \in V(L(G)) - D \text{ such that } \langle V(L(G)) - D - \{u\} \rangle \text{ yields more than one component. Clearly, } D \cup \{u\} \text{ forms a minimal } \gamma_{sl} \text{- set of } G. \text{ Therefore, it follows that } |D \cup \{u\}| \cup |S_i| \leq |J| \cup |S| \cup |C| \text{ and hence } \gamma_{sl}(G) + \gamma_c(G) \leq diam(G) + \gamma(G) + \alpha_0(G).

In the following Theorems we give lower bounds to split line domination number of graphs.

**Theorem 5.7:** If every non end vertex of a tree \(T\) is adjacent to at least one end vertex with \(T\) containing at least two cut vertices, then \(\gamma_{sl}(T) \leq c - 1\), where \(c\) is the number of cut vertices in \(T\).

**Proof:** Let \(F = \{v_1, v_2, ..., v_m\} \subseteq V(T)\) be the set of all cut vertices in \(T\) with \(|F| = c\). Further, let \(A = \{e_1, e_2, ..., e_k\}\) be the set of edges which are incident with the vertices of \(F\). Now by the definition of line graph, suppose \(D = \{u_1, u_2, ..., u_t\} \subseteq A\) be the set of vertices which covers all the vertices in \(L(T)\). Clearly, \(D\) forms a minimal split line dominating set of \(L(T)\). Therefore, it follows that \(|D| \leq |F| - 1\) and hence \(\gamma_{sl}(T) \leq c - 1\).
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**Theorem 5.8:** For any connected \((p,q)\)-graph \(G\), \(\gamma_{sl}(G) \leq \left\lfloor \frac{p}{2} \right\rfloor\).

**Proof:** Let \(D = \{v_1, v_2, \ldots, v_n\} \subseteq V(L(G))\) be the minimal split line dominating set of \(G\). Suppose \(|V(L(G)) - D| = 0\). Then the result follows immediately. Further, if \(|V(L(G)) - D| \geq 2\), then \(V(L(G)) - D\) contains at least two vertices such that \(2n < p\). Clearly, it follows that \(\gamma_{sl}(G) = n < \left\lfloor \frac{p}{2} \right\rfloor\).

**Theorem 5.9:** For any connected \((p,q)\)-tree \(T\), \(\gamma_{sl}(T) \leq q - \Delta'(T)\).

**Proof:** Let \(A = \{v_1, v_2, \ldots, v_n\} \subseteq V(L(T))\) be the set of all support vertices. Suppose there exists a set of vertices \(A_i = \{u_1, u_2, \ldots, u_m\} \subseteq V(L(T)) - A\) such that \(\text{dist}(u_i, v_j) \geq 2\), \(\forall u_i \in A_i, \ v_j \in A, \ 1 \leq i \leq m, \ 1 \leq j \leq n\). Then, clearly \(S = A \cup A_i\) forms a split line dominating set of \(T\). Otherwise, if \(A \nsubseteq V(L(T))\), then select the set of vertices \(S = A_i\) such that \(N[S] = V(L(T))\) and the subgraph \(\langle V(L(T)) - S \rangle\) is disconnected. Clearly, in any case \(S\) forms a minimal split line dominating set of \(T\). Since for any tree \(T\), there exists at least one edge \(e \in E(T)\) with \(\deg(e) = \Delta'(T)\), we obtain \(|S| \leq |E(T)| - \Delta'(T)\). Therefore, \(\gamma_{sl}(T) \leq q - \Delta'(T)\).
Theorem 5.10: For any connected unicyclic graph $G = (V,E)$, $\gamma_{sl}(G) \leq q - \Delta(G) + 1$, if one of the following conditions hold:


b. $G = C_3(u_1, u_2, \ldots, u_n)$, $\deg(u_1) \geq 3$, $\deg(u_2) = \deg(u_3) = 2$, $\text{diam}(u_1, w) \leq 2$ for all vertices $w$ not on $C_3$ and $\deg(w) \geq 3$ for at most one vertex $w$ not on $C_3$.

c. $G = C_3$, $\deg(u_1) \geq 3$, $\deg(u_2) \geq 3$, $\deg(u_3) = 2$, all vertices not on $C_3$ adjacent to $u_1$ have degree at most 2 and all vertices whose distance from $u_1$ is 2 are end vertices.

d. $G = C_3$, $\deg(u_1) = 3$, $\deg(u_2) \geq 3$, $\deg(u_3) \geq 3$ and all vertices not on $C_3$ are end vertices.

e. $G = C_4$, either exactly one vertex of $C_4$ or two vertices of $C_4$ have degree at least 3 and all vertices not on $C_3$ are end vertices.

Proof: Assume $\gamma_{sl}(G) = q - \Delta(G) + 1$. Let $A$ denote the set of all end vertices of $L(G)$ with $|A| = m$. Since $V(L(G)) - (A \cup \{v_1\})$ is a split line dominating set for any vertex $v_1$ of $C$, $\gamma_{sl}(G) \leq q - m$ so that $m \leq \Delta(G)$. Let $e$ be an edge of maximum degree $\Delta(G)$. Analogously in $L(G)$, $e = u \in V(L(G))$ such that $|u| = \Delta(L(G))$. If $u$ is not on $C$, then $m = \Delta(G)$ and there exists vertices $v_1$ and $v_2$ on $C$ such that $V(L(G)) - (A \cup \{v_1, v_2\})$ is a split line dominating set of cardinality $q - \Delta(G)$, which is a contradiction. Hence $u$ lies on $C$ and $m \geq \Delta(G) - 1$, we now consider the following cases.
Case 1: \( m = \Delta(G) - 1 \). In this case, all vertices other than \( u \) and \( v \) have degree either one or two. Hence \( C = C_3 \) or \( C_4 \) and \( G \) is isomorphic to one of the graphs described in (a) to (e).

Case 2: \( m = \Delta(G) \). In this case, there exists a unique vertex \( u \) on \( C \) such that \( V(L(G)) - (A \cup \{u\}) \) is a minimum split line dominating set of \( G \). It follows that \( C = C_3 \) and \( G \) is isomorphic to the graph described in (d).
REFERENCES

1. F. Harary, Graph theory, Adison Wesley, Reading mass, 1972.
