CHAPTER 4

ESTIMATION OF THE PARAMETERS OF THE GLSD

4.1 INTRODUCTION

Most of the biological and ecological count data are well fitted by decapitated negative binomial distribution or logarithmic series distribution (LSD) (Fisher et. al. [19], Williams [62, 63], Anscombe [4], Bliss and Fisher [5], Bliss [6] etc.). But there are some instances (Kendall and Stuart [42]) where the fits by these distributions are worse. Recently Jain and Gupta [25] and Jani [27] have generalized the logarithmic series distribution and obtained a new discrete distribution called the GLSD and defined as in (1.4). In this chapter we will study a few properties and different methods of estimating the parameters of the
The maximum likelihood (M.L.) method is the most efficient method for estimating the parameters but, some times, it involves so complicated forms of M.L. equations that they do not converge for unique solution. In such cases the methods which are equally efficient to the M.L. method are always to be thought of. We will discuss here five different methods for estimating the parameters:

(i) The M.L. method

(ii) The method of moments (Method A)

(iii) The method of approximate moment estimators (Method B)

(iv) The modified method of moments (Method C)

(v) The method using the first cell frequency and the sample mean (Method D).

We will also derive the asymptotic variances and covariances of the estimators obtained by the above methods and hence the asymptotic efficiencies of the methods A to D relative to the M.L. method. All the results have been expressed explicitly in terms of the parameters \( \theta \) and \( \beta \). The asymptotic efficiencies have been computed for a wide range of values of parameters.
4.2 **MOMENTS OF THE GLSD**

Using (1.8), (1.9) and (1.10), we derive the mean, the variance and the second to the sixth raw moments of the GLSD (1.4) as follows:

\[(4.2.1) \quad \mu = \alpha \theta \delta^{-1}\]

\[(4.2.2) \quad \mu_2 = \alpha \theta \delta^{-3} (1 - \theta - \alpha \theta \delta)\]

\[(4.2.3) \quad \mu_2^* = \alpha \theta \delta^{-3} (1 - \theta)\]

\[(4.2.4) \quad \mu_3 = \alpha \theta (1-\theta) \delta^{-5} \left[ 3(1-\theta) - \delta(2-\theta) \right]\]

\[(4.2.5) \quad \mu_4 = \alpha \theta (1-\theta) \delta^{-7} \left[ -15(1-\theta)^2 - 10\delta(1-\theta)(2-\theta) \right.\]
\[\left. + \delta^2(6-6\theta+\theta^2) \right]\]

\[(4.2.6) \quad \mu_5 = \alpha \theta (1-\theta) \delta^{-9} \left[ 105(1-\theta)^3 - 105\delta(1-\theta)^2(2-\theta) \right.\]
\[\left. + 5\delta^2(1-\theta)(26 - 26\theta + 5\theta^2) \right.\]
\[\left. - \delta^3(2-\theta)(12 - 12\theta + \theta^2) \right]\]
where

\[(4.2.8) \quad \delta = 1 - \beta \theta\]

4.3 **PROPERTIES OF THE GLSD**

In the following we shall study a few properties of the GLSD:

(i) For \( \beta = 1 \), the GLSD \((i.4)\) reduces to the usual LSD with probability function

\[P(x=x) = p_\mu(x) = \alpha \theta^x / x, \quad x \in \mathbb{N}\]

(ii) The probability function of the GLSD \((i.4)\) can be written as
\[ P(x=x) = p_x(\beta, \theta) = (x \beta - \theta \omega - \theta) (x \beta - 2\theta) \ldots \]
\[ = (x \beta - x \theta + \theta)(1 - \omega)^{x \beta - 1} \]
\[ = (x! \log(1 - \omega)^{1/0})^{-1} \]

Taking limits as \( \beta \to \infty \) and \( \theta \to 0 \) such that \( \beta \omega \) remains constant, equal to say \( \bar{\omega} \), we get

\[ P(x=x) = p_x(\bar{\omega}) = (x \bar{\omega})^{x-1} e^{-x \bar{\omega}} / x!, \quad x \in N \]

which is the probability function of the Borel distribution with one parameter \( \bar{\omega} \).

