CHAPTER 6

ESTIMATION OF THE PARAMETERS OF THE GPD

5.1  INTRODUCTION

Though the equality of mean and variance, in general, characterizes the Poisson distribution, it has been observed that in a population supposed to be Poissonian the probability of the occurrence of an event can change with time and/or previous occurrences, resulting in unequal mean and variance in data such as suicides and accidents. The different values of the parameter in the Poisson distribution represent the different values of the accident rates. Many authors have, therefore, obtained the mixture and compound distributions by compounding the Poisson distribution in different ways. With this idea, Consul and Jain ([16]) obtained a discrete distribution, as a limiting case of the GBD, called the GPD and defined as in (1.6). They obtained moment estimators of the parameters and
used these in fitting the distribution and showed that the GPD gives a good fit to numerical data which are strictly Poisson as well as to data for which different statisticians have suggested various modified forms of the Poisson distribution.

Since the GPD (1.6) defined by Consul and Jain [16] is not a MPSD, they used a long method to obtain the moments. Moreover, they used only the moment estimators, without considering the sampling distributions, for fitting purpose which are not always efficient. For certain estimates of the parameters, the moment estimators are found inefficient (see Table 5.5.1) and do not give a satisfactory fit to the data where the GPD is fitted (see Table 5.6.1).

We shall use the form (1.7) of the GPD which is a MPSD. As Nelson [45] remarked for the form (1.6), it is also true for the form (1.7) that for \(-1 < \beta < 0\), there are only a finite number of non-zero terms in the series \(\sum p_x = 1\), because of the condition \(p_x = 0\) whenever \(1 + x\beta \leq 0\), and hence except for some negligible proportion of choices for \(\beta < 0\) and \(\delta\), the series \(\sum p_x\) will not converge to unity.
Infact, \( \sum_{x=0}^{\infty} p_x = 1 \) for \((1 + x\beta) > 0\). The family of distributions (1.7) may be classified according as 
\( \beta = 0 \), \( \beta < 0 \) or \( \beta > 0 \). If \( \beta = 0 \), it reduces to Poisson. 
If \( \beta < 0 \), the variance is exceeded by the mean, and 
the distributions may be called 'Sub Poisson'. If 
\( \beta > 0 \), the variance exceeds the mean, and the distributions 
may be called 'Super Poisson'.

In this chapter we shall study the maximum likelihood (M.L.) method, along with the method of moments (Method A), for estimating the parameters \( \theta \) and \( \beta \). Since the M.L. equations are in series form and complicated to solve for the parameters, we shall suggest and discuss an alternative method for estimation which makes use of the zero-cell frequency and the first sample moment (Method B). We shall derive the asymptotic variances and covariances of the estimators obtained by all three methods and hence the asymptotic efficiencies of the Methods A and B relative to the M.L. method. We present a table of efficiencies which have been computed for different sets of values of the parameters \( \theta \) and \( \beta \). As an illustration, we consider an example where the method of moments is found inefficient while Method B is found fairly efficient.
5.2 THE M.L. METHOD

Consider a random sample of size \( n \) from the population (1.7) and let \( N_x \) be the observed frequency in the sample corresponding to \( X = x \). Then, the likelihood function \( L \) is given by

\[
L = \prod_{x=0}^{n} \frac{N_x}{p_x^x},
\]

where \( p_x = p_x(\beta, \theta) \) as given in (1.7). Taking the natural logarithm of \( L \) and then differentiating w.r.t. \( \theta \) and \( \beta \), we get

\[
\begin{align*}
\frac{\partial \ln L}{\partial \theta} & = N \cdot M'_1 \left( \frac{1}{\theta} - \beta \right) - N \\
\frac{\partial \ln L}{\partial \beta} & = \sum_{x=2}^{n} \frac{(x - 1) \cdot x \cdot N_x}{1 + x^\beta} - N \cdot M'_1 \cdot \theta
\end{align*}
\]

where \( N = \sum_{x=0}^{n} N_x \) and \( M'_1 = \sum_{x=0}^{n} x \cdot N_x / N \).

