4.1 INTRODUCTION

Multivariate discrete distributions like multinomial, negative multinomial, multivariate hypergeometric, multivariate Poisson and multivariate LSD have created interest in the researchers to use them as multivariate models but whenever a choice of an appropriate model is not easy then the generalized multivariate models like multivariate GNBD or multivariate GLSD are used. For generalizing multivariate NBD or multivariate LSD we introduce a new parameter vector $\mathbf{\theta}$ in the distributions. Whenever $\mathbf{\theta} = 1$ these generalized forms reduce to simple ungeneralized ones. These generalized distributions may be used as models for representing number of accidents suffered by the same individual in $k$ separate periods, for spatial distribution of galaxies, military aviation and to theory of epidemics.

Structural properties of multivariate GNBD have been given by Khatri and Mitra (1968), Sibuyu et al. (1964) whereas those of multivariate LSD have been given by Patil and Bildikar (1967). The problem of estimation of
multivariate LSD has been studied by Patil and Bildikar (1967). In this chapter we intend to study the structural properties and the problem of estimation of multivariate GNBD and multivariate GLSD. In Section 4.2 we derive the marginal and conditional distributions and relations for moments of multivariate GNBD and multivariate GLSD. Section 4.3 provides regression and correlation analysis. Section 4.4 deals with the estimation of parameters by the method of maximum likelihood and unbiased minimum variance.

4.2 STRUCTURAL PROPERTIES OF MULTIVARIATE GNBD AND GLSD

(i) Multivariate GNBD.

The sf of k-variate GNBD defined in (25) is

\[ f(x) = (1 - e \lambda t)^{-m} \] 

\[ = \sum_{x \in \mathbb{N}} x^m e^{-\lambda t} \]

and hence (25) is rewritten as

\[ P(X = x) = a(x_1, \ldots, x_k) \prod_{i=1}^k x_i^m e^{-\lambda t} \]

with probability generating function

\[ P(s_1, s_2, \ldots, s_k) = \prod_{i=1}^k \sum_{x_i=0}^{s_i} x_i^m e^{-\lambda t} \]
where $\phi_i = \phi_i (1 - \sum \phi_i)^{\beta_i - 1}$, $i = 1, 2, \ldots, k$.

(64) being same as (23), $k$-variate NBD we get the following results for the marginal and conditioned distributions and moments of $k$-variate GNBTD from those of $k$-variate NBD given in Johnson and Kotz (1969) without proof.

If a rv $X = (X_1, X_2, \ldots, X_k)$ has $k$-variate GNBTD defined in (25) with pv $(m, \varnothing, \beta)$

(A) the rv $(X_1, X_2, \ldots, X_r)$, $r \leq k$ has the $r$-variate GNBTD with pv $(m, \varnothing_1, \varnothing_2, \ldots, \varnothing_r, \beta_1, \ldots, \beta_r)$

(B) the distribution of $(X_1, X_2 + \ldots + X_k)$ is the bivariate GNBTD with pv $(m, \varnothing_1, \varnothing_2 + \ldots + \varnothing_k, \beta_1, \beta_2 + \ldots + \beta_k)$

(C) the conditional distribution of $(X_1, X_2, \ldots, X_r)$ given $(X_{r+1}, \ldots, X_k) = (x_{r+1}, \ldots, x_k)$ depends only on the sum $y = x_{r+1} + \ldots + x_k$ and not on the individual components and it is $r$-variate GNBTD with pv $(m + y, p_1, \ldots, p_r, \beta_1, \ldots, \beta_r)$,

where $p_i = \varnothing_i / (1 - \sum_{i=r+1}^{k} \varnothing_i)$, $i = 1, 2, \ldots, r$

(D) the components of the mean vector $\mu = (\mu_1, \ldots, \mu_k)$ are given by

$$\mu_i = E(X_i) = m \varnothing_i / M_i,$$

where $\varnothing_i = i - 1, 2, \ldots, k$. \hfill \ldots (65)
where \( \gamma = 1 - \theta_1 - \ldots - \theta_k \), \( M_i = \gamma - \beta \theta_i + \theta_i \), \( i = 1, 2, \ldots, k \)