(iii) Since the parameter \( \beta \) is connected with the mean and the variance of the GISP, from (4.2.1) and (4.3.2) we deduce that the mean is equal to, greater or smaller than the standard deviation (s.d.) according to \( \beta \) is equal to, smaller or greater than the term

\[ f(\theta) = (1 - (1 - \omega) / 2\omega) / \theta \]

respectively. Table - 4.3.1 shows the values
of $f(\theta)$ for different values of $\theta$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(\theta)$</td>
<td>5.25678</td>
<td>2.76856</td>
<td>1.94626</td>
<td>1.54220</td>
<td>1.30685</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(\theta)$</td>
<td>1.15762</td>
<td>1.06001</td>
<td>0.99852</td>
<td>0.96898</td>
</tr>
</tbody>
</table>

(iv) (4.2.2) shows that $\mu_2 < 0$ if

$$\beta < (1 - (1 - \theta) / \theta) / \theta = 2 f(\theta) - 1 / \theta$$

(v) The identity $\sum_{x=1}^{\infty} p_x = 1$ is, in general, exact only when the series is infinite. However, for the GLSD (1.4), due to the condition $p_x(\beta, \theta) = 0$ whenever $x\beta - x + 1 < 0$, for $\beta < 1$, the series is finite and for some values of $\beta$ and $\theta$, $\sum p_x > 1$. For example, with $\beta \geq 0.5$ and $0 < \theta < 1$, there will be only one non-zero term i.e. $p_1$ and in all such cases, except with very small $\theta$, $\sum p_x > 1$. Table - 4.3.2 shows the
approximate values of $\sum_{x=1}^{n} p_x$ calculated correct up to 10 decimal places. Hence, for $\beta < 1$ only those values are admissible for which $\sum p_x = 1$.

### Table - 4.3.2

Approximate values of the series $\sum_{x=1}^{n} p_x$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$0.1$</th>
<th>$0.5$</th>
<th>$0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.1$</td>
<td>1.04553</td>
<td>1.34608</td>
<td>5.19475</td>
</tr>
<tr>
<td>$(n=1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.3$</td>
<td>1.02177</td>
<td>1.17135</td>
<td>1.95697</td>
</tr>
<tr>
<td>$(n=1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.5$</td>
<td>1.00046</td>
<td>1.02014</td>
<td>1.23612</td>
</tr>
<tr>
<td>$(n=1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.7$</td>
<td>1.00002</td>
<td>1.00369</td>
<td>1.10607</td>
</tr>
<tr>
<td>$(n=3)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.9$</td>
<td>1.00000</td>
<td>1.00080</td>
<td>1.00783</td>
</tr>
<tr>
<td>$(n=9)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(vi) For $\beta > 1$, $\sum_{x=1}^{\infty} p_x < 1$. For example, with $\beta = 2$ and $\theta = 0.9$, $p_1 = 0.03209$, $p_2 = 0.00528$, $p_3 = 0.00195$, $p_6 = 0.00002$ and so on and hence $\sum p_x$ will never converge to 1.
(vii) For $p > 1$ and large value of $\theta$, the distribution will have longer tails. For example, for $p = 1.4 \quad n$ and $\theta = 0.6$, $\sum_{x=1}^{n} p_x = 0.99999$ when $n = 161$. Perhaps, for this reason the observed data with short tails are better fitted by the GLSD with $p < 1$ rather than the LSD where $p = 1$.

4.4 THE M.L. METHOD

Let us write the probability function of the GLSD (4.4) as

\[ P(X=x) = p_x(\beta, \theta) = (x\beta - 1)(x\beta - 2) \ldots (x\beta - x + 1) \]

\[ \times \frac{\alpha (\theta(1-\theta)^{\beta-1})^x}{x!} , \]

where $x \in N$, $\alpha = ( - \log(1 - \theta) )^{-1}$, $0 < \theta < 1$,
$p \theta < 1$ and $p_x = 0$, whenever $x\beta - x + 1 \leq 0$.

Now, consider a random sample of size $N$ from the population (4.4.1) and let $N_x$ be the observed frequency in the sample corresponding to $X = x$.

