CHAPTER 4

STRONGLY PRIME SPECTRUM OF Γ-NEAR RINGS

4.1 Introduction

Gabriel developed a systematic theory of rings and modules of quotients with respect to a filter of right ideals of an associative ring. In this Chapter, we introduce the notion of Gabriel topology for Γ-near rings. In Section 4.2, we establish the one-to-one order preserving correspondence between topologies(Gabriel) on a Γ-near ring and topologies(Gabriel) on its left operator near-ring.

Several authors such as Booth [8,10] and Selvaraj et.al. [41] studied the ideal theory of Γ-near rings. In Section 4.3, we show that there is a one-to-one order preserving correspondence between bases on the set of strongly prime ideals of a Γ-near ring and the set of strongly prime ideals of its left operator near-ring.
4.2 Gabriel topology for Γ-near rings

Throughout this section by a Γ-near ring \( N \) we mean a zero-symmetric Γ-near ring with left unity.

In this section, we introduce a Gabriel topology for Γ-near ring and we prove that if \( N \) is a subcommutative Γ-near ring with a right unity and a strong left unity, and if \( L \) is the left operator near-ring of \( N \), then there is a one-to-one order preserving correspondence between topologies (Gabriel) on \( N \) and topologies (Gabriel) on \( L \).

**Definition 4.2.1.** Let \( I \) be a left ideal in a Γ-near ring \( N \) and \( P \) a left ideal in \( L \). Then for each \( x \in N \) and \( \alpha \in \Gamma \), we define.

\[
(I : x)_{\alpha} = \{ y \in N \mid y\alpha x \in I \} \\
I^{(x)} = \{ A \in L \mid Ax \in I \} \\
P^{(\alpha)} = \{ y \in N \mid [y, \alpha] \in P \}
\]

**Lemma 4.2.2.** Let \( I \) be a left ideal of a Γ-near ring \( N \). Then

a) \((I : x)_{\alpha}\) is a left ideal of \( N \).

b) \(I^{(x)}\) is a left ideal in \( L \).

c) \(P^{(\alpha)}\) is a left ideal in \( N \).

**Proof.**

a) For \( A, m \in (I, x)_{\alpha}, (A - m)\alpha x = A\alpha x - m\alpha x \in I \) since \( I \) is a left ideal of \( N \). Therefore \( A - m \in (I : x)_{\alpha} \). For \( m \in (I : x)_{\alpha}, n \in N, (n + m - n)\alpha x = n\alpha x + m\alpha x - n\alpha x \in I \) since \( I \) is a left ideal of \( N \). Therefore, \( n + m - n \in (I : x)_{\alpha} \). Thus \((I : x)_{\alpha}\) is a normal subgroup of \((N, +)\). For all \( a, b \in N, i \in (I : x)_{\alpha}\) and \( \beta \in \Gamma, (a\beta(b + i) - a\beta b)\alpha x = a\beta(b + i)\alpha x - a\beta b\alpha x \)
\[ a\beta(b + i\alpha x) - a\beta bax = a\beta(bax + i\alpha x) - a\beta(bax) \in I. \]

Thus \( a\beta(b + i) - a\beta b \in (I : x)_{\alpha} \). This implies that \((I : x)_{\alpha}\) is a left ideal of \( N \).

b) For \( A, m \in I^{(x)} \), \((A - m)x = Ax - mx \in I\). This implies that \( A - m \in I^{(x)} \).

For \( A \in L \) and \( i \in I^{(x)} \), \((A + i - A)x = Ax + ix - Ax \in I\) since \( I \) is a left ideal of \( N \). Therefore \( I^{(x)} \) is a normal subgroup of \((L, +)\). For \( x \in N \), \((b + i)x = bx + ix \in bx + I\) since \( ix \in I \). By Lemma 1.3.26, \( a(b + i)x + I \leq abx + I \), i.e., \( a(b + i)x - abx \in I \). This implies that \( (a(b + i) - ab)x \in I \). Thus \( a(b + i) - ab \in I^{(x)} \). Therefore \( I^{(x)} \) is a left ideal of \( L \).

c) Let \( x, y \in P^{(\alpha)} \). Then \([x - y, \alpha] = [x, \alpha] - [y, \alpha] \in P\). Therefore \( x - y \in P^{(\alpha)} \). For \( n \in N \) and \( x \in P^{(\alpha)} \), \([n + x - n, \alpha] = [n, \alpha] + [x, \alpha] - [n, \alpha] \in P\) since \( P \) is a left ideal in \( L \). Therefore \( n + x - n \in P^{(\alpha)} \). Thus \( P^{(\alpha)} \) is a normal subgroup of \( N \).

