Integrating above inequality from $\alpha$ to $t$ gives

$$z(t)\exp\left(-\int_{\alpha}^{t} f(s) ds\right) - z(\alpha) \leq 0 \quad (1.2.10)$$

Using $z(\alpha) = c$ and $u(t) \leq z(t)$ in above inequality we get desired inequality.

Inequalities of this type are known in literature ‘Gronwall inequality’, ‘Bellman’s lemma or inequality’, the ‘Gronwall-Bellman inequality’. Gronwall inequality, like the fundamental inequalities as, the Arithmetic mean and Geometrical mean inequality, Holders inequality and the Minkowskosi inequality caught the fancy of number of research workers and large number of papers which deal with various generalizations, extensions and numerous variants have appeared in the literature see [1-7,10-16,18,20,21,23-24,26-81] and the references cited therein. In 1956, Bihari [10] gave a nonlinear generalization of Gronwall inequality of fundamental importance in the study of nonlinear problems and is known as Bihari’s inequality. At the same time Wendroff has given some important extensions of Gronwall inequality in two independent variables, see [6, p.154]

**Lemma 1.3. (Wendroff [6]).** Let $u(x, y), c(x, y)$ be nonnegative continuous functions defined for $x, y \in R_+$. If

$$u(x, y) \leq a(x) + b(y) + \int_{0}^{x} \int_{0}^{y} c(s, t)u(s, t)dtds, \quad (1.2.11)$$

for $x, y \in R_+$, where $a(x)$ and $b(y)$ are positive continuous functions
for \( x, y \in R_+ \) having derivatives such that \( a'(x) \geq 0 \) and \( b'(y) \geq 0 \) for all \( x, y \in R_+ \), then

\[
u(x, y) \leq E(x, y) \exp \left( \int_0^x \int_0^y c(s, t) dt ds \right)
\]

for \( x, y \in R_+ \), where

\[
E(x, y) = \frac{[a(x) + b(0)][a(0) + b(y)]}{[a(0) + b(0)]}
\]

for \( x, y \in R_+ \)

The above inequality has its origin in field of partial differential equations and provides a very useful and inspiring integral inequality of fundamental importance. Since the publication of the book[6] in 1965, a great interest in such kinds of inequalities has certainly contributed to the development of the theory of certain partial differential and integral equations, see[40, 42] and the references given therein.

In the study of qualitative behavior of solutions of certain nonlinear differential and integral equations some specific types of inequalities are need in various situations. The following inequality which provides an explicit bound on unknown function has played a very important role in the study of various classes of differential and integral equations, see [40, 42].
Lemma 1.4. (Ou-Iang [40]) If $u$, $f$ are nonnegative continuous functions on $\mathbb{R}_+$, $u_0 \geq 0$, is constant and

$$ u^2(t) \leq u_0^2 + 2 \int_0^t f(s)u(s)ds $$  \hspace{1cm} (1.2.14)

for $t \in \mathbb{R}_+$, then

$$ u(t) \leq u_0 + \left( \int_0^t f(s)ds \right) $$  \hspace{1cm} (1.2.15)

for $t \in \mathbb{R}_+$.

Differential equations with retarded arguments have been studied by many investigators and various methods and ideas have been proposed for the study of their different aspects. The fundamental role played by the integral inequalities which provide explicit bounds on unknown functions in the development of the theory of differential and integral equations is well known. It is natural to expect that some new generalizations and variants of such inequalities would also be equally important in certain new applications. Motivated by a desire to apply integral inequalities which provide explicit bounds on unknown functions, in the development of the theory of differential and integral equations with retarded arguments, recently some new inequalities have been developed to achieve a diversity of desired goals, see [1-3,12,13,18,23-25,28-34,43,52,54,58,60-62,64,67,68,70,71,76,77,80,81].

In 2000, Lipovan prove a retarded version of Gronwall inequality, which stated as follows,
Lemma 1.5. (Lipovan [29]) Let \( u, f \in C([t_0, T), R_+) \), further, let \( \alpha \in C^1([t_0, T), [t_0, T)) \), be nondecreasing with \( \alpha(t) \leq t \) on \([t_0, T)\), and let \( k \) be a nonnegative constant. Then the inequality

\[
u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)ds, \quad t_0 \leq t \leq T \tag{1.2.16}\]

implies that,

\[
u(t) \leq k \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} f(s)ds\right), \quad t_0 \leq t \leq T \tag{1.2.17}\]

The nonlinear generalization of retarded Gronwall inequality which used to prove the global existence and boundedness of solutions to certain integral equations and which is stated as follows.