(E) the covariance \( \sigma_{ij} \) between \( X_i \) and \( X_j \) is given by

\[
\sigma_{ij} = \frac{\gamma \mu_i \mu_j}{m_i}, \quad i, j = 1, 2, \ldots, k, i \neq j \ldots \quad (66)
\]

and variance \( \sigma_{ii} \) of \( X_i \) is given by

\[
\sigma_{ii} = \frac{\gamma \mu_i}{M_i} \left( \frac{\beta \mu_i}{m} + 1 \right), \quad i = 1, 2, \ldots, k \ldots \quad (67)
\]

(F) denoting \( r = (r_1, r_2, \ldots, r_k) \), \( r_i \) being non-negative integral valued for \( i = 1, 2, \ldots, k \) the raw moment of order \( r \) denoted by

\[
m_r = E(X_1^{r_1} \ldots X_k^{r_k})
\]

satisfies the recurrence relation

\[
m_{r+e_i} = \mu_i m_r + \frac{\gamma \theta_i}{M_i} \frac{\partial}{\partial \theta_i} m_r, \quad i = 1, 2, \ldots, k, \quad (68)
\]

where \( \mu_i \) is same as given in (65) and \( \gamma', M_i \) are same as defined in (D) of this section

(G) the factorial moment of order \( r = (r_1, \ldots, r_k) \) denoted by
\[ m_r = E(x_1^{r_1} \cdots x_k^{r_k}), \text{ where} \]

\[ z^{(k)} = z(z-1) \cdots (z-k+1) \text{ satisfies the recurrence relation} \]

\[ m_{r+e_i} = (\mu_i - r_i) m_r + \frac{\gamma_i}{M_i} \frac{3}{3 \theta_1} m_r, \quad i=1,2,\ldots,k, \]

... (69)

where \( \mu_i \) is same as given in (65) and \( \gamma_i, M_i \) are same as defined in (D) of this section.

(H) the central moment of order \( r = (r_1, \ldots, r_k) \) denoted by

\[ \gamma^r_r = E((x_1 - \mu_i)^{r_1} \cdots (x_k - \mu_k)^{r_k}), \]

where \( \mu_i (i = 1,2,\ldots,k) \) is same as given in (65) satisfies the recurrence relation

\[ \gamma^r_{r+e_i} = \frac{\gamma_i}{M_i} \frac{3}{3 \theta_1} \gamma^r_r + \sum_{j=1}^{k} r_j \sigma_{ij} \gamma^r_{r-e_j}, \quad i=1,2,\ldots,k \]

... (70)

where \( \gamma_i \) and \( M_i \) are same as defined in (D) of this section, \( \sigma_{ij} \) is same as given in (66).

We note that for \( \beta_i = 1, i = 1,2,\ldots,k \), the results (65) to (70) reduce to those for k-variate NBD defined in (23) and for \( \beta_i = 0, i=1,2,\ldots,k \) they reduce to those for k-variate MD defined in (21).
(11) **Multivariate G-LSD.**

The sf of k-variate G-LSD defined in (26) can be expressed in another form such as

\[ f(\underline{x}) = -\log (1 - \sum_{i=1}^{k} \varphi_i) = \sum_{x \in \Omega} a(x) \prod_{i=1}^{k} \varphi_i = G(\underline{\varphi}) \]

and hence (26) can be rewritten as

\[ P(\underline{X} = \underline{x}) = a(\underline{x}) \prod_{i=1}^{k} \varphi_i / \sum_{\underline{x} \in \Omega} \prod_{i=1}^{k} \varphi_i \]

with probability generating function

\[ P(\underline{s}) = \frac{G(\underline{s})}{G(\underline{\varphi})}, \]

where

\[ \varphi_i = \theta_i (1 - \theta_1) \theta_i^{\beta_i - 1}, \quad i = 1, 2, \ldots, k. \]

(71) being same as (24), k-variate LSD with sf \((\varphi_1, \ldots, \varphi_k)\) we get the following results for the k-variate G-LSD from those of k-variate LSD obtained by Patil and Bildikar (1967) without proof.