For \( a, b \in N \) and \( x \in P^{(\alpha)} \),

\[
[a\beta(b + x) - a\beta b, \alpha]y = [a\beta(b + x), \alpha]y - [a\beta b, \alpha]y
\]

\[
= a\beta(b + x)\alpha y - a\beta b\alpha y
\]

\[
= a\beta(b\alpha y + x\alpha y) - a\beta(b\alpha y)
\]

\[
= a\beta([b, \alpha]y + [x, \alpha]y) - a\beta[b, \alpha]y
\]

\[
= [a, \beta][[b, \alpha] + [x, \alpha]]y - [a, \beta][b, \alpha]y
\]

\[
= ([a, \beta][[b, \alpha] + [x, \alpha]) - [a, \beta][b, \alpha])y.
\]

Since \( P \) is a left ideal, \([a, \beta][[b, \alpha] + [x, \alpha]) - [a, \beta][b, \alpha] \in P \). This implies
that \( a\beta(b + x) - a\beta b \in P(\alpha) \). Therefore \( P(\alpha) \) is a left ideal in \( N \).

\[ \square \]

**Lemma 4.2.3.** Let \( I, J \) be left ideals in \( N \) and \( Q \) a left ideal in \( L \). Then for all \( x, y \in N \) and \( \alpha, \beta \in \Gamma \),

a) \( (I : x)_\alpha = N \) for all \( \alpha \in \Gamma \) if and only if \( x \in I \).

b) \( (I \cap J : x)_\alpha = (I : x)_\alpha \cap (J : x)_\alpha \).

c) \( ((I : x)_\alpha : y)_\beta = (I : y\alpha x)_\beta \)

**Proof.**  a) Suppose that \( (I : x)_\alpha = N \) for all \( \alpha \in \Gamma \) and \( x \in N \). Let \( n \in N \).

Then \( n \in (I : x)_\alpha \). This implies that \( n\alpha x \in I \). Then \( n\alpha x = n\alpha(0 + x) - n\alpha 0 \in I \). This implies that \( x \in I \) since \( I \) is a left ideal of \( N \). Conversely, suppose that \( x \in I \). \( (I : x)_\alpha \subseteq N \) is obvious. Suppose \( N \notin (I : x)_\alpha \). Then there exists some \( n \in N \) such that \( n \notin (I : x)_\alpha \). This implies that \( n\alpha x \notin I \). But \( n\alpha x = n\alpha(0 + x) - n\alpha 0 \in I \) since \( I \) is a left ideal of \( N \) and \( x \in I \), a contradiction. Therefore \( N = (I : x)_\alpha \).

b) Let \( n \in (I \cap J : x)_\alpha \). This implies that

\[
n\alpha x \in I \cap J \iff n\alpha x \in I \text{ and } n\alpha x \in J
\]

\[
\iff n \in (I : x)_\alpha \text{ and } n \in (J : x)_\alpha
\]

\[
\iff n \in (I : x)_\alpha \cap (J : x)_\alpha.
\]

c) Let \( n \in ((I : x)_\alpha : y)_\beta \). This implies that

\[
n\beta y \in (I : x)_\alpha \iff n\beta y\alpha x \in I
\]

\[
\iff n \in (I : y\alpha x)_\beta.
\]

\[ \square \]
Lemma 4.2.4. Let $N$ be a subcommutative $\Gamma$-near ring. Let $I$ and $Q$ be left ideals in $N$ and $L$, respectively. Then, for all $x, y, \in N, A \in L$ and $\alpha \in \Gamma$,

a) $(I^+ : [x, \alpha]) = ((I : x)_\alpha)^+$

b) $(Q : [x, \alpha])^\Gamma = (Q^\Gamma : x)_\alpha$

c) $(f(x) : [y : \alpha]) = ((f : x)_\alpha(y)$

d) $(I \cap f)^{(x)} = f(x) \cap f^{(x)}$

e) $f^{(Ax)} = (f(x) : A)$

Proof. a) 

$A \in ((I : x)_\alpha)^+ \iff AN \subseteq (I : x)_\alpha$

$\iff ANx_\alpha \subseteq I$

$\iff Ax_\alpha N \subseteq I \quad [\because N \text{ is subcommutative}]$

$\iff A[x, \alpha] \in I^+$

$\iff A \in (I^+ : [x, \alpha])$.

b) 

$y \in (Q^\Gamma : x)_\alpha \iff y_\alpha x \in Q^\Gamma$

$\iff [y_\alpha x, \Gamma] \subseteq Q$

$\iff [y, \alpha][x, \Gamma] \subseteq Q$

$\iff [y, \alpha][x, \alpha] \in Q$ for any $\alpha \in \Gamma$

$\iff [y, \alpha] \in (Q : [x, \alpha])$ for any $\alpha \in \Gamma$

$\iff [y, \Gamma] \subseteq (Q : [x, \alpha])$

$\iff y \in (Q : [x, \alpha])^\Gamma$.