Then, the likelihood function $L$ is given by
where \( p_x = p_x(\theta, \phi) \) as given in (4.4.1). Taking the natural logarithm of \( L \) and differentiating w.r.t. \( \theta \) and \( \phi \) and writing
\[
\sum_{x=1}^{n} x = N \quad \text{and} \quad \sum_{x=1}^{n} x \cdot N_x / N = N',
\]
we obtain

\[
(4.4.2) \quad \frac{\partial \ln L}{\partial \theta} = N(1-\theta)^{-1} \left( M' \delta / \theta - \alpha \right)
\]

\[
(4.4.3) \quad \frac{\partial \ln L}{\partial \phi} = \sum_{x=2}^{n} \sum_{j=1}^{x-1} \frac{x \cdot N_x}{x \cdot \beta - j} + N \cdot N' \ln(1-\theta)
\]

Equating (4.4.2) and (4.4.3) with zero we get the M.L. equations

\[
(4.4.4) \quad (N \cdot N')^{-1} \sum_{x=2}^{n} \sum_{j=1}^{x-1} \frac{x \cdot N_x}{x \cdot \beta - j} = \frac{1}{\hat{\alpha}}
\]

\[
(4.4.5) \quad \frac{1}{\hat{\theta}} - \frac{\hat{\alpha}}{N'} = \hat{\beta}
\]

which can be solved for \( \hat{\theta} \) and \( \hat{\beta} \), the M.L. estimators of \( \theta \) and \( \phi \) respectively, by using an iterative technique such as the method of scoring (Rao [55], p. 370).
M.I. equations (4.4.4) and (4.4.5) do not yield explicit expressions for the corresponding M.L. estimators of \( \theta \) and \( \beta \). In practice it is found that the M.I. equations do not converge for unique solution, even tried on computer. For some problems, an approximate solution is achieved but it is less efficient. This necessitates the search for other efficient estimators.

The Fisher information matrix \( U \) of the M.I. estimators can be found to be

\[
(4.4.6) \quad U = N \begin{bmatrix} u_{ij} \end{bmatrix},
\]

where the elements

\[
u_{ij} = E \left[ -N^{-1} \frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right],
\]
i, \( j = 1, 2 \) and \( (x_1, x_2) = (0, \beta) \), are derived as

\[
(4.4.7) \quad u_{11} = \frac{\alpha (1 - \theta - \alpha \theta)}{\theta (1 - \theta)^2}
\]

\[
(4.4.8) \quad u_{12} = u_{21} = \frac{\alpha \theta}{\theta (1 - \theta)}
\]
\[(4.4.9) \quad u_{22} = \sum \sum \frac{x^2 p_x}{x} \] 

The asymptotic variance-covariance matrix \( U^{-1} \) can be obtained by inverting \( U \) i.e.

\[(4.4.10) \quad U^{-1} = \frac{N^{-1}}{|U|} \begin{bmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{bmatrix} = N^{-1} \left[ u^{ij} \right]. \]

\(|U|\) is the determinant of the matrix \( U \) which is given by

\[(4.4.11) \quad |U| = N^2 \left( u_{11} u_{22} - u_{12}^2 \right) \]

\[= N^2 B^{-1} \left[ A'u_{11} - B \right]^{-1} \]

\[= N^2 B^{-1} \left[ A' u_{22} - B^{-1} \right]^{-1} \]

where

\[(4.4.12) \quad B = \frac{\delta(1-\theta)}{\sigma^2} \]

\[(4.4.13) \quad A = \frac{1-\theta - \alpha\delta\theta}{\sigma^2(1-\theta)}. \]
To compute the probabilities one may use the recurrence relation

\[(4.4.14) \ p_{x+1} = \beta \theta (1 - \theta)^{\beta-1} \frac{x(x\beta + \beta)}{(x+1)(x\beta)_x} \cdot p_x\]

where \((r)_x = r(r - 1) \cdot \ldots \cdot (r - x + 1)\).