Equating (5.2.2) and (5.2.1) to zero, we get the M.L. equations

\[
(5.2.3) \quad (N \cdot M'_1)^{-1} \sum_{x=2}^{n} \frac{n \cdot (x - 1) \cdot x \cdot N_x}{1 + x^\beta} = \hat{\theta}
\]
which can be solved for $\hat{\theta}$ and $\hat{\beta}$ by using the method of iterations or the Newton-Raphson method, although both methods may fail because of non-convergence. Another approach is to combine both the equations into a single equation for one estimator $\hat{\theta}$, as

\begin{equation}
(5.2.4) \quad \frac{1}{\hat{\theta}} - \frac{1}{M_1} = \hat{\beta} \tag{5.2.4}
\end{equation}

We then find two values of $\theta$ for which the left hand side (l.h.s.) of (5.2.5) gives negative and positive values. Then, by linear interpolation we obtain a new value of $\theta$ and for that value of $\theta$ we find the value of the l.h.s. of (5.2.5). Continuing in this way, we can find estimate of $\theta$. The estimate of $\theta$ obtained by Method A or B described in Sections 5.3 and 5.4 will be helpful as an initial value.

The Fisher information matrix $U$ of the M.L. estimators can be found to be

\begin{equation}
(5.2.6) \quad U = N \sum_{ij} u_{ij} \tag{5.2.6}
\end{equation}
where the elements

\[ u_{ij} = \mathbb{E} \left[ -u^{-1} \cdot \sigma^2 \ln \left( \sigma r_i \sigma r_j \right) \right], \]

\( i, j = 1, 2 \) and \((x_1^*, x_2^*) = (\theta, \beta)\), are derived as

(5.2.7) \( u_{11} = 1 / (\theta \delta) \)

(5.2.8) \( u_{12} = u_{21} = \theta / \delta \)

(5.2.9) \( u_{22} = \sum_{x=2}^{\infty} \frac{(x-1) x^2 p_x}{(1 + x^\beta)^2} \)

and

(5.2.10) \( \delta = 1 - \beta \theta. \)

Using (1.7), (5.2.9) can be written as

(5.2.11) \( u_{22} = \sum_{x=2}^{\infty} x(1 + x\beta)^{x-3} (\theta e^{-\beta \theta}) x^x e^{-\beta} / (x - 2)! \)

\[ = \sum_{x=3}^{\infty} (1 + x\beta)^{x-2} \theta^x e^{-(1+x\beta)\theta} / (x - 3)! \]

\[ + 2 \sum_{x=2}^{\infty} (1 + x\beta)^{x-3} \theta^x e^{-(1+x\beta)\theta} / (x - 2)! \]
Now,

\[(5.2.12) \sum_{x=3}^{\infty} (1 + x\beta)^{x-3} e^{x-3} e^{(1+x\beta)\theta}/(x-3)! \]

\[= \theta^3 \sum_{j=0}^{\infty} (1 + j\beta^\prime) (1 + j\beta^\prime)^{j-1} e^{j} e^{(1+j\beta^\prime)\theta^\prime}/j! \]

\[= \theta^3 \left[ 1 + \beta^\prime \cdot u_1^j (\theta^\prime, \beta^\prime) \right], \]

where \(\beta^\prime = \beta/(1 + 3\beta)\), \(\theta^\prime = (1 + 3\beta)\theta\) and

\[(5.2.13) \quad u_1^j (\theta^\prime, \beta^\prime) = \frac{\theta^\prime}{1 - \beta^\prime \theta^\prime} = \frac{(1 + 3\beta)\theta}{1 - \beta \theta}. \]

Using (5.2.13) in (5.2.12) and applying the same treatment to the second term in r.h.s. of (5.2.11) we get \(u_{22}\) in explicit form as

\[(5.2.14) \quad u_{22} = \frac{\theta^3 (2 + \theta)}{\delta \left( \theta + 2(1 - \delta) \right)}, \]

where \(\delta\) is as given in (5.2.10).

The determinant \(|U|\) of the matrix \(U\) is given by

\[(5.2.15) \quad |U| = \frac{2 \theta^2 \delta^2}{\delta \left( \theta + 2(1 - \delta) \right)} (u_{11}u_{22} - u_{12}^2)\]
and hence the asymptotic variances and covariances of the M.L. estimators $\hat{\theta}$ and $\hat{\beta}$ (the elements of the matrix $V^{-1}$) are derived as

(5.2.16) $u_{11} = V(\hat{\theta}) = \theta(2 + \theta) / (2N)$

(5.2.17) $u_{12} = \text{Cov}(\hat{\theta}, \hat{\beta}) = (\theta - u_{11}) / (\theta^2 N)$

