# CHAPTER - 2

RELATED ASPECTS AND DERIVATION OF MODIFIED REYNOLDS EQUATION

## 2.1 Introduction

2.1.1 Inertia and Turbulent Effects in Lubrication 038

2.1.2 Thermal Effects in Lubrication 040

2.1.3 Non-Newtonian Lubricants 041

2.1.4 Surface Roughness Effects in Lubrication 043

## 2.2 Mathematical Modelling of a Bearing System

2.2.1 Equation of State 047

2.2.2 Constitutive Equation of Lubricant 048

2.2.3 Continuity Equation 050

2.2.4 The Equations of Motion 051

2.2.5 Energy Equation 054

2.2.6 Elastic Considerations 055

2.2.7 Surface Roughness 056

## 2.3 Boundary Conditions

## 2.4 Basic Assumptions of Hydrodynamic Lubrication

## 2.5 Modified Reynolds Equation

## 2.6 The Bearing Performance Characteristics

037

038

040

041

043

045

047

048

050

051

054

055

056

057

058

061

069
In this chapter, we propose to develop the generalized Reynolds equation.

2.1 INTRODUCTION:

The basic equation employed in the analysis of fluid film lubrication is called the Reynolds equation which was first presented by Reynolds in 1886 which was formed by combining the equation of motion and the equation of continuity. In deriving this equation Reynolds neglected fluid inertia and gravitational effects in relation to viscous action restricting his analysis to a thin film of isoviscous, incompressible fluid. The adequacy of these assumptions was demonstrated by the satisfactory explanation of the performance of a number of non-porous bearings. However, the application of hydrodynamic theory within the assumptions made by Reynolds is valid only over a much narrower field than is generally supposed [Halton (1958)]; in particular, surface-roughness, high speed variable viscosity, thermal effects etc. emphasize the need to generalize the Reynolds equation accordingly. Moreover, the increased severity of bearing operating conditions, the greater use of gas bearings, the porous bearings and several limitations pertaining to lubricant properties etc. have also necessitated the generalization of Reynolds equation to account for the various effects. However, to improve upon the bearing performance which otherwise suffers on many counts, several attempts have been made incorporating the various factors such as surface roughness etc. arising out of recent technological advancements.
For the sake of completeness we discuss, below, the gradual development of the Reynolds equation incorporating the various effects that come to be accounted for and gradual relaxing of the various assumptions made in the theoretical investigations to approach near the more realistic situations.

2.1.1 Inertia and Turbulent Effects in Lubrication:

In most hydrodynamic bearings the lubricant flow is laminar and as such is governed by the Navier-Stokes equations which relate the pressure and viscous forces acting on the lubricant to the lubricant inertia. The importance of the inertia terms relative to the viscous terms in this equation can be characterized by a parameter known as Reynolds number which increases as the inertia effect increases. In the operation of most bearings the Reynolds numbers are small enough, so that the inertia effect can be ignored safely, reducing the governing relationship to the familiar Reynolds equation. However, with the continuing trend in machine design for which higher speeds as well as the use of unconventional lubricants such as water or liquid metals, the question of how important inertia will be at high Reynolds numbers in the laminar regime itself is of growing interest [Slezkin and Targ (1946); Kahlert (1948)]. If the Reynolds number becomes sufficiently high, turbulence may develop and the previous governing equation will no longer apply even with the inertia terms included. Several contributions towards including the inertia effects and their re-examinations have been made [Brand (1955); Osterle and Saibel (1955.a); Osterle, Chou and Saibel (1957);
Inertia effects are also important because of the current interest in assessing the importance of possible visco-elastic effects in lubricant behaviour. Both visco-elasticity and inertial effects are likely to become important in highly unsteady conditions. Therefore, it is necessary to have a through understanding of inertia effects in order to adequately isolate them as not to inadvertently attribute them to visco-elasticity.

The relative importance of the various fluid inertia terms in the inertia force has been based on the order of magnitude analysis and consequently with varying degree of approximations for simplifying the analysis; various studies have been made [Tichy and Winer (1970); Pinkus and Sternlicht (1961); Saibel and Macken (1974); Jones and Wilson (1975)]. The retention of inertial terms in the Navier-Stokes equations gives rise to non-linearity, resulting in the analysis becoming quite complicated. However, the method of averaged inertia and the method of iteration have been used to account for inertia terms [Slezkin and Targ (1946); Kahlert (1948); Agrawal (1969.a, 1969.b, 1970.c, 1970.d)]. Tichy and Winer (1970) used the method of regular perturbation taking Reynolds number as the perturbation parameter, while Rodkiewicz and Anwar (1971) used the series-expansion method.
2.1.2 Thermal Effects in Lubrication:

Research into the thermodynamics of fluid film bearings got an impetus by the experimental work of Fogg (1946) who observed that a parallel surface thrust bearing can support a load and gave an explanation for this by introducing the concept of thermal wedge i.e. the expansion of fluid due to heating. Later, Cope (1949) modified the classical Reynolds equation by introducing viscosity and density variation along the fluid film and obtained a corresponding form of the energy equation for determining temperature in the film. He coupled the energy balance equation in the fluid film with the momentum and continuity equations to obtain the temperature and the pressure distribution. This form of the equation differed from those of Christopherson (1941), and Cameron and Wood (1959) and this discrepancy was due to the neglect of the work done by the pressure forces in the energy equation [Charnes, Osterle and Saibel (1952)].