4.5 **METHOD A**

In the following, we will obtain the moment estimators of \(\theta\) and \(\beta\). Since moments do not exist for \(\beta \theta > 1\) and (4.2.2) shows that \(\mu_2 > 0\) if and only if \(\beta \theta > 1 - (1 - \theta) / \alpha \theta\), we will obtain the moment estimators of the parameters for the parametric space \(1 - (1 - \theta) / \alpha \theta \leq \beta \theta < 1\). Using (4.2.1) and (4.2.3) and replacing \(\mu\) and \(\mu_2'\) by their respective estimators, the sample mean \(\bar{M}_1\) and the second sample raw moment \(\bar{M}_2'\), we get the equations for moment estimators \(\theta^*\) and \(\beta^*\) of \(\theta\) and \(\beta\) as:

\[(4.5.1) \ \frac{(\theta^* \theta^*)^2}{1 - \theta^*} = \frac{(\bar{M}_1')^3}{\bar{M}_2'}\]

\[(4.5.2) \ \beta^* = \frac{1}{\theta^*} - \frac{\alpha^*}{\bar{M}_1'}\]
where \( \alpha = ( - \log(1 - \theta))^{-1}. \) The equation (4.5.1) can easily be solved by using the method of iterations. The values of \( (\alpha \theta) \) for different values of \( \theta = 0.01(0.01) \) \( 0.99 \) have been computed and given in the table in appendix.

Using the following differential formulae (Kendall and Stuart [42] § 10.6):

\[
V(\theta^*) = \sum_{i,j} \left[ \frac{\partial \theta^*}{\partial M_i} \cdot \frac{\partial \theta^*}{\partial M_j} \right] M_i = \mu_{1i}, \quad M_j = \mu_{2j}
\]

\[
V(\beta^*) = \sum_{i,j} \left[ \frac{\partial \beta^*}{\partial M_i} \cdot \frac{\partial \beta^*}{\partial M_j} \right] M_i = \mu_{1i}, \quad M_j = \mu_{2j}
\]

\[
\text{Cov}(\theta^*, \beta^*) = \sum_{i,j} \left[ \frac{\partial \theta^*}{\partial M_i} \cdot \frac{\partial \beta^*}{\partial M_j} \right] M_i = \mu_{1i}, \quad M_j = \mu_{2j}
\]

\[
(4.5.3) \quad V(M_i) = n^{-1} \delta^{-3} \alpha \theta (1 - \theta - \alpha \theta \delta)
\]

and
\[(4.5.4) \quad V(M_2) = N^{-1} \delta^{-7} \alpha \delta (1 - \delta) \left[ 18(1 - \delta)^2 - 10 \delta (1 - \delta)(2 - \delta) \right]
+ \delta^2(6 - 6\theta + \delta^2) - \alpha \delta (1 - \delta) \right] \]
\[(4.5.5) \quad \text{Cov}(M_1, M_2) = N^{-1} \delta^{-5} \alpha \delta (1 - \delta) \left[ 3(1 - \theta) - \delta (2 - \theta) - \alpha \delta \right], \]

we obtain the asymptotic variance-covariance matrix \( V \) of the moment estimators \( \theta^* \) and \( \beta^* \), to the order \( N^{-1} \), as

\[(4.5.6) \quad V = N^{-1} \sum v_{ij} 1, \quad i, j = 1, 2, \]

where the elements \( v_{ij} (i, j = 1, 2) \) are derived as

\[(4.5.7) \quad v_{11} = N \cdot V(\theta^*) = \frac{\theta (1 - \theta)}{\alpha \delta (2 - \theta - 2\alpha \theta)} \left[ 6(1 - \theta)^2 
- 4 \delta(1 - \theta)(2 - \theta + \alpha \theta) + \delta^2(6 - 6\theta + \theta^2) \right] \]
\[(4.5.8) \quad v_{12} = v_{21} = N \cdot \text{Cov}(\theta^*, \beta^*) = -A \cdot v_{11} + B \]
\[(4.5.9) \quad v_{22} = N \cdot V(\beta^*) = -A \cdot v_{12} \]

where \( B \) and \( A \) are as given in (4.4.12) and (4.4.13).
The determinant $|V|$ of the matrix $V$
is given by

$$(4.5.10) \quad |V| = N^{-2} \left( v_{11} \cdot v_{22} - v_{12}^2 \right)$$

$$= N^{-2} \cdot B \left( A \cdot v_{11} - B \right).$$

The asymptotic efficiency $E_A$ of the Method $A$
relative to the M.L. method is the joint asymptotic
efficiency of the moment estimators ($\hat{\theta}^*, \hat{\beta}^*$) relative
to the M.L. estimators ($\hat{\theta}, \hat{\beta}$) and is given by

$$E_A = 1 / \left( |U| \cdot |V| \right),$$

which after using (4.4.11) and (4.5.10) gives

$$(4.5.11) \quad E_A = \frac{1}{(A \cdot u_{22} - B^{-1}) (A \cdot v_{11} - B)}$$

where $u_{22}$ is as given by (4.4.9) and $v_{11}$ as
given by (4.5.7).
Method A for obtaining the moment estimators does not give the estimators in explicit form. We will now see a method which gives the approximate, but efficient, moment estimators in explicit forms.