If a rv \(X = (X_1, X_2, \ldots, X_k)\) has k-variate G-LSD defined in (26) or (71) with pv \((\underline{\varphi}, \underline{\varphi})\) then
(I) the rv \((X_1, \ldots, X_r), r < k\) has the \(r\)-variate modified GLSD defined in (27) with pv 
\[(\delta; p_1, \ldots, p_r, \beta_1, \ldots, \beta_r), \text{ where}
\]
\[
p_i = \phi_i / (1 - \sum_{i=r+1}^{k} \phi_i)
\]
\[
\delta = -\log (1 - \sum_{i=r+1}^{k} \phi_i) / L,
\]
\[
L = -\log (1 - E_i)
\]

(J) the distribution of \((X_1, X_2 + \cdots + X_k)\) is the bivariate GLSD with pv \((\phi_1, \phi_2 + \cdots + \phi_k, \beta_1, \beta_2 + \cdots + \beta_k)\)

(K) the conditional distribution of \((X_1, \ldots, X_r)\) given \((X_{r+1}, \ldots, X_k) = (x_{r+1}, \ldots, x_k)\) depends only on the sum \(y = x_{r+1} + \cdots + x_k\) and not on the individual components and whenever \(x_{r+1} + \cdots + x_k = 0\) the conditional distribution is the \(r\)-variate GLSD with pv \((\phi_1, \ldots, \phi_r, \beta_1, \ldots, \beta_r)\) whereas whenever \(x_{r+1} + \cdots + x_k > 0\) it is \(r\)-variate GBD defined in (25) with pv \((y, \phi_1, \ldots, \phi_r, \beta_1, \ldots, \beta_r)\)

(L) the components of the mean vector \(\mu = (\mu_1, \ldots, \mu_k)\) are given by 
\[
\mu_i = E(X_i) = \theta_i / L M_i,
\]  \(\ldots (72)\)
where $L = -\log \gamma$, $\gamma$ and $M_1$ being same as defined in (D) of this section.

(M) the covariance between $X_i$ and $X_j$ is given by

$$
\sigma_{ij} = \mu_j \left( \frac{\gamma \theta_i}{M_i^2} \right) - \mu_i, \quad i \neq j, \quad i, j = 1, 2, \ldots, k
$$

and variance $\sigma_{ii}$ of $X_i$ is given by

$$
\sigma_{ii} = \mu_i \left( \frac{\gamma}{M_i} \right) - \mu_i + \frac{\theta_i M_i^2}{\gamma M_i^2}, \quad i = 1, 2, \ldots, k
$$

where $\mu_i (i = 1, 2, \ldots, k)$ is same as given by (72), $\gamma$ and $M_1$ are same as defined in (D) of this section.

(N) the recurrence relations for raw moments, factorial moments and central moments of the multivariate G-LSD defined in (26) are same as given by (68), (69), (70) respectively with $\mu_i$, $\sigma_{ij}$ and $\sigma_{ii}$ are same as in (72), (73) and (74) respectively.

We remark that for $\beta_i = 1$, $i = 1, 2, \ldots, k$, the results (68), (69), (70), (72), (73) and (74) reduce to those for k-variate LSD and they agree with the results of Patil and Bildikar (1967).
4.3 REGRESSION AND CORRELATION FOR THE MULTIVARIATE GNBD and G-LSD.

In view of results (c) and (k) of Section 4.2, without any loss of generality we study the regression of \( X_1 \) on \( X_{r+1} + \ldots + X_k \) in order to study the regression of \( X_i \) on \( X_{r+1}, \ldots, X_k \) for \( k \)-variate GNBD or \( k \)-variate G-LSD. Thus in order to predict the value of \( X_i \), no weighing of the individual components of the conditioning vector \( (X_{r+1}, \ldots, X_k) \) is necessary. We may also observe that while similar property holds for \( k \)-variate NBD and MD, it is not necessarily true for the multivariate normal distribution.

**Theorem 3.** If the rv \( X = (X_1, \ldots, X_k) \) has the \( k \)-variate GNBD defined in (25) with \( p \text{ rv } (m, \theta_1, \theta_2) \) then for each \( i = 1, 2, \ldots, r \) the regression of \( X_i \) on \( X_{r+1}, \ldots, X_k \) is given by

\[
E(X_i / x_{r+1}, \ldots, x_k) = (m + y)/(1 - \beta_i \theta_i), \quad y = x_{r+1} + \ldots + x_k,
\]

where \( i = 1, 2, \ldots, r \).