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c) \[ A \in (I^{(x)} : [y, \alpha]) \iff A[y, \alpha] \in I^{(x)} \]
\[ \iff A[y, \alpha]x \in I \]
\[ \iff Ayax \in I \]
\[ \iff Ay \in (I : x)_{\alpha} \]
\[ \iff A \in ((I : x)_{\alpha})^{(\nu)}. \]

d) \[ A \in (I \cap J)^{(x)} \iff Ax \in I \cap J \]
\[ \iff Ax \in I \text{ and } Ax \in J \]
\[ \iff A \in I^{(x)} \text{ and } A \in J^{(x)} \]
\[ \iff A \in I^{(x)} \cap J^{(x)}. \]

e) \[ m \in I^{(Ax)} \iff mA_{x} \in I \]
\[ \iff mA \in I^{(x)} \]
\[ \iff m \in (I^{(x)} : A). \]

Lemma 4.2.5. Let \( I \) be a left ideal in \( N \) and \( P, Q \) left ideals in \( L \). Then for all \( x \in N \) and \( \alpha, \beta \in \Gamma \),

a) \( (P \cap Q)^{(\alpha)} = P^{(\alpha)} \cap Q^{(\alpha)} \)

b) \( (P : [x, \alpha])^{(\beta)} = (P^{(\alpha)} : x)_{\beta} \)
c) \((P^{(\alpha)})(x) = (P : [x, \alpha])\)

d) \((I^{(x)})(\alpha) = (I : x)_{\alpha}\)

Proof.

a)

\[ x \in (P \cap Q)^{(\alpha)} \iff [x, \alpha] \in P \cap Q \]
\[ \iff [x, \alpha] \in P \text{ and } [x, \alpha] \in Q \]
\[ \iff x \in P^{(\alpha)} \text{ and } x \in Q^{(\alpha)} \]
\[ \iff x \in P^{(\alpha)} \cap Q^{(\alpha)}. \]

b)

\[ y \in (P : [x, \alpha])^{(\beta)} \iff [y, \beta] \in (P : [x, \alpha]) \]
\[ \iff [y, \beta][x, \alpha] \in P \]
\[ \iff [y\beta x, \alpha] \in P \]
\[ \iff y \in (P^{(\alpha)} : x)_{\beta}. \]

c)

\[ A \in (P^{(\alpha)})^{(x)} \iff Ax \in P^{(\alpha)} \]
\[ \iff [Ax, \alpha] \in P \]
\[ \iff A[x, \alpha] \in P \]
\[ \iff I \in (P : [x, \alpha]). \]

d)

\[ y \in (I^{(x)})(\alpha) \iff [y, \alpha] \in I^{(x)} \]
\[ \iff [y, \alpha]x \in I \]
\[ y \alpha x \in I \]
\[ \iff y \in (I : x)_\alpha. \]

**Definition 4.2.6.** A nonempty family \( F(N) \) of left ideals of \( N \) is said to be a **pretopology** on \( N \) if

1. \( I \in F(N) \) implies \( (I : x)_\alpha \in F(N) \) for all \( x \in N \) and \( \alpha \in \Gamma \);
2. \( I \in F(N), I \subseteq J \) implies \( J \in F(N) \) for all left ideals \( J \) of \( N \);
3. \( I, J \in F(N) \) implies \( I \cap J \in F(N) \).

A pretopology on \( N \) is said to be a **(Gabriel) topology** on \( N \) if

4. \( (I : x)_\alpha \in F(N) \) for all \( \alpha \in \Gamma \) and \( x \in J \) for some \( J \in F(N) \) implies \( I \in F(N) \).

**Example 4.2.7.** Let \( Z \) be the ring of integers,

\[
\begin{align*}
N_{2,1} &= \begin{bmatrix} 0 \end{bmatrix} /a \in Z, \text{ and } \Gamma_{1,2} = \{(b, c) / b, c \in Z\}. \\
\end{align*}
\]

Then \( N_{2,1} \) is a \( \Gamma_{1,2} \)-near ring in a natural way. Moreover the left ideals of \( N_{2,1} \)

are of the form

\[
\begin{align*}
0 &= \begin{bmatrix} 0 \end{bmatrix} /a \in Z, \text{ and } \Gamma_{1,2} = \{(b, c) / b, c \in Z\}. \\
mZ &= \begin{bmatrix} ma \end{bmatrix} /a \in Z, m \text{ a fixed integer} \\
\end{align*}
\]

Then \( F_1 = \begin{bmatrix} 0, 0, 0, 0, 0 \end{bmatrix} \)

is a (Gabriel) topology on \( N_{2,1} \).
Proposition 4.2.8. If $F(N)$ is a topology on $N$ and $I, J \in F(N)$ then

$I \cap J \in F(N)$.