The effect of viscosity variation due to pressure and temperature on the characteristics of the slider bearing have been studied by Charnes, Osterle and Saibel (1953.a, 1953.b, 1955); Osterle and Saibel (1955.b). The variation of viscosity across the film thickness among others has been studied by Zienkiewicz (1957); Cameron (1960), who concluded that the temperature gradients and viscosity variation across the film should not be ignored. Dowson (1962) generalized the Reynolds equation by taking into consideration variation of fluid characteristics across the film thickness. Dowson and Hudson (1963.a) used the
generalized Reynolds equation, allowing for the heat transfer to the bearing solids to determine the fluid film thermal boundaries. They also [Dowson and Hudson (1963.b)] studied the case of parallel surface bearing by taking into account the heat transfer to the bearing surfaces and found that their investigation completely reversed the earlier predictions of the thermal and viscosity wedge effects. The effect of thermal distortions of the bearing solids was considered by Hahn and Kettleborough (1967, 1968, 1969) in an analysis of the one dimensional slider bearing. They concluded that while thermal distortions are responsible for the load carrying capacity exhibited by parallel surface sliders, they have little effect on the performance of inclined sliders when compared to leakage effects.

Gould (1967) discussed the thermohydrodynamic performance of fluid film between two surface approaching each other at a constant velocity, the effect of prescribed thermal gradients was also studied for journal bearings [Hagg (1944); Hughes and Osterle (1958); Motosh (1964); Tipei and Nica (1967)]. McCallion et. al. (1970) presented a thermohydrodynamic analysis of a finite journal bearing. The analysis indicated that the bearing load carrying capacity is insensitive to heat transfer in the solids. It also showed that the thermohydrodynamic load is not necessarily bounded by either the adiabatic or the isothermal solutions.

2.1.3 Non-Newtonian Lubricants:

In the development of better lubrications required to meet the needs of advancing scientific technology, it was found that by adding polymer additives to
the base lubricant, the viscosity of the fluid is increased considerably and is relatively temperature independent. This increase in viscosity brings about an increase in load carrying capacity. It was observed that the viscosity of the modified lubricant is, however, no longer constant, but decreases as the rate of strain increases. This particular phenomenon is known as pseudoplastic behaviour. It can be quantitatively explained by assuming that the initially random molecules of the polymer additives in the lubricant become realigned in the direction of motion when placed in a shear field. For example, in order to have engine performance fairly uniform over a wide range of temperature and pressure, engine oils have been treated with additives.

Steidler and Horowitz (1960) analyzed mathematically the effect of non-Newtonian lubrication on the slider bearing with side leakage. The corresponding experimental analysis was carried out by Dubois et al. (1960). Using a perturbation technique and taking \( n = 3 \), Saibel et al. (1962) gave a solution of the slider bearing with side leakage and found that the load carrying capacity is reduced by about ten percent.

Tipei and Rohde (1974) gave a new theological model for lubricants containing additives having long molecules. The viscosity of the fluid depends on the angle between viscous forces and the velocity vector at each point in the lubricating film. They discussed the finite slider bearing lubricated with such a lubricant. Kodnir et al. (1975) obtained an approximate solution of the stationary,
isothermal elastohydrodynamic problem for a Ree-Eyring fluid model, also the
solution's algorithm is described for a non-Newtonian fluid of an arbitrary model.
A similar problem has been solved by Chow and Saibel (1971), Bell (1972)
studied the effect of lubrication on rolling surfaces when the lubricant used was a
 Ree-Eyring fluid.

2.1.4 Surface Roughness Effects in Lubrication:

Halton (1958) recognized that, in bearings working with small film
thickness it was unrealistic to assume that the bearing surfaces can be represented
by smooth mathematical planes. A more realistic representation of engineering
rubbing surfaces was presented by Burton (1963) by critically examining the
Reynolds equation. However, his method was inappropriate for the general type of
roughness. Tzeng and Saibel (1967.a) studied the effect of surface roughness by
introducing stochastic concepts related to roughness; while deriving the Reynolds
equation. This method also fell short of approximating the roughness. It was the
proposed a statistical analysis in order to obtain a modified Reynolds equation and
analyzed the surface roughness effect. They concluded that the bearing
performance could decrease or increase depending on the type of roughness. In
1975 Christensen et al. (1975) obtained a generalized Reynolds equation
applicable to rough surfaces by assuming that the film thickness function is a
stochastic process. Christensen and Tonder's approach was employed by Gupta
and Deheri (1996) in order to study the effect of transverse surface roughness on the behaviour of squeeze film in a spherical bearing. In this paper it was concluded that the bearing suffers on account of transverse roughness. Subsequently, Andharia, Gupta and Deheri (1997) extended the method of Christensen and Tonder by incorporating the measure of symmetry and mean of roughness besides the standard deviation, while deriving the modified Reynolds equation, and analyze the problem of the effect of this generalized roughness on the performance of slider bearings. It has been established in these investigations that the bearing performance suffers mostly because of transverse roughness, while the standard deviation of roughness may enhance the performance of a longitudinally rough slider bearing.