Using (4.2.1) and (4.2.3) we have

\begin{equation}
(4.6.1) \; \alpha^2 \theta^2 - c (1 - \theta) = 0
\end{equation}

where

\begin{equation}
(4.6.2) \; c = \mu^3 / \mu_2^1.
\end{equation}

Expanding \( \alpha = [- \log(1 - \theta)]^{-1} \) into power series expansion and neglecting \( \theta^3 \) and terms higher powers of \( \theta \) and replacing \( \mu \) and \( \mu_2^1 \) by their respective estimators \( M_1^1 \) and \( M_2^1 \) we get

\[ \theta^{**^2} - 12 (\theta^{**} - 1) (1 - \theta^{**}) = 0, \]

which gives for \( \theta^{**} > 0 \), the approximate moment estimator \( \theta^{**} \) of \( \theta \) as
(4.6.3) \( \theta^{**} = 2 \sqrt[3]{3\varphi^{**} - 3} \left[ \sqrt[3]{3\varphi^{**} - 2} - \sqrt[3]{3\varphi^{**} - 3} \right] \)

provided \( \varphi^{**} > 1 \), where

(4.6.4) \( \varphi^{**} = \left( \frac{M_1}{M_2} \right)^3 \).

Using (4.2.1) we get the approximate moment estimator \( \beta^{**} \) of \( \beta \) as

(4.6.5) \( \beta^{**} = 1 / \theta^{**} - \alpha^{**} / M_1 \).

Here, \( \theta^{**} \) and \( \beta^{**} \) are in explicit forms and easily be computed.

Following the method used in Section 4.5 for \( \theta^{**} \) and \( \beta^{**} \), we obtain the asymptotic variance-covariance matrix \( W \) of the approximate moment estimators \( \theta^{**} \) and \( \beta^{**} \), to the order \( N^{-1} \), as

(4.6.6) \( W = N^{-1} \left[ w_{ij} \right] \), \( i, j = 1, 2 \),

where the elements \( w_{ij} \) are derived as
(4.6.7) \( w_{11} = N.V(\theta^{**}) = \frac{36 \alpha^3 \theta^3 (1-\theta)}{\delta \left[ 6 \alpha^2 \theta^2 - (1-\theta)(6-\theta) \right]^2} \]

\[ \left[ 6(1-\theta)^2 - 48(1-\theta)(2-\theta+\theta^2) \right] + \delta^2(2 - 6\theta + \theta^2) \]

(4.6.8) \( w_{12} = w_{21} = N.Cov(\theta^{**}, \beta^{**}) = -A w_{11} + B. D \)

(4.6.9) \( w_{22} = N.V(\beta^{**}) = -A w_{12} + (D - 1) B \]

where

(4.6.10) \( D = \frac{6 \alpha^2 \theta^2 (2 - \theta - 2\alpha\theta)}{6 \alpha^2 \theta^2 - (1-\theta)(6-\theta)} \)

and B and A are as given in (4.4.12) and (4.4.13).

Comparing (4.5.7) with (4.6.7), in virtue of (4.6.10), \( w_{11} \) can be written in terms of \( v_{11} \) as

(4.6.11) \( w_{11} = D^2 v_{11} \).

Using (4.6.11) in (4.6.8) we get
\((4.6.12) \quad w_{12} = D \left[ -A.D.v_{11} + B \right] \)

or, in virtue of \((4.5.8)\), equivalently

\[(4.6.13) \quad w_{12} = D \left[ D.v_{12} + B(1 - D) \right].\]

Similarly using \((4.6.13)\) in \((4.6.9)\), in virtue of \((4.5.9)\), \((4.6.9)\) reduces to

\[(4.6.14) \quad w_{22} = D^2.v_{22} + A.B(1 - D)^2.\]

\((4.6.11), (4.6.13)\) and \((4.6.14)\) give \(w_{11}, w_{12}\) and \(w_{22}\) in terms of \(v_{11}, v_{12}\) and \(v_{22}\) respectively.