(5.2.18) $u_{22} = V(\hat{\beta}) = -u_{12} / \theta^2$

5.3 METHOD A

Since the GPD (1.7) is a MFSQ, using (1.8), (1.9) and (1.11), the first four moments are derived as:

(5.3.1) $\mu = 0 / \delta$

(5.3.2) $\mu_2 = 0 / \delta^3$

(5.3.3) $\mu_3 = 0(3 - 2\delta) / \delta^5$

(5.3.4) $\mu_4 = 0(15 - 20\delta + 6\delta^2 + 30\delta) / \delta^7$

where $\delta$ is as given in (5.2.10).
Using (5.3.1) and (5.3.2) and replacing \( \mu \) and \( \mu_2 \) by their respective sample estimates \( M_1' \) and \( M_2' \), we get the moment estimators \( \theta^* \) and \( \beta^* \) as

\[
(5.3.5) \quad \theta^* = \left( \frac{M_1'^3}{M_2'} \right)^{1/2}
\]

\[
(5.3.6) \quad \beta^* = \frac{1}{\theta^*} - \frac{1}{M_1'}.
\]

Using the differential formulae

\[
V(\theta^*) = \sum_{i,j} \left[ (\partial \theta^*/\partial R_i) \cdot (\partial \theta^*/\partial R_j) \right] \bar{R} = \mu \cdot \text{Cov}(R_i, R_j)
\]

\[
V(\beta^*) = \sum_{i,j} \left[ (\partial \beta^*/\partial R_i) \cdot (\partial \beta^*/\partial R_j) \right] \bar{R} = \mu \cdot \text{Cov}(R_i, R_j)
\]

\[
\text{Cov}(\theta^*, \beta^*) = \sum_{i,j} \left[ (\partial \theta^*/\partial R_i) \cdot (\partial \beta^*/\partial R_j) \right] \bar{R} = \mu \cdot \text{Cov}(R_i, R_j)
\]

where \( i, j = 1, 2 \), \( \bar{R} = (R_1, R_2) = (M_1', M_2') \), \( \mu = (\mu, \mu_2) \)

and

\[
(5.3.7) \quad V(M_1') = \theta / (N \sigma^3)
\]

\[
(5.3.8) \quad V(M_2) = \theta (15 - 205 + 6\delta^2 + 20\delta) / (N \sigma^7)
\]

\[
(5.3.9) \quad \text{Cov}(M_1', M_2) = \theta (3 - 2\delta) / (N \sigma^5)
\]
we obtain the asymptotic variance-covariance matrix \( V \) of the estimators \( \theta^* \) and \( \beta^* \), to the order \( N^{-1} \), as

\[
(5.3.10) \quad V = N^{-1} \sum_{i,j=1}^{2} v_{ij}, \quad i,j = 1, 2.
\]

The elements \( v_{ij} \) \( (i,j = 1, 2) \) are derived as

\[
(5.3.11) \quad v_{11} = N V(\theta^*) = \frac{\theta (3 - 4\delta + 3\delta^2 + \delta)}{(2\alpha)}
\]

\[
(5.3.12) \quad v_{12} = v_{21} = N \text{Cov}(\theta^*, \beta^*) = \frac{\theta \delta - v_{11}}{\theta^2}
\]

\[
(5.3.13) \quad v_{22} = N V(\beta^*) = \frac{-v_{12}}{\theta^2}
\]

The determinant \( |V| \) of the matrix \( V \) is given by

\[
(5.3.14) \quad |V| = N^{-2}(v_{11}v_{22} - v_{12})^2
\]

\[
= N^{-2} \left[ (1 - \delta)(3 - \delta) + 3\delta \right]/(2\theta^2)
\]

and hence the asymptotic efficiency \( E_A \) of Method A relative to the M.L. method is derived as

\[
(5.3.15) \quad E_A = \frac{1}{|U| \cdot |V|} = \frac{\delta (\theta + 2(1 - \delta))}{(1 - \delta)(3 - \delta) + 3\delta}
\]
5.4 METHOD B

Now, we consider a method which makes use of the zero-call frequency and the first sample moment.