Proof. For all $x \in J$ and $\alpha \in \Gamma$, $I \alpha x \subseteq I \cap J$ implies $I \subseteq (I \cap J : x)_\alpha$. By 4.2.6(T2), $(I \cap J : x)_\alpha \in F(N)$ for all $x \in J$ and $\alpha \in \Gamma$. By 4.2.6(T4), $I \cap J \in F(N)$. $\square$

Lemma 4.2.9. Let $F(N)$ be a pretopology on $N$. Then $F(L) = \{\text{left ideals } P \text{ of } L / P^{(\alpha)} \in F(N) \text{ for all } \alpha \in \Gamma \}$ is a pretopology on $L$.

Proof. Let $P \in F(L)$. Then $P^{(\beta)} \in F(N)$ for all $\beta \in \Gamma$ implies $(P^{(\beta)} : y)_\alpha = (P : [y, \beta])^{(\alpha)} \in F(N)$ for all $\alpha \in \Gamma$ and $y \in N$, by Lemma 4.2.5(b), and so $(P : [y, \beta]) \in F(L)$. If $P \in F(L)$ and $Q$ is any left ideal of $L$ such that $P \subseteq Q$, then for all $\alpha \in \Gamma$, $P^{(\alpha)} \subseteq Q^{(\alpha)}$ implies $Q^{(\alpha)} \in F(N)$ and so $Q \in F(L)$. If $P, Q \in F(L)$ then $P \cap Q \in F(L)$ by Lemma 4.2.5(a). $\square$

Lemma 4.2.10. Let $L$ be the left operator near-ring of $N$ and $F(L)$ a pretopology on $L$. Then $F(N) = \{\text{left ideals } I \text{ of } N / I^{(x)} \subseteq F(L) \text{ for all } x \in N \}$ is a pretopology on $N$.

Proof. Let $I \in F(N)$. Then for all $x, y \in N$ and $\alpha \in \Gamma$, $I^{(x)} \in F(L)$ implies $(I^{(x)} : [y, \alpha]) = ((I : x)_\alpha)^{(y)} \in F(L)$ by Lemma 4.2.4(c) and so $(I : x)_\alpha \in F(N)$. If $I \in F(N)$ and $J$ is any left ideal of $N$ such that $I \subseteq J$, then $I^{(x)} \subseteq J^{(x)}$ for all $x \in N$ implies $J^{(x)} \in F(L)$ and so $J \in F(N)$. If $I, J \in F(N)$ then $I \cap J \in F(N)$ by Lemma 4.2.4(d). Thus $F(N)$ is a pretopology on $N$. $\square$

Lemma 4.2.11. Let $N$ be a subcommutative $\Gamma$-near ring. If $F(N)$ is a pretopology on $N$, then $F(L) = \{I^+ / I \in F(N) \}$ is a pretopology on $L$. 

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Proof. Let $I^+ \in F(L)$. We have to prove that $(I^+ : [x, \alpha]) \in F(L)$ for all $x \in N$ and $\alpha \in \Gamma$. But $(I^+ : [x, \alpha]) = ((I : x)\alpha)^+ \in F(L)$ by Lemma 4.2.4(a), since $(I : x)\alpha \in F(N)$. Let $I$ and $J$ be left ideals of $N$ and $I \subseteq J$. Since $F(N)$ is a pretopology on $N$, $J \in F(N)$. Since $I \subseteq J \Rightarrow I^+ \subseteq J^+$ and $J \in F(N)$, $J^+ \in F(L)$. Since $I \cap J \in F(N)$, $(I \cap J)^+ \in F(L)$. Also $(I \cap J)^+ = I^+ \cap J^+$. Thus $I^+ \cap J^+ \in F(L)$. Therefore $F(L)$ is a pretopology on $L$. 

Lemma 4.2.12. Let $F(L)$ be a pretopology on $L$. Then $F(N) = \{P^+ / P \in F(L)\}$ is a pretopology on $N$.

Proof. Let $I^+ \in F(N)$. We have to prove that $(I^+ : x)\alpha \in F(N)$. By Lemma 4.2.4(b), $(I^+ : x)\alpha = (I : [x, \alpha])^+$. Since $I \in F(L)$, $(I : [x, \alpha])^+ \in F(N)$.