The determinant \(|W|\) of the matrix \(W\) given in \((4.6.6)\) is derived as

\[(4.6.15) \quad |W| = N^{-2} (w_{11}w_{22} - w_{12}^2) = N^{-2}.B \left[ A.w_{11} - B.D^2 \right]\]

which after using \((4.6.11)\) and \((4.5.10)\) reduces to

\[(4.6.16) \quad |W| = D^2. |V|\]
and hence the asymptotic efficiency $E_B$ of Method B relative to the M.L. method is given by

\[(4.6.17) \quad E_B = \frac{1}{\left( |U| |W| \right)}\]

\[= \frac{1}{\left( D^2 |U| |V| \right)}\]

\[= \left( \frac{1}{D^2} \right) \cdot E_A.\]

(4.6.17) shows that Method B is less, equal to or more efficient than Method A depending on the value of $D$ which is greater, equal to or less than unity.

4.7 METHOD C

We will now discuss a method which uses the first three raw moments. Use of an additional moment avoids the iterations and it is possible to write the estimators of $\theta$ and $\beta$ explicitly in terms of the sample moments. We call this method as the method of modified moments (Method C).

Consider

\[(4.7.1) \quad k = \frac{\mu_3 - 3 \mu_2^2}{\left( \mu_2^3 \right)^{1/2}},\]
which after using (4.2.1), (4.2.3) and (4.2.4) in right hand side give

\[(4.7.2) \quad k = \frac{(\theta - 2)}{(1 - \theta)^{1/2}}.\]

Replacing \(\mu_i^\dagger (i = 1,2,3)\) by their respective estimators \(M_i^\dagger (i = 1,2,3)\) in (4.7.1) and then using in (4.7.2) we get

\[(4.7.3) \quad k = \frac{\theta^{**} - 2}{(1 - \theta^{**})^{1/2}},\]

where

\[(4.7.4) \quad k = \frac{M_i^1 M_i^3 - 3 M_i^2}{(M_i^1 M_i^3)^{1/2}}.\]

(4.7.3) can be written in a quadratic equation

\[(4.7.5) \quad (\theta^{**})^2 - (K^2 - 4)(1 - \theta^{**}) = 0\]

which gives

\[(4.7.6) \quad \theta^{**} = \left( \pm K \cdot \sqrt{(K^2 - 4) - (K^2 - 4)} \right) / 2.\]
Since \( 0 < \theta < 1 \), (4.7.3) shows that \( K < 0 \) and (4.7.5) shows that \( K^2 > 4 \). Hence, for \( \theta^{***} > 0 \)

\[(4.7.7) \quad \theta^{***} = \frac{4 - K \sqrt{K^2 - 4} - K^2}{2} \]

Using (4.2.1) we can have

\[(4.7.8) \quad \beta^{***} = (\theta^{***})^{-1} - \alpha^{***} (M_1^{**})^{-1} \]

\( \theta^{***} \) given by (4.7.7) and \( \beta^{***} \) given by (4.7.8) are the modified moment estimators of \( \theta \) and \( \beta \) respectively.

Using the following differential formulae

(Kendall and Stuart [42] § 10.6):

\[
\varphi(\theta^{***}) = \sum_{i j} \left( \frac{\partial \theta^{***}}{\partial M_i^j} \right) \left( \frac{\partial \theta^{***}}{\partial M_j^i} \right) \sum_{M} = \mu \cdot \text{Cov}(M_i^j, M_j^i)
\]

\[
\varphi(\beta^{***}) = \sum_{i j} \left( \frac{\partial \beta^{***}}{\partial M_i^j} \right) \left( \frac{\partial \beta^{***}}{\partial M_j^i} \right) \sum_{M} = \mu \cdot \text{Cov}(M_i^j, M_j^i)
\]