From (1.7) we have

\[(5.4.1) \quad p_0 = P(X = 0) = e^{-\theta}\]

from which, after replacing \(p_0\) by its sample estimator \(N_0 / N\), we get the estimator \(\hat{\theta}\) as

\[(5.4.2) \quad \hat{\theta} = \ln(N / N_0).\]

The estimator \(\hat{\beta}\) can be obtained by using (5.3.1) and replacing \(\mu\) by \(M'\) as

\[(5.4.3) \quad \hat{\beta} = 1 / \hat{\theta} - 1 / M'.\]

Using the differential formulae

\[
V(\hat{\theta}) = \sum_i \sum_j \left[ \left( \frac{\partial \hat{\theta}}{\partial F_j} \right) \cdot \left( \frac{\partial \hat{\theta}}{\partial F_j} \right) \right] = \mathbb{E} \cdot \text{Cov}(F_i, F_j)
\]

\[
V(\hat{\beta}) = \sum_i \sum_j \left[ \left( \frac{\partial \hat{\beta}}{\partial F_j} \right) \cdot \left( \frac{\partial \hat{\beta}}{\partial F_j} \right) \right] = \mathbb{E} \cdot \text{Cov}(F_i, F_j)
\]

\[
\text{Cov}(\hat{\theta}, \hat{\beta}) = \sum_i \sum_j \left[ \left( \frac{\partial \hat{\theta}}{\partial F_j} \right) \cdot \left( \frac{\partial \hat{\beta}}{\partial F_j} \right) \right] = \mathbb{E} \cdot \text{Cov}(F_i, F_j)
\]
where $i, j = 1, 2$, $F = (F_1, F_2) = (N_0 / N, N_1')$, $F = (p_0, \mu)$, $V(F_2) = V(N_1')$ as given in (5.3.7) and

\begin{align*}
(5.4.4) \quad & V(F_1) = e^\theta \left(1 - e^{-\theta}\right) / N \\
(5.4.5) \quad & \text{Cov}(F_1, F_2) = -\theta e^\theta / (\delta N),
\end{align*}

we obtain the asymptotic variance-covariance matrix $W$ of the estimators $\tilde{\delta}$ and $\tilde{\beta}$, to the order $N^{-1}$, as

\begin{align*}
(5.4.6) \quad & W = N^{-1} \sum w_{ij}, \quad i, j = 1, 2.
\end{align*}

The elements $w_{ij}$ are derived as

\begin{align*}
(5.4.7) \quad & w_{11} = N.V(\tilde{\delta}) = e^\theta - 1 \\
(5.4.8) \quad & w_{12} = w_{21} = N.V(\tilde{\delta}, \tilde{\beta}) = (3 \delta - w_{11}) / \theta^2 \\
(5.4.9) \quad & w_{22} = N.V(\tilde{\beta}) = -w_{12} / \theta^2,
\end{align*}

from which we have the value of the determinant $|W|$ of the matrix $W$ as

\begin{align*}
(5.4.10) \quad & |W| = N^{-2}(w_{11}w_{22} - w_{12}^2) = N^{-2} \delta(\theta^3 - 1 - 6\delta) / \theta^3.
\end{align*}
The asymptotic efficiency of the method B relative to the M.L. method is the joint asymptotic efficiency of the estimators (\( \hat{\theta}, \hat{\beta} \)) relative to the M.L. estimators (\( \hat{\theta}, \hat{\beta} \)) and is given by

\[
E_B = \frac{1}{\left( |U| \cdot |W| \right)}
\]

which after the use of (5.2.15) and (5.4.10) give

\[
(5.4.11) \quad E_B = \frac{\theta \left( \theta + 2(1 - \delta) \right)}{2 \left( e^\theta - 1 - \theta \delta \right)}
\]

5.5 COMPARISON OF ASYMPTOTIC EFFICIENCIES

Using the results (5.3.15) and (5.4.11), the asymptotic efficiencies \( E_A \) and \( E_B \) have been computed for different sets of values of the parameters \( \theta \) and \( \beta \) and the results are shown in Table 5.5.1. When \( \beta = -0.5 \), \( E_A = E_B = 0 \) for all \( \theta \). The table shows that:

(i) Method A i.e. the method of moments is efficient only for small values of the product \( |\beta \theta| \).

(ii) Method B i.e. the method using the zero cell frequency and the first sample mean is fairly efficient except for \( \theta > 1 \).
The relative asymptotic efficiencies $E_A$ and $E_B$ of Methods A and B.