2.2 MATHEMATICAL MODELLING OF A BEARING SYSTEM:

The mathematical modelling of the bearing system is closely linked to the research developments in the field of fluid dynamics of real fluids which started in nineteenth century. Hydrodynamic film lubrication was effectively used before it was scientifically understood. The process of lubrication is basically a part of an overall phenomena of hydrodynamics whose scientific analysis was initiated during nineteenth century. Adams (1853) first attempted, developed and patented several rather good designs for railway axle bearings in 1847. The understanding of hydrodynamic lubrication began with the classical experiments of Tower (1883, 1884, 1885) in connection with the investigation of friction of the railway partial journal bearing when he measured the lubricant pressure in the bearing. Petrov (1883) in his separate independent studies reached the same conclusion from friction measurements. This work was closely followed by Reynolds (1886). He applied hydrodynamic laws to the bearing problem and was able to explain Tower's results satisfactorily. He derived and employed an equation for the analysis of fluid film lubrication which has by now become a basic governing equation and is named after him as Reynolds equation. He has combined Navier-Stokes equations with continuity equation to generate a second order differential equation for lubricant pressure. This equation is derived under certain assumptions, such as neglect of inertia and gravitational effects in comparison to
viscous action, lubricating film to be a thin one of isoviscous incompressible fluid etc. This equation can be deduced from first principles also provided same set of assumptions is made. Although, the adequacy of these assumptions was demonstrated by the satisfactory explanation of the performance of a number of bearings, however, subsequently it was realized that the Reynolds equation is valid only over a much narrower field than is generally supposed. The so called conventional Reynolds equation contains viscosity, density and film thickness as parameters. These parameters both determine and depend on the temperature and the pressure fields and on the elastic behaviour of the bearing surfaces. Besides these, sometimes surface roughness, porosity and other increased severity of bearing operating conditions etc. may demand the need to generalize Reynolds equation accordingly to account for these effects. Likewise, consistent with these effects and the requirement of the particular bearing problems, it may become necessary to relax few of the assumptions used for derivation of the Reynolds equation. Thus, study of hydrodynamic lubrication is from a mathematical point of view is infact, the study of a particular form of Navier-Stokes equations compatible with the system. Since the Reynolds time, researches in the field of lubrication have made much progress and with the rapid advancement of machines, manufacturing process and materials in which lubrication plays an important role, the study of lubrication has gained considerable importance and
has become, from analytical point of view, an independent branch of fluid mechanics. From practical point of view it remains a part of TRIBOLOGY.

Mathematical modelling of a bearing system consists of various conservation laws of fluid dynamics such as conservation of mass, momentum, energy and equation describing various aspects characterizing the bearing problem such as constitutive equation of lubricant, viscosity dependence on pressure - temperature, equation of state, elastic deformations, surface roughness etc.

2.2.1 Equation of State:

Phenomenological considerations require specification of the state of fluid which is given by an equation which is called equation of state. For an incompressible fluid it is given by

\[ \rho = \text{constant} \]  \hspace{1cm} (2.2.1)

while for a perfect gas for isothermal variations in pressure it is given by Boyle-Mariotte law as

\[ P = \rho RT \]  \hspace{1cm} (2.2.2)

where \( R \) is the universal gas constant.

For constant compressibility fluids under isothermal conditions, equation of state is

\[ \rho = \rho_0 \exp [C(P - P_0)] \]  \hspace{1cm} (2.2.3)
where $\rho_0$ is the value of $\rho$ at the reference atmospheric pressure $P_0$ and $C$ is the compressibility. This particular equation of state applies rather well to most liquids.

For ideal gas flow under adiabatic condition, the equation of state is

$$P = \rho_0 \left( \frac{\rho}{P_0} \right)^\gamma \quad (2.2.4)$$

where $\gamma = c_p / c_v$ is the ratio of specific heats at constant pressure and constant volume of the fluid.

For many thermodynamic processes the Eq. (2.2.4) is written as

$$P = \rho_0 \left( \frac{\rho}{P_0} \right)^n \quad (2.2.5)$$

where $1 \leq n \leq \gamma$, this equation is called polytropic law.

Gas lubricating films are essentially isothermal because the ability of common bearing materials to dissipate heat is much greater than the rate of film heat generation. Both liquid and gas films may be considered incompressible under light load and at low speeds.