\[
\text{Cov}(\theta^{***}, \beta^{***}) = \sum_{i j} \left( \frac{\partial \theta^{***}}{\partial M_i^j} \right) \left( \frac{\partial \beta^{***}}{\partial M_j^i} \right) \sum_{M} = \mu \cdot \text{Cov}(M_i^j, M_j^i)
\]
where $i, j = 1, 2, 3$, $\mathbf{M} = (M_1', M_2', M_3')$, $\mu = (\mu_1', \mu_2', \mu_3')$ and $V(M_1'), V(M_2')$ and $\text{Cov}(M_1', M_2')$ are as given in (4.5.3) to (4.5.5) and

\begin{align*}
(4.7.9) \quad V(M_3') &= N^{-1} \delta^{-11} \alpha(1-\delta) \left[ 945(1-\delta)^4 - 1260\delta(1-\delta)^3(2-\delta) 
+ 70 \delta^2 (1-\delta)^2 (34 - 349 + 76^2) 
- 14 \delta^3 (1-\delta)(2-\delta)(33 - 359 + 49^2) 
+ \delta^4(120 - 240\delta + 150\delta^2 - 39\delta^3 + \delta^4) 
- \alpha\delta(1-\delta)(3(1-\delta) - \delta(2-\delta))^2 \right] \\
(4.7.10) \quad \text{Cov}(M_1', M_3') &= N^{-1} \delta^{-7} \alpha(1-\delta) \left[ 15(1-\delta)^2 
- 10\delta(1-\delta)(3-\delta) + \delta^2(6 - 6\delta + \delta^2) 
- \alpha\delta \left( 3(1-\delta) - \delta(2-\delta) \right) \right]
\end{align*}
\[(4.7.11) \quad \text{Cov}(N_2', M_3') = N^{-1} \delta^{-9} \alpha \beta (1-\delta) \left[ 105 (1-\delta)^5 - 210 \delta (1-\delta)^4 (2-\delta) + 56 \delta^2 (1-\delta) (26 - 269 + 502) - 8 \delta^3 (2-\delta) (12 - 129 + \delta^2) - 3 \delta^4 (1-\delta) (3(1-\delta) - 5(3-\delta)) \right] \]

We obtain the asymptotic variance-covariance matrix \( R \) of the modified moment estimators \( \theta^{**} \) and \( \beta^{**} \), to the order \( N^{-1} \), as

\[(4.7.12) \quad R = N^{-1} \left[ \begin{array}{cc} r_{ij} & \end{array} \right], \quad i, j = 1, 2. \]

The elements \( r_{ij} \) (i, j = 1, 2) are derived as

\[(4.7.13) \quad r_{11} = \text{N.V}(\theta^{**})
= (\alpha \beta \delta^3)^{-1} (1-\delta) \left[ 864 (1-\delta)^4 - 1248 \delta (1-\delta)^3 (2-\delta) + 18 \delta^2 (1-\delta)^2 (140 - 1400 + 276^2)
- 56 \delta^3 (1-\delta) (2-\delta) (14 - 149 + \delta^2)
+ \delta^4 (120 - 2490 + 1369^2 - 189^3 + \delta^4) \right] \]
\[(4.7.14) \quad r_{12} = r_{21} = \text{N.Cov}(\theta^{**}, \beta^{**}) = -A \cdot r_{11} + B\]

\[(4.7.15) \quad r_{22} = \text{N.V}(\beta^{**}) = -A \cdot r_{12}\]

where \(B\) and \(A\) are same as given in \((4.4.12)\) and \((4.4.13)\) respectively.

The determinant \(|R|\) of the matrix \(R\) is given by

\[(4.7.16) \quad |R| = N^{-2}(r_{11} \cdot r_{22} - r_{12}^2)\]

\[= N^{-2} \cdot B \left( A \cdot r_{11} - B \right)\]

The asymptotic efficiency \(E_C\) of the method \(C\) relative to the M.L. method is the joint asymptotic efficiency of the modified moment estimators \((\hat{\theta}^{***}, \hat{\beta}^{**})\) relative to the M.L. estimators \((\hat{\theta}, \hat{\beta})\) and is given by

\[E_C = 1 / \left( \left| U \right| \cdot \left| R \right| \right)\]

which after using \((4.4.11)\) and \((4.7.16)\) gives
where $u_{22}$ is as given in (4.4.9) and $r_{11}$ is as given in (4.7.13). In practice these estimators are found less efficient than other estimators.