Lemma 1.6. (Lipovan [29]). Let \( u, f \in C([t_0, T), R_+) \). Let \( \omega \in C([t_0, T), R_+) \) be nondecreasing with \( \omega(u) > 0 \) on \((0, \infty)\) and let \( \alpha \in C^1([t_0, T), [t_0, T]) \), be nondecreasing with \( \alpha(t) \leq t \) on \([t_0, T)\) and let \( k \) be a nonnegative constant. If

\[
u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} f(s)\omega(u(s))ds, \quad t_0 \leq t \leq T \tag{1.2.18}\]

where \( k \) be a nonnegative constant, then for \( t_0 \leq t \leq T \)

\[
u(t) \leq G^{-1}\left(G(k) + \int_{\alpha(t_0)}^{\alpha(t)} f(s)ds\right), \quad t_0 \leq t \leq T \tag{1.2.19}\]

where

\[
G(r) = \int_1^r \frac{ds}{\omega(s)}, \quad r > 0. \tag{1.2.20}
\]
and \( t_1 \in [t_0, T) \) is chosen so that

\[
G(k) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds \in \text{Dom}(G^{-1}) \tag{1.2.21}
\]

for all \( t \) lying in the interval \([t_0, T]\)

Agarwal et al. generalized the above lemma proved by Lipovan. It is as follows

**Lemma 1.7.** (Agarwal et al. [2]) Let \( \varphi \in C(R_+, R_+) \) be increasing functions, \( u, a, f \in C([t_0, T), R_+) \), \( a(t) \) be increasing function and \( \alpha \in C^1([t_0, T), [t_0, T)) \), be nondecreasing with \( \alpha(t) \leq t \) on \([t_0, T)\), where \( T \in (0, \infty) \). Then the inequality

\[
u(t) \leq a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) \varphi(u(s)) ds, \quad t_0 \leq t < T \tag{1.2.22}
\]

implies that

\[
u(t) \leq W^{-1}\left(W(a(t)) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds\right), \quad t_0 \leq t < T \tag{1.2.23}
\]

where

\[
W(r) = \int_{1}^{r} \frac{dt}{\varphi(t)}, \quad t > 0 \text{ and } t_1 \in [t_0, T) \tag{1.2.24}
\]

\( W^{-1} \) is the inverse of the function \( W \), \( T_1 \) is the largest number such that

\[
W(a(T_1)) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds \leq \int_{1}^{\infty} \frac{dt}{\varphi(t)} ds \tag{1.2.25}
\]
There can be found a lot of generalizations of Gronwall-Bellman inequalities in various cases from literature. One of the Gronwall-Bellman type retarded integral inequality established by Wang[76]. Inequality by Wang is stated in chapter 4 in section two in the form of lemma. This inequality is used to prove the existence of all solutions of nonlinear differential equations. The retarded version of Ou-Iang inequality in lemma is proved by Lipovan [30]. It is as follows.

**Lemma 1.8.** *(Lipovan [30])* If $u, f$ are nonnegative continuous functions on $R_+, c \geq 0$, is constant and if let $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$, on $R_+$ and

\[
 u^2(t) \leq c^2 + 2 \int_0^{\alpha(t)} g(s)u(s)ds \quad (1.2.26)
\]

for $t \geq 0$, then

\[
 u(t) \leq c + \left( \int_0^{\alpha(t)} g(s)ds \right) \quad (1.2.27)
\]

for $t \geq 0$

Again, Pachpatte in [61] establishes further generalization of theorem in [30] which is handy in the study of global existence of solution of integral equations and functional differential equation. Similarly Agarwal et al. [3] again extend the result proved by Pachpatte [61]. Again, the nonlinear generalization of Wendroff’s inequality in Lemma 1.3 established by Pachpatte in [61] which can be used in the qualitative analysis of retarded partial differential equations with
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retarded arguments. Similarly Cheng [12] established some new nonlinear integral inequalities of the Gronwall-Bellman-Ou-Iang type in generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of differential equations. Kim in [24] proved a number of two-dimensional nonlinear retarded integral inequalities of some Gronwall-Bellman type which can be used as ready and powerful tools in the analysis of various classes of partial differential, integral and integrodifferential equations. Nonlinear retarded integral inequality by Kim in [24] is stated in chapter 3 in section two in the form of Lemma 3.2.2.

1.3 PROBLEM OF THE THESIS

In recent years inequalities are playing a very active role in many parts of analysis. Integral inequalities which provide explicit bounds on unknown functions play an important role in the development of the theory of the differential, integral and integrodifferential equations. However in certain situations the bounds provided by the integral inequalities are not directly applicable and it is desirable to find some new estimates which will equally important in order to achieve a diversity of desired goal.

The main purpose of the thesis is to develop the nonlinear retarded integral inequality in one as well as two independent variables. These
retarded integral inequalities are used to study the qualitative properties such as existence, uniqueness and asymptotic behavior of various types of retarded differential, integral and integrodifferential equations.

1.4 PLAN OF THE THESIS

The aim of the present work is to develop retarded integral inequalities in one and two independent variables which can be used as handy tools in the theory of integrodifferential equations with several retarded arguments, partial differential equations. The areas of applications of these inequalities are global existence, uniqueness, asymptotic behavior and other problems in the theory of integrodifferential and partial differential equations.

Chapter 2 deals with one dimensional extension of the integral inequalities [3], which claims their origin in the field of integrodifferential equations. The inequalities proved here can be used in the analysis of solutions of integrodifferential equations with several retarded arguments with initial condition.

In chapter 3, we prove some new two independent variables extension of the retarded integral inequality [24]. The bounds provided by these inequalities are useful in the theory of partial integrodifferential equations. Some applications of these inequalities are also given to illustrate their usefulness.
Chapter 4, deals with generalization of retarded integral inequalities [76]. This inequality is used to prove existence of nonlinear partial differential equation with suitable initial boundary conditions. Chapter 5 devoted to prove the retarded integral inequality which is used to study the boundedness and asymptotic behavior of solutions of second order integrodifferential equations with retarded argument.