2.2.2 Constitutive Equation of the Lubricant:

The mathematical equation relating the viscous contribution to the stress tensor with the rate of deformation tensor is called constitutive equation applicable to the description of rheological behaviour of the lubricant. The constitutive equations are of three types namely integral type, rate type, and differential type.
Many lubricating fluids are generally Newtonian and in such case, shearing stress is directly proportional to rate of strain tensor, constant of proportionality being the dynamic viscosity of the lubricant. From mathematical point of view the assumption of the lubricant being Newtonian in character greatly simplifies the mathematical analysis. The lubricants which exhibit a relationship other than that exists for a Newtonian lubricant are generally called non-Newtonian lubricant.

In case of ideal plastic like grease, some initial stress must be imposed upon the lubricant before the flow begins. Minimum stress necessary to cause the flow is called yield stress. A real plastic behaves in a non-Newtonian manner upto a certain shearing stress and then starts to behave as if it is Newtonian. A number of fluids have been classified on the basis of their constitutive equation and given various names e.g. Maxwell fluid, Second order fluid, Walter's fluid, Oldroyd fluid, etc. A general class of non-Newtonian fluids for which stress tensor is expressed as directly proportional to some power n of deformation tensor is called power law fluid. If \( n > 1 \), the fluid is called dilatant and for \( n < 1 \) it is termed as pseudo-plastic. There are certain lubricants like paints which show time dependent behaviour i.e. their viscosity changes with time. If viscosity increases with time then the lubricants are called thixotropic and in the case of viscosity decreasing with time they are called rheopectic. Lai, Kuei and Mow (1978) have given a list of constitutive equations for the rheological behaviour of the synovial fluid which acts as lubricant in synovial joints of human or animal body.
Significant lubricating fluid properties are viscosity, density, specific heat and thermal conductivity. Amongst these fluid properties, viscosity plays a more prominent role. Viscosity varies with temperature as well as pressure and this variation is important in lubrication mechanics. There is a general rule that more viscous the lubricant it is more susceptible to change. In general, viscosity increases with pressure and decreases with temperature for most liquid lubricants. Viscosity of common gases increases with temperature but are comparatively insensitive to moderate temperature and pressure changes. A number of viscosity pressure, temperature relationship have been proposed.

The viscosity of heavily loaded lubricating film is generally treated as a function of both pressure and temperature. According to Barus (1893) Viscosity may be approximated for limited ranges by

\[ \mu = \mu_0 \exp \left[ a (P - P_0) + b (T - T_0) \right] \]  

\[ (2.2.6) \]

where a and b are called pressure and temperature viscosity coefficients and the subscript '0' refers to atmospheric conditions. Over reasonably large ranges of temperature and pressure, the linear relation.

\[ \mu = \mu_0 \left[ 1 + a (P - P_0) + b (T - T_0) \right] \]  

\[ (2.2.7) \]

is useful.

2.2.3 Continuity Equation:

All fluid flow problems satisfy the basic law of conservation of mass, besides laws of conservation of momentum and energy. The equation expressing
law of conservation of mass is called continuity equation. It expresses the condition that for any fixed volume of source-sink free region, the mass of entering fluid must equal the mass of fluid leaving plus the accumulated mass.

If the fluid is compressible, the continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \quad (2.2.8)$$

where $\mathbf{q}$ is the velocity vector of the flowing fluid and $\rho$ is the density. If the flow is steady $\frac{\partial \rho}{\partial t} = 0$, and hence the continuity equation becomes

$$\nabla \cdot (\rho \mathbf{q}) = 0 \quad (2.2.9)$$

The equation of continuity for homogeneous, incompressible fluid takes the form

$$\nabla \cdot \mathbf{q} = 0 \quad (2.2.10)$$

A comparison of Eqs. (2.2.8) and (2.2.10) shows that the density of the fluid does not appear in the continuity equation for incompressible fluids whereas it does appear in the corresponding equation for compressible fluids. Thus the continuity equation for incompressible fluids is a purely kinematical equation whereas for compressible fluids it is a dynamical one.

2.2.4 The Equations of Motion:

Principle of conservation of momentum when applied to fluid contained in a control volume states that forces acting on the fluid in the control volume equal
the rate of outflow of momentum from the control volume through the closed surface enclosing it. The mathematical equation expressing this condition for Newtonian, isoviscous, laminar, continuum and compressible fluid flow for which body forces such as gravitational forces or electromagnetic forces etc. are considered negligible, is :

\[
\frac{\partial q}{\partial t} + \rho \left( q \cdot \nabla \right) q = -\nabla p + (\lambda + \mu) \nabla (\nabla \cdot q) + \mu \nabla^2 q \tag{2.2.11}
\]