Therefore $(I^+ : x)\alpha \in F(N)$. Let $I^+$ and $J^+$ be the left ideals in $N$ and $I^+ \subseteq J^+$. Then $I^+ \subseteq J^+ \Rightarrow I \subseteq J$. Since $I \in F(L)$ and $I \subseteq J$ and $F(L)$ is a pretopology on $L$, $J \in F(L)$. This implies that $J^+ \in F(N)$. Let $I^+ \in F(N)$. This implies that $I \in F(L)$ and $J \in F(L)$. Since $F(L)$ is a pretopology on $L$, $I \cap J \in F(L)$. Thus $(I \cap J)^+ \in F(N)$. It is clear that $(I \cap J)^+ = I^+ \cap J^+$. Therefore $I^+ \cap J^+ \in F(N)$. Hence $F(N)$ is a pretopology on $N$. 

Proposition 4.2.13. Let $N$ be a subcommutative $\Gamma$-near ring with a right unity and a strong left unity. Then there is a one-to-one order preserving correspondence between pretopologies on $N$ and pretopologies on $L$.

Proof. Starting with a pretopology $F(N)$ on $N$, we get a pretopology $F(L)$ on $L$ given by $F(L) = \{I^+ / I \in F(N)\}$. This in turn induces a pretopology $F_1(N)$ on $N$ given by $F_1(N) = \{(I^+)^+ / I^+ \in F(L)\} = F(N)$ since $(I^+)^+ = I$ by Proposition 1.3.27.
On the other hand, if we start with a pretopology \( F(L) \) on \( L \) we get a pretopology \( F(N) \) on \( N \) given by \( F(N) = \{ P^+ / P \in F(L) \} \). This in turn induces a pretopology \( F_1(L) \) on \( L \) given by \( F_1(L) = \{ (P^+)^+ / P^+ \in F(N) \} = F(L) \), since \( (P^+)^+ = P \) by Proposition 1.3.27. Thus this correspondence is order preserving and the proof is complete.

**Lemma 4.2.14.**

a) If \( P \) is an essential left ideal in \( L \) then \( P^{(\alpha)} \) is an essential left ideal in \( N \) for all \( \alpha \in \Gamma \).

b) If \( I \) is an essential left ideal in \( N \) then \( I^{(\alpha)} \) is an essential left ideal in \( L \) and \( (I : x)_{\alpha} \) is an essential left ideal in \( N \) for all \( x \in N \) and \( \alpha \in \Gamma \).

**Proof.**

a) Let \( J \) be a nonzero left ideal in \( N \). Then \( [J, \alpha] \) is a left ideal in \( L \). If \( [J, \alpha] = 0 \), then \( J \subseteq P^{(\alpha)} \). If \( [J, \alpha] = 0 \) then since \( P \) is essential, \( [J, \alpha] \cap P = 0 \). Therefore there exists \( x \in J \) such that \( 0 = [x, \alpha] \subseteq P \), i.e., \( P^{(\alpha)} \cap J = 0 \) and so \( P^{(\alpha)} \) is essential.

b) Let \( P \) be any nonzero left ideal in \( L \). If \( Px = 0 \) then \( P \subseteq I^{(\alpha)} \). If \( Px = 0 \) then \( Px \cap I = 0 \) implies that there exists \( 0 = r \in P \) such that \( rx \in I \), i.e., \( r \in P \cap I^{(\alpha)} \). Thus \( I^{(\alpha)} \) is essential in \( L \). Moreover, since \( (I : x)_{\alpha} = (I^{(\alpha)})_{(\alpha)} \) by (a), \( (I : x)_{\alpha} \) is essential in \( N \).

**Lemma 4.2.15.**

a) \( P \) is an essential left ideal in \( L \) if and only if \( P^{(\alpha)} \) is essential in \( N \) for all \( \alpha \in \Gamma \).

b) \( I \) is an essential left ideal in \( N \) if and only if \( I^{(\alpha)} \) is an essential left ideal in \( L \).

**Proof.**

a) One implication was proved in Lemma 4.2.14. Conversely, let \( P^{(\alpha)} \) be essential in \( N \) for all \( \alpha \in \Gamma \). Let \( Q \) be a non-zero left ideal in \( L \). Since
$P^{(\alpha)}$ is an essential in $N$, $P^{(\alpha)} \cap Q^{(\alpha)} = 0$. But $(P \cap Q)^{(\alpha)} = P^{(\alpha)} \cap Q^{(\alpha)} = 0$
by Lemma 4.2.5(a). This implies that there exists $0 = y \in N$ such that $[y, \alpha] \in P \cap Q$. This implies that $P \cap Q = 0$. Therefore $P$ is an essential left ideal in $L$.

b) One implication was proved in Lemma 4.2.14. Conversely, let $I^{(x)}$ be an essential left ideal in $L$. Let $J$ be a non-zero left ideal in $N$. Since $I^{(x)}$ is essential, $I^{(x)} \cap J^{(x)} = 0$. But $I^{(x)} \cap J^{(x)} = (I \cap J)^{(x)} = 0$, by Lemma 4.2.4(d). This implies that there exists $0 = l \in L$ such that $Ax \in I \cap J$. Therefore $I \cap J = 0$. Thus $I$ is essential in $N$.

\[ \square \]

**Lemma 4.2.16.** Let $N$ be a $\Gamma$-near ring with a right unity and a strong left unity. Then

a) $I$ is an essential left ideal in $N$ if and only if $I^+$ is an essential left ideal in $L$.

b) $Q$ is an essential left ideal in $L$ if and only if $Q^+$ is an essential left ideal in $N$.