**Proof.** Follows from (c) of Section 4.2 and result (15).

**Theorem 4.** If the rv \( X = (X_1, \ldots, X_k) \) has the \( k \)-variate G-LSD defined in (26) with \( p \text{ rv } (\theta, \phi) \) then for each \( i = 1, 2, \ldots, r \)
the regression of $X_i$ on $X_{r+1}, \ldots, X_k$ is given by

$$E(X_i \mid X_{r+1}, \ldots, X_k) = \frac{\theta_i}{(1 - \beta_i \theta_i)(-\log(1 - \theta_i))}, \text{if } x_{r+1} + \cdots + x_k = 0$$

$$= \frac{\theta_i + x_{r+1} + \cdots + x_k}{1 - \beta_i \theta_i}, \text{if } x_{r+1} + \cdots + x_k > 0.$$  

**Proof.** Follows from result (K) of Section 4.2 and results (15) and (20).

**THEOREM 5.** If the rv $X = (X_1, \ldots, X_k)$ has the $k$-variate GNBD defined in (25) with pv ($m$, $\varnothing$, $\beta$) then the multiple correlation coefficient $\rho_{1(23\ldots k)}$ of $X_1$ on $X_2, \ldots, X_k$ is given by

$$\rho_{1(23\ldots k)} = \frac{\mu_1}{m} \left( \beta_1 - \frac{A_1}{\nu_{k-1}} \right), \quad \ldots \quad (75)$$

where $\mu_1$ is same as given in (65),

$$\nu_{k-1} = \sum_{j=2}^{k-1} \frac{\mu_1}{m} A_j + (\frac{\mu_k}{m} \beta_k + 1) A_k,$$

$$A_j = \frac{k-1}{j} \prod_{i=1}^{j} \left( \frac{\mu_1}{m} (\beta_1 - 1) + 1 \right), \quad j = 1, 2, \ldots, k.$$
Proof. Let $\Sigma$ be variance-covariance matrix of a rv $X = (X_1, \ldots, X_k)$ which follows $k$-variate GNBD defined in (25). $\sigma_{ij}$, the $(ij)$th elements of $\Sigma$ is given by (66) and $\sigma_{ii}$, the $(ii)$th element of $\Sigma$ is given by (67). Then we know from the general theory that

$$P(23 \ldots k) = 1 - \frac{1}{\sigma_{ii} | \Sigma_{11} |},$$

... (76)

where $| \Sigma_{11} |$ is the cofactor of $\sigma_{11}$ in $| \Sigma |$.

Using (66) and (67) we get

$$| \Sigma | = \frac{\gamma^k \mu_1 \cdots \mu_k}{m_1 \cdots m_k} | V_k |,$$

where $\gamma$, $m_i$ are same as defined in (D) of Section (4.2),

$$| V_k | = \begin{vmatrix}
\frac{\mu_1}{m} + 1 & \frac{\mu_1}{m} & \cdots & \frac{\mu_1}{m} \\
\frac{\mu_2}{m} & \frac{\beta_2 \mu_2}{m} + 1 & \cdots & \frac{\mu_2}{m} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\mu_k}{m} & \frac{\mu_k}{m} & \cdots & \frac{\beta_k \mu_k}{m} + 1 \\
\end{vmatrix},$$

... (77)

$$= \sum_{j=1}^{k} \frac{\mu_j}{m} A_j + \left( \frac{\mu_k}{m} \beta_k + 1 \right) A_k,$$

$$A_j = \sum_{i=1}^{k-1} \left( \frac{\mu_i}{m} ( \beta_i - 1 ) + 1 \right), \quad j = 1, 2, \ldots, k.$$
Further noting that
\[ | \Sigma_{11} | = \frac{\gamma^{k-1} \mu_2 \cdots \mu_k}{M_2 \cdots M_k} | v_{k-1} |, \]
where \( | v_{k-1} | \) is the cofactor of \( \frac{\beta_1 \mu_1}{m} + 1 \) in \( | v_k | \)
given by (77) and using (77) in (76) we get (75).

Particularly for \( \beta_i = 1, i = 1, 2, \ldots, k \) (75) will reduce to the square of the multiple correlation coefficient of \( X_1 \) on \( X_2 \ldots X_k \) for \( k \)-variate NBD defined in (23).