where \( \mu \) is called the coefficient of shear viscosity of the fluid and \( \lambda \) is called coefficient of bulk viscosity. It is often assumed that they are related by \( 3\lambda + 2\mu = 0 \). The Eqs. (2.2.11) were first obtained by Navier in 1821 and later independently by Stokes in 1845. Hence these are known as Navier-Stokes equations. The first term on the left hand side of Eq. (2.2.11) is temporal acceleration term while the second is convective inertia term. The first term on right hand side is due to pressure and the other terms are viscous forces. If, however, the fluid is incompressible, as is the case with most liquid lubricants, then

\[
\nabla \cdot q = 0 \quad \text{and Eq. (2.2.11) simplifies to}
\]

\[
\frac{\partial q}{\partial t} + \rho \left( q \cdot \nabla \right) q = -\nabla p + \mu \nabla^2 q \tag{2.2.12}
\]
In obtaining Eqs. (2.2.11) and (2.2.12) the viscosity coefficients have been
taken to be constants. But in certain situations \( \lambda \) and \( \mu \) may very with temperature
and pressure, hence they will both be functions of spatial coordinates. Taking this
variation of \( \lambda \) and \( \mu \) into account the Eq. (2.2.11) will take the form

\[
\frac{\partial q}{\partial t} + \rho (q \cdot \nabla) q = - \nabla p + (\lambda + \mu) \nabla (\nabla \cdot q) + \mu \nabla^2 q + (\nabla \cdot q) \nabla \lambda + \nabla \mu \cdot \text{def} \ q
\]

\[(2.2.13)\]

where \( \text{def} \ q \) is the deformation tensor.

In case of Non-Newtonian lubricants, suitable modifications of the Navier-
Stokes equations are required to be made.

Non-laminar flow may occur in bearings for two main reasons. The first is
high speed operation. The second is the use of unconventional lubricants of low
kinematic viscosity such as water, synthetic lubricants or liquid metals or
cryogenic fluids. A high velocity combined with a low kinematic viscosity leads to
a high Reynolds number resulting in either a super-laminar flow or turbulence
[Kumar and Rao (1992.a, 1992.b)]. For such a case, the necessary mathematical
changes in the analysis are called for.

When a large external electromagnetic field through the electrically
conducting lubricant is applied it gives rise to induced circulating currents, which
in turn interacts with the magnetic field and creates a body force called Lorentz
force. This extra electromagnetic pressurization pumps the fluid between the
bearing surfaces. In such a case Navier-Stokes equations for an incompressible, isoviscous liquid get modified as

\[
\begin{align*}
\rho \frac{\partial \mathbf{q}}{\partial t} + \rho (\mathbf{q} \cdot \nabla) \mathbf{q} &= -\nabla p + \mu \nabla^2 \mathbf{q} + \mathbf{J} \times \mathbf{B} \\
\end{align*}
\]

(2.2.14)

where \( \mathbf{J} \) is the electric current density and \( \mathbf{B} \) is the magnetic induction vector. In this case, Maxwell's equations and Ohm's law should also be taken into account. These are:

\[
\begin{align*}
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \cdot \mathbf{B} &= 0 \\
\mathbf{J} &= \sigma [\mathbf{E} + \mathbf{q} \times \mathbf{B}] \\
\nabla \times \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{E} &= 0
\end{align*}
\]

(2.2.15)

where \( \mathbf{E} \) is the electric field intensity vector, \( \sigma \) is the electrical conductivity and \( \mu_0 \) is the magnetic permeability of the lubricant.

2.2.5 Energy Equation:

This equation formulates the conservation of energy principle for a fluid element in which there is no heat source or sink. The equation is:

\[
\begin{align*}
\rho g \left[ \frac{\partial (C_v T)}{\partial t} + (\mathbf{q} \cdot \nabla)(C_v T) \right] + \mathbf{PV} \cdot q = \nabla \cdot (K \nabla T) + \Phi \\
\end{align*}
\]

(2.2.16)
where $C_v =$ Specific heat at constant volume per unit weight,

\[ T = \text{Absolute temperature}, \]

\[ K = \text{Coefficient of thermal conductivity}, \]

\[ \Phi = \text{Viscous dissipation function}. \]

The above equation states that within an element volume the rate of change of internal energy plus compression work must be balanced by the energy conducted by the fluid and dissipated by friction. The first term on the left hand side denotes abiaabatic compression work and the first term on right hand side is the conductive heat transfer. For incompressible flow of lubricant adiabatic compression work energy becomes zero. The energy equation does not apply when the film lubricated bearing during operation is isothermal.

2.2.6 Elastic Considerations:

In some cases, thermal stresses or high loading of the bearing surfaces may distort the film shape and consequently may effect the pressure distribution. The study of this aspect of lubrication is called elastohydrodynamic lubrication. This effect is particularly important in the lubrication of gear and roller bearings where very high pressure can be developed. In order to mathematically model such a system, which involves the interaction of elastic and fluid phenomena, additional equation to account for elastic deformations is needed. This equation is called elasticity equation; it relates the displacements of the solid surfaces to the stress system.
2.2.7 Surface Roughness:

In most of the theoretical studies of film lubrication, it has more or less explicitly been assumed that the bearing surfaces can be represented by smooth mathematical planes. It has, however, long been recognized that this might be an unrealistic assumption, particularly, in bearing working with small film thickness i.e. in boundary and mixed lubrication regimes. Several mathematical ways such as postulating a sinusoidal variation in film thickness have been introduced in order to seek a more realistic representation of rubbing surfaces. However, this method is perhaps, more appropriate in an analysis of the influence of waviness rather than roughness.