**Proof.** a) Let $I$ be an essential left ideal in $N$. Let $P$ be a non-zero left ideal in $L$. If $PN = 0$, then $P \subseteq I^+$. If $PN = 0$, then $PN \cap I = 0$ implies that there exists $0 = r \in P$ such that $rN \subseteq I$. That is $r \in P \cap I^+$. Thus $I^+$ is essential in $L$.

Conversely, let $I^+$ be an essential left ideal in $L$. Let $J$ be a non-zero left ideal in $N$. Then $J^+$ is a non-zero left ideal in $L$. Since $I^+$ is essential, $I^+ \cap J^+ = 0$. Then there exists $0 = l \in L$ such that $lx \in I \cap J$. This implies that $I \cap J = 0$. Thus $I$ is an essential in $N$. 

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b) Let $Q$ be an essential left ideal in $L$. Since $Q = (Q^+)^+$ by Proposition 1.3.27, $(Q^+)^+$ is an essential left ideal in $L$. Thus, by (a), $(Q^+)$ is an essential left ideal in $N$. The converse is similar.

\[ \square \]

**Proposition 4.2.17.** The family of all essential left ideals in $N$ is a pretopology on $N$.

**Proof.** Let $I \in F(N)$, the family of all essential left ideal in $N$. Then by Lemma 4.2.14(b), $(I : x)_{\alpha}$ is an essential left ideal in $N$. Since 4.2.6(72) and (73) are obvious, $F(N)$ is a pretopology on $N$.

\[ \square \]

**Theorem 4.2.18.** Let $N$ be a subcommutative $\Gamma$-near ring with a right unity and a strong left unity. Let $L$ be a commutative left operator near-ring of $N$. Then there is a one-to-one order preserving correspondence between topologies on $N$ and topologies on $L$.

**Proof.** Let $F(N)$ be a topology. Suppose $I$ is a left ideal in $N$ such that $(I^+ : A) \in F(L)$ for all $A \in J^+$ for some $J \in F(N)$. Then by Lemma 4.2.4(a), $((I : x)_{\alpha})^+ = (I^+ : [x, \alpha]) \in F(L)$ for all $x \in J$ and $\alpha \in \Gamma$. This shows that $(I : x)_{\alpha} \in F(N)$ and so $I \in F(N)$ by (74) of 4.2.6. Therefore $I^+ \in F(L)$ and hence $F(L)$ is a topology on $L$.

Conversely, let $F(L)$ be a topology on $L$ and $F(N)$ the corresponding pre-topology on $N$. We have to prove that $F(N)$ satisfies (74) of 4.2.6. Let $Q$ be a left ideal in $L$ such that $(Q^+ : x)_{\alpha} \in F(N)$ for all $\alpha \in \Gamma$ and $x \in P^+$ for some $P \in F(L)$. Then for all $A \in P$, $y \in L$ and $\alpha \in \Gamma$,

\[
(Q : A[y, \alpha])^+ = (Q : [Ay, \alpha])^+ = (Q^+ : Ay)_{\alpha} \in F(N) \quad \text{by Lemma 4.2.4(b)}
\]
since $Ay \in P^+$. 

Since $L$ is commutative, $(Q : A[y, \alpha]) = (Q : [y, \alpha])$ and $[y, \alpha]^A = [y, \alpha][0 + A] - [y, \alpha]0 \in P$ since $P$ is a left ideal. Therefore $(Q : A[y, \alpha]) = (Q : A_1) \in F(L)$ for all $A_1 \in P$. Hence $Q \in F(L)$ by $(T4)$ of 4.2.6. Therefore $Q^+ \in F(N)$ by Lemma 4.2.16(b). Thus $F(N)$ is a topology on $N$. □

**Proposition 4.2.19.** Let $N$ be a subcommutative $\Gamma$-near ring. Let $\frac{n}{j=1} [\gamma_j, \beta_j]$ be left unity in $N$. Then a non empty family $F(N)$ of left ideals in $N$ is a topology on $N$ if and only if

(i) $I \in F(N)$ implies $(I : x)_{\gamma_j} \in F(N)$ for all $x \in N$ and $j = 1, 2, \cdots n$.