**Theorem 6.** If the rv \( X = (X_1, \ldots, X_k) \) has the \( k \)-variate GLSD defined in (26) with \( p_V \) \((\mathcal{G}, \mathcal{L})\) then the multiple correlation coefficient \( \rho_{1(23\ldots k)} \) of \( X_1 \) on \( X_2, \ldots, X_k \) is given by

\[
\rho_{1(23\ldots k)}^2 = 1 - (a_1^2 \mu_1 + \gamma M_1)^{-1} (1 - (a_1 \mu_1) A_1 / | v_{k-1} |),
\]

where \( \gamma, M_1 \) are same as defined in (D) of Section (4.2),

\[
a_i = \gamma \theta_i / M_i^2, \quad i = 1, 2, \ldots, k,
\]

\[
| v_{k-1} | = \sum_{j=2}^{k-1} (a_j - \mu_j) A_j + (a_k \beta_k - \mu_k + \gamma / M_k) A_k,
\]

\[
A_j = \prod_{i=1}^{k} \frac{a_i (\beta_i - 1) + \gamma / M_i}{\prod_{i \neq j}^{k} (a_i (\beta_i - 1) + \gamma / M_i)}, \quad j = 1, 2, \ldots, k.
\]
**Proof.** Being similar to that of Theorem 3, it is left.

Particularly for \( \beta_i = 1 \), \( i = 1, 2, \ldots, k \), (78) reduces to the square of the correlation coefficient for \( k \)-variate LSD obtained by Patil and Bildikar (1967).

**THEOREM 7.** If the rv \( (X_1, X_2) \) has the bivariate GNBD with \( pv (m, \theta_1, \theta_2, \beta_1, \beta_2) \) or has the bivariate G-LSD with \( pv (\theta_1, \theta_2, \beta_1, \beta_2) \) then the ordinary correlation coefficient \( r_{12} \) between \( X_1 \) and \( X_2 \) is given by

\[
r_{12} = (1 - \mu_1 / \sigma_{11}) (1 - \mu_2 / \sigma_{22})
\]

**Proof.** On simplifying \( \sigma_{12}^2 / (\sigma_{11} \sigma_{22}) \) and right hand side of (79) with the use of the results (65), (66), (67), (72), (73), (74) we get the proof.

**THEOREM 8.** If the rv \( X=(X_1, X_2, \ldots, X_k) \) has the \( k \)-variate GNBD defined in (25) or the \( k \)-variate G-LSD defined in (26) then the multiple correlation coefficient of \( X_1 \) on \( X_2, \ldots, X_k \) is equal to the ordinary correlation coefficient of \( X_1 \) on \( X_1 + \ldots + X_k \).

**Proof.** This follows from Theorems 3, 4 and 5 with results (B) and (J) of Section (4.2).
4.4 ESTIMATION FOR MULTIVARIATE GNBD AND MULTIVARIATE GLSD

(a) Multivariate GNBD.

Observing that k-variate GNBD defined in (25) is a special case of k-variate MPSD defined in (6) with
\[ f(\theta) = (1 - \theta \beta_j)^{-m}, \quad \theta_i = \theta_i(1 - \theta \beta_j)^{-1}, \quad i=1,2,\ldots,k \]
using the results of Section (3.3) and (65) we get maximum likelihood estimators \( \hat{\theta}_1, \ldots, \hat{\theta}_k \) of \( \theta_1, \ldots, \theta_k \) for known \( \beta_1, \ldots, \beta_k \) by solving the equation

\[
\hat{\theta}_i \sum_{j=1}^{k} \left( \bar{x}_j (\beta_j - 1) \right) - \bar{x}_i \left( 1 - \sum \hat{\theta}_i \right) + m \hat{\theta}_i = 0, \quad i=1,2,\ldots,k,
\]

where \( \bar{x}_j = \frac{1}{n} \sum_{m=1}^{n} x_{ij} \) is the sample mean of \( n \) observations \( x_{ij} (i=1,2,\ldots,k, j=1,2,\ldots,n) \) from k-variate GNBD defined in (25). The information matrix of \( (\hat{\theta}_1, \ldots, \hat{\theta}_k) \) is

\[
I = \left( n \frac{M_i}{\gamma(\theta)} \cdot \frac{M_j}{\gamma(\theta)} \cdot \sigma_{ij} \right),
\]

where \( M_i \) and \( \gamma \) are same as defined in (D) of Section (4.2), \( \sigma_{ij} \) is same as given in (66), \( \sigma_{ii} \) is same as given in (67).