Earlier attempts for mathematical modelling of the rough bearing surfaces used the postulation by a saw-tooth curve [Davies (1963)] or representation by waviness approximated by a Fourier type series [Burton (1963)]. Tzeng and Saibel (1967.a), used a method of random analysis and assumed the one dimensional film thickness to be of the form

\[ h(x, e) = h(x) + h(x) \]  \hspace{1cm} (2.2.17)

where \( h(x) \) constitutes the stochastic variation from the smooth film thickness \( h(x) \). Thickness \( h(x) \) is regarded as a random variable whose probability density function is either a Gaussian normal probability distribution function or beta-distribution function given by
This distribution function is used to average out the physical quantities in the Reynolds equation with respect to film thickness. Besides this mathematical form, many other mathematical approaches have also come up in a number of investigations.

2.3 BOUNDARY CONDITIONS:

The mathematical description of lubrication problem formulated above is in the form of differential equations which is required to be solved and hence until a set of boundary conditions compatible with the physical system is not prescribed, the mathematical description of the model would not be complete. If the film conditions are steady, the momentum equations may be combined with the continuity equation to give equation governing the film pressure - this equation is called REYNOLDS equation which is a single differential equation relating pressure, density, surface velocities and film thickness. However, before these equations can be integrated, it is necessary to establish the boundary conditions. The combination of momentum and continuity equations allows the distributed film velocity to be eliminated and replaced by the film surface velocities. Unless slip is present these film surface velocities are identical to the velocities of the adjacent bearing surfaces - this condition is called no-slip condition. For the values
of Knudsen number, $K_n (= l/h)$, where $l$ is the mean free molecular path and $h$ is the film thickness) less than 0.01, flow may be treated as continuum and no-slip conditions may be applied. When $0.01 \leq K_n \leq 15$ slip flow becomes significant and for $K_n > 15$ fully developed molecular flow results. As regards the boundary conditions for the film pressure governed by Reynolds equation the entrance and exit effects of a self-acting slider bearing may usually be ignored. Pressure is therefore taken to be ambient along the boundary. Since the hydrodynamic pressures generated in the film are very large compared to the ambient pressures, the pressure conditions are usually assumed to be zero. The pressure condition at the source inlet of an externally pressurized film is given by the value of the supply pressure.

However, form of the boundary conditions for a particular problem depends upon the peculiarities of the particular situation.

2.4 BASIC ASSUMPTIONS OF HYDRODYNAMIC LUBRICATION:

The general mathematical model described above is highly non-linear in character besides being a coupled one. Thus, the severe complexity of the mathematical system describing the general problem of lubrication, theoretically, does not lend it at all straight to analytical study. A number of simplifications resulting from the physical considerations compatible with the system are required to be made before attempting to proceed to solve the system. Simplifications may be of great value if their limitations are clearly specified. It is of prime importance
that all assumptions or simplifications be justified and that the limitations imposed
thereby be understood in interpreting the results. Likewise, in certain situations
certain idealizations may be required to be made and consequently the limits of
their applicability must be recognized. Order of magnitude analysis may be
attempted to estimate the relative effects of various terms in the equations and
hence to simplify it. Assumptions that are to be made and the simplifications
resulting therefrom would depend upon the nature of the problem and the aspect of
the problem to be studied.

For the analysis that follows to derive the modified Reynolds equation
following assumptions are usually made:

(1) The lubricant is considered to be incompressible, non-conducting and non-
magnetic with constant density and viscosity, unless and otherwise stated.
Most lubricating fluids satisfy this condition.

(2) Flow of the lubricant is laminar, unless and otherwise stated. A moderate
velocity combined with a high kinematic viscosity gives rise to a low
Reynolds number, at which flow essentially remains laminar.

(3) Body forces are neglected, i.e. there are no external fields of force acting on
the fluid. While magnetic and electrical forces are not present in the flow of
non-conducting lubricants, forces due to gravitational attraction are always
present. However, these forces are small compared to the viscous force
involved, they are usually neglected in lubrication mechanics without causing any significant error.

(4) Flow is considered steady, unless and otherwise stated, i.e. velocities and fluid properties do not vary with time. Temporal acceleration due to velocity fluctuations are small enough in comparison with lubricant inertia, hence may usually be ignored.

(5) Boundary layer is assumed to be fully developed throughout the lubricating region so that entrance effects at the leading edge and the film discontinuity at the trailing edge from which vortices may be shed, are neglected.