(ii) If $(I : x)_{\beta_j} \in F(N)$ for all $x \in J$ and for some $J \in F(N)$, $j = 1, 2, \cdots n$,

then $I \in F(N)$.

**Proof.** If $F(N)$ is a topology, then clearly (i) is satisfied. Let $F(N)$ be a topology on $N$. If $I$ is a left ideal in $N$ such that $(I : x)_{\beta_j} \in F(N)$ for all $x \in J, J \in F(N)$, $j = 1, 2, \cdots n$, then for all $A \in J^+, (I^+ : [Ay_j, \beta_j]) = ((I : Ay_j)_{\beta_j})^+ \in F(L)$ by Lemma 4.2.4(a) and since $Ay_j \in J$. Since $A = \frac{n}{j=1} [Ay_j, \beta_j]$, $(I^+ : A) \in F(L)$. This shows that $I^+ \in F(L)$ since $F(L)$ is a topology on $L$ and so $I \in F(N)$. This proves (ii).

Conversely, if $F(N)$ satisfies (i) and (ii), then $(T4)$ of 4.2.6 is automatically satisfied. Also if $I \in F(N)$ then for all $x, y \in N$ and $\beta \in \Gamma$,

$((I : x)_{\beta} : y)_{\beta_j} = (I : y\beta x)_{\beta_j} \in F(N)$ for $j = 1, 2, \cdots n$

and so $(I : x)_{\beta} \in F(N)$ by (ii). □
4.3 Topology on the set of all strongly prime ideals in $\Gamma$-near rings

In this section, we prove that there is a one-to-one order preserving correspondence between bases on $\text{SSpec}(N)$, the set of all strongly prime ideals on $N$ and bases on $\text{SSpec}(L)$, the set of all strongly prime ideals of $L$.

**Definition 4.3.1.** Let $N$ be a $\Gamma$-near ring. A **basis for a topology** on $\text{SSpec}(N)$ is a collection $B_x(N)$ of subsets of $\text{SSpec}(N)$ such that

1. For each $P \in \text{SSpec}(N)$, there exists an element $B \in B_x(N)$ containing $P$;
2. If $P \in B_1 \cap B_2$, where $B_1, B_2 \in B_x(N)$, then there is a basis element $B_3$ containing $P$ such that $B_3 \subseteq B_1 \cap B_2$.

**Proposition 4.3.2.** If $P \in \text{SSpec}(N)$, then $P^+ \in \text{SSpec}(L)$, where $P^+ = \{A \in L | AN \subseteq P \}$.

*Proof.* Let $A \notin P^+$. Then $Ax \notin P$ for some $x \in N$. Since $P \in \text{SSpec}(N)$, there exists finite subsets $F = \{f_j | j = 1, 2, \cdots m \} \subseteq N$ and $\Delta = \{\alpha_i | i = 1, 2, \cdots n \} \subseteq \Gamma$ such that for any $x \in N$

$$Ax\alpha_i f_j \alpha_k y \in P \text{ for all } \alpha_i, \alpha_k \in \Delta, f_j \in F \text{ implies } y \in P.$$  \hspace{1cm} (4.3.1)

Let $G = \{x\alpha_i f_j, \alpha_k \} | 1 \leq i, k \leq n, 1 \leq j \leq m \}$ and $A' \in L$ such that $A'G \subseteq P^+$, i.e., $A[x\alpha_i f_j, \alpha_k]A' \in P^+$ and so that $A[x\alpha_i f_j, \alpha_k]A' N \subseteq P$. Hence $A\alpha_i f_j \alpha_k A' N \subseteq P$ for all $\alpha_i, \alpha_k \in \Delta, f_j \in F$.

By (4.3.1), $A' N \subseteq P$. Therefore $A' \in P^+$. Thus $P^+ \in \text{SSpec}(L)$. \hfill $\Box$
Proposition 4.3.3. Let $N$ be a distributive $\Gamma$-near ring. If $Q \in SS\text{pec}(L)$, then $Q^+ \in SS\text{pec}(N)$, where $Q^+ = \{x \in N / [x, \Gamma] \subseteq Q\}$.