For the problem of MVU estimation of the k-variate GNBD defined in (25) we first note that

\[
(1 - \sum \theta_i) = \sum b(z; n) \theta_1 \cdots \theta_k.
\]
where summation extends over $z_1 = 0, 1, 2, \ldots$ for $i = 1, 2, \ldots, k$, such that

$$
\sum_{b(z; n)} = \frac{mn(mn + \sum \beta_i z_i - 1)!}{(mn + \sum \beta_i z_i - \sum z_i)! z_1 \cdot \ldots \cdot z_k} \quad \ldots \quad (80)
$$

where $z_i = \sum_{j=1}^{n} x_{ij}$, $i = 1, 2, \ldots, k$, $x_{ij}$ being the $i$th component of the $j$th observation on the rv $(X_1, X_2, \ldots, X_k)$ having $k$-variate GNBD defined in (25).

By the use of result (56) with $b(z; n)$ given in (80), for known $\beta_i$, $i = 1, 2, \ldots, k$, whenever it exists, the MVU estimator of $\Pi \phi_1^{r_i}$, $r_i (i = 1, 2, \ldots, k)$ being non-negative integer, can be explicitly written as

$$
\sum (mn + a - b - 1)! (mn + a - c)! (mn + a - c - b + d)! (mn + a - 1)! \Pi (z_i'/(z_i - r_i)!),
$$

where $\sum \beta_i z_i = a$, $\sum \beta_i r_i = b$, $\sum z_i = c$, $\sum r_i = d$.

For $\beta_i = 1$, $i = 1, 2, \ldots, k$ the expression given in (80 a) will be the MVU estimator of $\Pi \phi_1^{r_i}$ of the $k$-variate NBD defined in (23) and for $\beta_i = 0$, $i = 1, 2, \ldots, k$, the expression given in (80 a) will be the MVU estimator of $\Pi (\theta_i / (1 - \theta_i))^{r_i}$ for the $k$-variate MD defined in (21).
Further for known \( p^{(i=1,2,...,k)} \) with \( b(z; n) \) given, by (80) by using the result (63) the MVU estimator \( p(z; n) \) of probability function \( p(z, r; n) \) of k-variate GNBD defined by (25), \( r = (r_1, ..., r_k) \in T \), a known vector, always exists and it is given by

\[
p(z, r; n) = \frac{m(n-1)}{n} \frac{(m-b-1)! (mn+a-b-1)! (mn+a-c)!}{(m+b-d)! (mn+a-b-c+d)! (mn+a-l)!} \prod_{i=1}^{k} \left( \frac{z_i}{r_i} \right)
\]

... (81)

where \( a, b, c, d \) are same as defined under (80 a),

Particularly for \( \beta_i = 0, i = 1, 2, ..., k, \) (81) reduces to

\[
p(z, r; n) = \left( \frac{z_1}{r_1} \right) ... \left( \frac{z_k}{r_k} \right) \left( \frac{mn-\Sigma z_i}{m-\Sigma r_i} \right) / \left( \frac{mn}{m} \right)
\]

and for \( \beta_i = 1, i = 1, 2, ..., k, \) (81) reduces to

\[
p(z, r; n) = \frac{m}{m+\Sigma r_i} \left( \frac{z_1}{r_1} \right) ... \left( \frac{z_k}{r_k} \right) \left( \frac{mn-\Sigma z_i - 1}{m+\Sigma r_i} \right) / \left( \frac{mn}{m+\Sigma r_i} \right)
\]

... (83)

Results (82) and (83) are MVU estimators of \( p(z; \theta) \) of MD and k-variate NBD defined in (21) and (23) respectively and they agree with the results (28) and (29) of Patil (1966) respectively.
Multivariate GLSD.