(6) A fundamental assumption of hydrodynamic lubrication is that the thickness of the fluid film is considered very small in comparison with the dimensions of the bearings. As a consequence of this assumption:

(a) the curvature of the film may be neglected, so that bearing surfaces may be considered locally straight in direction.

(b) Fluid inertia may be neglected when compared with viscous forces.

(c) Since lubricant velocity along the transverse direction to the film is small, variation of pressure may also be neglected in this direction.

(d) Velocity gradients across the film predominate as compared to those in the plane of the film.

(7) The fluid behaves as a continuum which implies that pressure are high enough so that the mean free path of the molecule of the fluid are much
smaller than the effective pore diameter or any other dimension. No slip boundary condition is applicable at the bearing surfaces.

(8) Lubricant film is assumed to be isoviscous.

(9) Temperature changes of the lubricant are neglected.

(10) The bearing surfaces are assumed to be perfectly rigid so that elastic deformation of the bearing surfaces may be neglected.

(11) In case of bearings working with magnetic fluids, the lubricant is assumed to be free of charged particles.

(12) When bearings work under the influence of electromagnetic fields, it is assumed that the forces due to induction are small enough to be neglected.

2.5 MODIFIED REYNOLDS EQUATION:

The differential equation which is developed by making use of the assumptions of hydrodynamic lubrication in equations of motion and continuity equation and combining them into a single equation governing lubricant pressure is called Reynolds equation. The Reynolds equation when derived for more general situations like porous bearings or hydromagnetic bearings or bearings working with non-Newtonian or magnetic lubricant, etc. is called generalized Reynolds equation or modified Reynolds equation. This equation is the basic governing differential equation for the problems of hydrodynamic lubrication.

The differential equation originally derived by Reynolds is restricted to incompressible fluids. This, however is an unnecessary restriction, for the equation
can be formulated broadly enough to include effects of compressibility and
dynamic loading. We have called this the generalized Reynolds equation.

Consider that the upper surface of the bearing surfaces is $S_1$ and the lower
surface is $S_2$ which are in relative motion with uniform velocities $(U_1, V_1, W_1)$ and
$(U_2, V_2, W_2)$ respectively as shown in Fig. 2.5.1. The surfaces $S_1$ and $S_2$ enclose
the lubricant film. The lubricant velocities in the film region $F$, $S_1$ and $S_2$ are $(u, v,$
$w)$, $(u_1, v_1, w_1)$ and $(u_2, v_2, w_2)$ respectively. Lubricant pressures in $F$, $S_1$ and $S_2$
are $p$, $P_1$ and $P_2$ respectively. Film thickness $h$ is assumed the function of $x$.

1. The height of the fluid film $z$ is very small compared to the span and length
$x, y$. This permits us to ignore the curvature of the fluid film, such as in the
case of journal bearings, and replace rotational by translational velocities.

2. No variation of pressure across the fluid film. Thus,
\[
\frac{dp}{dz} = 0
\]

3. The flow is laminar; no vertex of flow and no turbulence occur any where
in the film.

4. No external forces act on the film. Thus,
\[
X = Y = Z = 0
\]

5. Fluid inertia is small compared to the viscous shear. These inertia forces
consist of acceleration of the fluid centrifugal forces acting in curved films
and fluid gravity. Thus,
Fig. 2.5.1 Configuration of Bearing
6. No slip at the bearing surfaces.

7. Compared with the two velocity gradients $du/\partial z$ and $dv/\partial z$, all other velocity gradients are considered negligible. Since $u$ and, to a lesser degree, $v$ are the predominant velocity and $z$ is a dimension much smaller than either $x$ or $y$, the above assumption is valid. The two velocity gradients $du/\partial z$ and $dv/\partial z$ can be considered shears, while all others are acceleration terms, and the simplification is also in line with assumption 5. Thus any derivatives of terms other than $du/\partial z$ and $dv/\partial z$ will be of a much higher order and negligible. We can thus omits all derivatives with the exception of $d^2u/\partial z^2$ and $d^2v/\partial z^2$.

The equation of motion under the assumptions stated above, takes the form

$$-\frac{dp}{dz} + \mu \frac{d^2u}{dz^2} = 0$$
(2.5.1)

$$-\frac{dp}{dz} + \mu \frac{d^2v}{dz^2} = 0$$
(2.5.2)

$$-\frac{dp}{dz} = 0$$
(2.5.3)
From Eq. (2.5.1) and Eq. (2.5.2)

\[ \frac{d^2u}{d\mu} = \frac{dp}{dx} \]  
\[ \frac{d^2v}{d\mu} = \frac{dp}{dy} \]  

and from Eq. (2.5.3)

\[ \frac{dp}{dz} = 0 \Rightarrow p = p(x, y) \]  

The no-slip boundary conditions are

\[ u = \text{U}_1, \ v = \text{V}_1, \ W = \text{W}_1 \text{ at } z = 0 \]

\[ u = \text{U}_1, \ v = \text{V}_1, \ W = \text{W}_1 \text{ at } z = h \]