<table>
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<tr>
<th>$E$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1.0</th>
<th>3.0</th>
<th>5.0</th>
</tr>
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<td>$E_A$</td>
<td>0.939</td>
<td>0.843</td>
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<td>0.719</td>
<td>0.670</td>
<td>0.658</td>
<td>0.484</td>
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<td>$E_B$</td>
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<td>0.787</td>
<td>0.678</td>
<td>0.588</td>
<td>0.512</td>
<td>0.473</td>
<td>0.135</td>
</tr>
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<td>-0.2</td>
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<td>0.946</td>
<td>0.917</td>
<td>0.891</td>
<td>0.863</td>
<td>0.857</td>
<td>0.727</td>
</tr>
<tr>
<td></td>
<td>$E_B$</td>
<td>0.946</td>
<td>0.846</td>
<td>0.760</td>
<td>0.631</td>
<td>0.611</td>
<td>0.579</td>
<td>0.189</td>
</tr>
<tr>
<td>-0.1</td>
<td>$E_A$</td>
<td>0.995</td>
<td>0.989</td>
<td>0.983</td>
<td>0.976</td>
<td>0.970</td>
<td>0.967</td>
<td>0.920</td>
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<tr>
<td></td>
<td>$E_B$</td>
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<td>0.808</td>
<td>0.740</td>
<td>0.677</td>
<td>0.647</td>
<td>0.237</td>
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<td>0.992</td>
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<td>0.982</td>
<td>0.976</td>
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<td></td>
<td>$E_B$</td>
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<td>0.3</td>
<td>$E_A$</td>
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<td>0.947</td>
<td>0.910</td>
<td>0.870</td>
<td>0.828</td>
<td>0.806</td>
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<td></td>
<td>$E_B$</td>
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<td></td>
<td>$E_B$</td>
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<td>0.840</td>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>$E_B$</td>
<td>0.955</td>
<td>0.957</td>
<td>0.927</td>
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<td>0.863</td>
<td>0.846</td>
<td></td>
</tr>
<tr>
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<tr>
<td></td>
<td>$E_B$</td>
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<td>0.937</td>
<td>0.909</td>
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</tr>
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</tr>
<tr>
<td></td>
<td>$E_B$</td>
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<td>0.941</td>
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<tr>
<td></td>
<td>$E_B$</td>
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<tr>
<td></td>
<td>$E_B$</td>
<td>0.997</td>
<td></td>
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5.6 EXAMPLE

To illustrate that the method of moments is, some times, less efficient we consider in Table 5.6.1 a data set from Katti and Gurland [41] who fitted the Poisson-Pascal distribution to the data. We fit a GPD to this data by all three methods. The fit by the method of moments is inferior while the fit by the method using the zero-cell frequency is comparable to that by the M.L. method. Moreover, for the M.L. estimates $\theta = 0.79581$ and $\beta = 0.73419$ of the example, the relative efficiencies $R_A$ and $R_B$ are found to be $R_A = 0.444$ and $R_B = 0.883$ which shows that the moment estimators are inefficient while the estimators obtained by Method B are fairly efficient.
### Observed frequencies of Leptinotarsa Decemlineata (Katti and Garland [41])

<table>
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<tr>
<th>Insects</th>
<th>Observed</th>
<th>M.L.Method</th>
<th>Method B</th>
<th>Method A</th>
</tr>
</thead>
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<td>0</td>
<td>33</td>
<td>31.59</td>
<td>33.00</td>
<td>27.38</td>
</tr>
<tr>
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<td>12</td>
<td>14.01</td>
<td>13.52</td>
<td>15.44</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7.67</td>
<td>7.24</td>
<td>9.08</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>4.72</td>
<td>4.43</td>
<td>5.65</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>3.11</td>
<td>2.93</td>
<td>3.63</td>
</tr>
<tr>
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<td>2.05</td>
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<td>0.51</td>
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<td>2</td>
<td>2.07</td>
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<td>1.42</td>
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**Total** | 70       | 70.00      | 70.00    | 70.00    |

**Mean** | 1.91429  | 1.91429    | 1.91429  | 1.91429  |

**Variance** | 7.96356  | 7.96356    | 7.96356  | 7.96356  |

\[ \chi^2 \]  |
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<tr>
<td>8.42</td>
<td>8.60</td>
<td>10.31</td>
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</table>

**Estimates**

- \( \hat{\theta} \): 0.79581, 0.75199, 0.93855
- \( \hat{\beta} \): 0.73419, 0.80742, 0.54309