Proof. Let $x \in Q^+$. Then there exists $\alpha \in \Gamma$ such that $[x, \alpha] \notin Q$. Since $Q \in SS\text{pec}(L)$, there exists

$$G = \left\{ \sum_{i=1}^{m} [y_{ik}, \beta_{ik}] k = 1, 2, \cdots, n \right\} \subseteq L$$

such that for any $A \in L$,

$$[x, \alpha]GA \subseteq G \text{ implies } A \in Q. \quad (4.3.2)$$

Let $F = \{y_{ik} / i = 1, 2, \cdots, m, k = 1, 2, \cdots, n\}$ and $\Delta = \{\beta_{ik}, \alpha / i = 1, 2, \cdots, m, k = 1, 2, \cdots, n\}$. Let $z \in M$ be such that $x\Delta F\Delta z \subseteq Q^+$. Then $x\alpha y_{ik} \beta_{ik} z \in Q^+$ for all $i = 1, 2, \cdots, m, k = 1, 2, \cdots, n$. Hence $[x\alpha y_{ik} \beta_{ik} z, \beta] \in Q$ for all $\beta \in \Gamma$, i.e., $[x, \alpha][y_{ik}, \beta_{ik}][z, \beta] \in Q$ for all $\beta \in \Gamma$. Thus $[x, \alpha] \left\{ \sum_{i=1}^{m} [y_{ik}, \beta_{ik}] z, \beta \right\} \subseteq Q$ for all $k = 1, 2, \cdots, n$ and for all $\beta \in \Gamma$. By (4.3.2), $[z, \beta] \subseteq Q$ for all $\beta \in \Gamma$. Therefore $z \in Q^+$ and consequently $Q^+ \in SS\text{pec}(N)$.

Definition 4.3.4. For any subset $A \subseteq N$. We define $B_A = \{P \in SS\text{pec}(N) / A \not\subseteq P \}$. In case $A = \{x\}$, we write $B_x = \{P \in SS\text{pec}(N) / x \not\subseteq P \}$. For any subset $U \subseteq L$, we define $B_U = \{I \in SS\text{pec}(L) / U \not\subseteq I \}$.

Lemma 4.3.5. For any $\Gamma$-near ring $N$, $B(N) = \{B_x / x \in N\}$ forms a basis for a topology on $SS\text{pec}(N)$.

Proof. For any $P \in SS\text{pec}(N)$, there exists $x \in N$ such that $x \not\subseteq P$, because $P = N$. From the definition of $B_x$, $P \in B_x$. If $P \in B_y \cap B_z$ for some $y, z \in N$, then $y \not\subseteq P$ and $z \not\subseteq P$. Since $P$ is strongly prime, there exist finite subsets
F \subseteq N and \Delta \subseteq \Gamma such that yafz \notin P for some \alpha, \beta \in \Delta and f \in F. Hence \( P \in B_{yafz}. \) We claim that \( B_{yafz} \subseteq B_y \cap B_z. \) Let \( Q \in B_{yafz}. \) Then \( yafz \notin Q, \) suppose \( y \in Q \) or \( z \in Q, \) we have \( yafz \notin Q, \) a contradiction. Therefore \( y \notin Q \) and \( z \notin Q \) and consequently, \( Q \in B_y \cap B_z. \)

Lemma 4.3.6. Let \( N \) be a distributive \( \Gamma \)-near ring. Then \( B(L) = \{ B_{[x, \Gamma]} \mid x \in M \} \) forms a basis for a topology on \( SSpec(L). \)

Proof. Let \( P \in SSpec(L). \) Then \( P^+ \in SSpec(N) \) by Proposition 4.3.3, since \( B(N) \) is a basis on \( SSpec(N), \) there exists \( B_x \in B(N) \) such that \( P^+ \in B_x. \)

Hence \( x \notin P^+, \) that is \([x, \Gamma] \notin P \) and so that \( P \notin B_{[x, \Gamma]} \). Let \( Q \in B_{[y, \Gamma]} \cap B_{[z, \Gamma]} \) for some \( y, z \in N. \) Then \([y, \Gamma] \notin Q \) and \([z, \Gamma] \notin Q. \) It means that \( y \notin Q^+ \) and \( z \notin Q^+. \) Hence \( Q^+ \subseteq B_y \cap B_z. \) Since \( B(N) \) is a basis, there is an element \( B_{x'} \in B(N) \) such that \( Q^+ \subseteq B_{x'} \subseteq B_y \cap B_z. \) It can be easily verified that \( Q \in B_{[x', \Gamma]} \subseteq B_{[y, \Gamma]} \cap B_{[z, \Gamma]} \). Thus \( B(L) \) forms a basis for a topology on \( SSpec(L). \)

Theorem 4.3.7. Let \( N \) be a distributive \( \Gamma \)-near ring with a right unity and a strong left unity. Then there is a one-to-one order preserving correspondence between the following:

(i) base for \( SSpec(N); \)

(ii) base for \( SSpec(L); \)

Proof. Since \( (P^+)^+ = P \) by Proposition 1.3.27, the mapping \( B_x \mapsto B_{[x, \Gamma]} \) defines a one-to-one order preserving correspondence between \( B(N) \) and \( B(L). \)