By integrating Eq. (2.5.4) twice with the above boundary conditions we have,

\[ \frac{du}{dz} = \frac{1}{\mu} \frac{dp}{dx} z + A \]

\[ u = \frac{1}{\mu} \frac{dp}{dx} z^2 + Az + B \]

Similarly,

\[ \frac{dv}{dy} = \frac{1}{\mu} \frac{dp}{dy} z^2 + A_1z + B_1 \]
We now make use of the continuity Eq. (2.2.9) with no source or sinks present and with the state of the lubricant independent of time, the continuity equation reads,

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0 \quad (2.5.8)$$

The equation for homogeneous incompressible fluid takes the form

$$\nabla \cdot \mathbf{q} = 0$$

OR

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad (2.5.9)$$

Solutions of Eqs. (2.5.7) with related boundary conditions are

$$u = \frac{1}{2\mu} \frac{dp}{dx} \left( z^2 - hz \right) + \frac{z}{h} \left( U_2 - U_1 \right) + U_1 \quad (2.5.10)$$

$$v = \frac{1}{2\mu} \frac{dp}{dy} \left( z^2 - hz \right) + \frac{z}{h} \left( V_2 - V_1 \right) + V_1$$
Substituting values of $u$ and $v$ in Eq. (2.5.9), we get

\[
\frac{dw}{dz} = \frac{d}{dx} \left[ \frac{1}{2\mu} \frac{dp}{dx} \right] \frac{z}{(z^2 - hz)} + \frac{(U_2 - U_1)}{h} + U_1
\]

\[
- \frac{d}{dy} \left[ \frac{1}{2\mu} \frac{dp}{dy} \right] \frac{z}{(z^2 - hz)} + \frac{(V_2 - V_1)}{h} + V_1
\]

By integrating across the film thickness i.e. from $z = 0$ to $z = h$, we get

\[W_2 - W_1 = \]

\[
\frac{d}{dx} \left[ \frac{h^3}{12\mu} \frac{dp}{dx} \right] + \frac{d}{dy} \left[ \frac{h^3}{12\mu} \frac{dp}{dy} \right] = 
\]

\[
\frac{d}{dx} \left[ \frac{h}{2} \frac{dp}{dx} \right] + \frac{d}{dy} \left[ \frac{h}{2} \frac{dp}{dy} \right] = 
\]

\[
\frac{d}{dx} \left[ (U_1 + U_2) \right] + \frac{d}{dy} \left[ (V_1 + V_2) \right] + (W_2 - W_1) \quad (2.5.11)
\]

This equation is known as Generalized Reynolds equation. It holds for incompressible fluid.
In this equation the viscosity \( \mu \) is still treated as a variable, being a function of both the \( x \) and \( y \) co-ordinates. The film thickness \( h \), too, in general enough and can be a function of both co-ordinates.

Eq. (2.5.11) is non homogeneous partial differential equation of two variables. It is difficult equation to solve, and the degree of complexity depend on the form of the parametric functions \( m \) and \( h \) and on the boundary conditions. Even for the simplest case of \( m = \) constant and \( W = 0 \) (\( W_1 = W_2 = 0 \)) when Eq. (2.5.11) reduces to

\[
\frac{d}{dp} \left[ h^3 \right] + \frac{d}{dy} \left[ h^3 \right] =
\]

\[
\frac{d}{dx} \frac{d}{dx} \frac{d}{dy} \frac{d}{dy}
\]

\[
12\mu \left\{ \frac{d}{dx} \left[ (U_1 + U_2) \right] + \frac{d}{dy} \left[ (V_1 + V_2) \right] \right\}
\]

(2.5.12)

In cylindrical co-ordinates, using the substitutions

\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta
\]

the generalized Reynolds equation Eq. (2.5.11) becomes

\[
\frac{d}{dr} \left[ \frac{rh^3}{12\mu} \right] + \frac{1}{r} \frac{d}{d\theta} \left[ \frac{h^3}{12\mu} \right] =
\]

\[
\frac{d}{d\theta} \left[ (U_1 + U_2) \right] + \frac{1}{r} \frac{d}{d\theta} \left[ (V_1 + V_2) \right] + r \left( W_2 - W_1 \right)
\]

(2.5.13)
2.6 THE BEARING PERFORMANCE CHARACTERISTICS:

The Reynolds equation is solved with appropriate boundary conditions to get the expression for lubricant film pressure \( p \). From this film pressure, the load carrying capacity \( W \) of the bearing may be derived by integrating it over the entire bearing surface. Likewise, the shearing frictional force \( F \) on both the stator as well the bearing, may be computed. The coefficient of friction \( f = F/W \) may also be computed. For the purpose of pivoting, the centre of pressure may be found. For squeeze film bearings the relationship between the film thickness and the response-time may be established. The other performance characteristics like mass flow rate, stiffness of the bearing, temperature rise, power requirement etc. may also be calculated depending upon the nature of the bearing.