Again we observe that k-variate GLSD defined in (26) is a special case of k-variate MPSD defined in (6) with

\[ f(\theta) = - \log (1 - \Sigma \theta_i) \]

\[ s_i(\theta) = \theta_i (1 - \Sigma \theta_i)^{\beta_i - 1}, \quad i = 1, 2, \ldots, k. \]

Using the results of Section (3.3) and result (72) we get maximum likelihood estimators \( \hat{\theta}_1, \ldots, \hat{\theta}_k \) of \( \theta_1, \ldots, \theta_k \) for known \( \beta_1, \ldots, \beta_k \) by solving the equations

\[ (1 - \Sigma \hat{\theta}_i) \bar{x}_i - \hat{\theta}_i \Sigma_{j=1}^k (x_{ij}(\beta_j - 1)) + \hat{\theta}_i / (\log (1 - \Sigma \hat{\theta}_i)) = 0 \]

where \( \bar{x}_j = \frac{\sum_{m=1}^n x_{jm}}{n} \), the sample mean of \( n \) observations \( x_{ij} \) \( (i = 1, 2, \ldots, k, j = 1, 2, \ldots, n) \) from the k-variate GLSD defined in (26). The information matrix of \( (\hat{\theta}_1, \ldots, \hat{\theta}_k) \) is

\[ I = (n \frac{M_i}{\gamma \hat{\theta}_i} \cdot \frac{M_j}{\gamma \hat{\theta}_j} \sigma_{ij}), \]

where \( \sigma_{ij} \) is same as defined in (73), \( \sigma_{ii} \) is same as defined in (74), \( \gamma \) and \( M_i \) are same as defined in (D) of Section (4.2).

For the problem of MVU estimation of the k-variate GLSD defined in (26) we first note that
(- \log (1 - \Sigma \theta_i))^n = \Sigma b(z; n) \phi_1^{z_1} \ldots \phi_k^{z_k},

where \( z_i = \sum_{j=1}^{n} x_{ij} \), \( i = 1, 2, \ldots, k \), \( x_{ij} \) being the \( i \)th component of the \( j \)th observation on the \( r \)v \( (X_1, X_2, \ldots, X_k) \) having \( k \)-variate GLSD defined in (26), summation extends over \( z_i = 0, 1, 2, \ldots \) such that \( c = \Sigma z_i \geq n \). \( b(z; n) \) is given by

\[
b(z; n) = \sum_{t=0}^{c-1} \frac{(c-1)!}{(\Sigma \beta_i z_i - c)!} \frac{\left| S(t + 1, n) \right|}{(\Sigma \beta_i z_i - t - 1)! z_1^{z_1} \ldots z_k^{z_k}},
\]

where \( |S(m, n)| \) is the signless Stirling number of the first kind with arguments \((m, n)\) (see Riordan (1958)).

(84) is the multivariate form of \( b(z; n) \) obtained for GLSD by Patel (1979b). We note that for \( \beta_i = 1, i = 1, 2, \ldots, k \), \( b(z; n) \) reduces to that of \( k \)-variate LSD given by Patil and Bildikar (1987).

With \( b(z; n) \) given by (84) for known \( \beta_i, i = 1, 2, \ldots, k \), the MVU estimator of \( \Pi \phi_{r_i}^{x_i} \), \( r_i (i = 1, 2, \ldots, k) \) being non-negative integer, can be explicitly expressed with the use of the result (56) as
where \( a, b, c, d \) are same as defined under (80 a).

For \( \beta_i = 1, i = 1, 2, \ldots, k \), (85) reduces to the MVU estimator of \( \Pi \theta_i \) for the \( k \)-variate LSD defined in (24) c and the result agrees with that of Patil and Bildikar (1967).

Further for known \( \beta_i, i = 1, 2, \ldots, k \) with \( b(\mathbf{z}; n) \) given by (84) and with the use of result (65) the MVU estimator of the probability function \( p(\mathbf{z}, \theta) \) at \( \mathbf{X} = \mathbf{r} \) of the \( k \)-variate GLSD defined by (26), \( \mathbf{r} = (r_1, \ldots, r_k) \in T \), a known vector, always exists and it is given by

\[
\sum_{t=0}^{c-1} \frac{(c-1)!}{(a-c)!} \frac{S(t+1,n)}{(a-t-1)!} \frac{(b - d)!}{(b-1)!} \frac{b(z - r; n-1)}{r_1! \cdots r_k! b(g; n)}\]

(85 a)

where \( b, d \) are same as defined under (80 a).

For \( \beta_i = 1, i = 1, 2, \ldots, k \) the expression given in (85 a) will be the MVU estimator of \( p(\mathbf{z}, \theta) \), the probability of \( k \)-variate LSD defined in (24) at \( \mathbf{X} = \mathbf{r} \) and the result agrees with Patil and Bildikar (1967).

The results of this chapter have been obtained by Patel (1979 c).