4.8 METHOD D

We now consider a fifth method for estimating the parameters $\theta$ and $\beta$ which makes use of the first cell frequency and the sample mean.

From (1.4) we have

\begin{equation}
(4.8.1) \quad p_1 = P(X = 1) = \alpha \theta (1 - \theta)^{\beta - 1}.
\end{equation}

Taking the natural logarithm of (4.8.1) and using (4.2.1) we get

\[ \ln(\alpha \theta) - (1 - \theta)/(\alpha \theta) = \ln p_1 - 1/\mu, \]

from which after replacing $p_1$ and $\mu$ by their respective estimators $\hat{N}_1/N$ and $\hat{\mu}$ we get
(4.8.2) \( \ddot{\theta} = 1 - \tilde{\alpha} \tilde{\theta} \left[ \ln (\tilde{\alpha} \tilde{\theta}) - P \right] \)

where

(4.8.3) \( P = \ln(N_1 / N) - 1 / M' \).

The equation (4.8.2) can easily be solved iteratively for \( \dot{\theta} \), an estimator for \( \theta \). The table of \( (\alpha\theta) \), given in appendix, may be found useful for this purpose. The estimate of \( \ddot{\beta} \) can be obtained from

(4.8.4) \( \ddot{\beta} = 1 / \ddot{\theta} - \tilde{\alpha} / M' \).

Using the following differential formulae:

\[
V(\ddot{\theta}) = \sum_{i,j} \left[ (\ddot{\theta}/\partial F_i) \cdot (\ddot{\theta}/\partial F_j) \right]_{\theta} \cdot \text{Cov}(F_i, F_j)
\]

\[
V(\ddot{\beta}) = \sum_{i,j} \left[ (\ddot{\beta}/\partial F_i) \cdot (\ddot{\beta}/\partial F_j) \right]_{\theta} \cdot \text{Cov}(F_i, F_j)
\]

\[
\text{Cov}(\ddot{\theta}, \ddot{\beta}) = \sum_{i,j} \left[ (\ddot{\theta}/\partial F_i) \cdot (\ddot{\beta}/\partial F_j) \right]_{\theta} \cdot \text{Cov}(F_i, F_j)
\]

where \( i, j = 1, 2 \), \( F = (F_1, F_2) = (N_1 / N, M') \), \( P = (P_1, \mu) \), \( V(F_2) = V(M') \) as given in (4.5.3) and
\( V(F_1) = N^{-1} \alpha \theta (1 - \theta)^{\beta - 1} \left[ 1 - \alpha \theta (1 - \theta)^{\beta - 1} \right] \)

\( \text{Cov}(F_1, F_2) = -N^{-1} \sigma^{-1} \alpha \theta (1 - \theta)^{\beta - 1} (\alpha \theta - \delta) \)

we obtain the asymptotic variance-covariance matrix \( S \) of the estimators \( \theta \) and \( \beta \), to the order \( N^{-1} \), as

\( S = N^{-1} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \quad i, j = 1, 2 \)

The elements \( s_{ij} \) are derived as

\[ s_{11} = N \cdot V(\tilde{\theta}) \]

\[ = \frac{\theta (1 - \theta)^2}{a(1 - \theta)^2 \sigma^2 \theta (1 - \theta)^{\beta - 1} - \alpha \theta} \]

\[ + \alpha \sigma^2 (\delta - 2\alpha \theta) + \delta(1 - \theta) \]

\( s_{12} = s_{21} = N \cdot \text{Cov}(\tilde{\theta}, \tilde{\beta}) = -A \cdot s_{11} + B \)

\( s_{22} = N \cdot V(\tilde{\beta}) = -A \cdot s_{12} \)

where \( B \) and \( A \) are same as given in (4.4.12) and (4.4.13).
The determinant $|S|$ of the matrix $S$ is given by

$$|S| = N^{-2} (s_{11} s_{22} - s_{12}^2) = B(s_{11} - B)$$

and hence the asymptotic efficiency $E_D$ of the method $D$, relative to the M.L. method, is given by

$$E_D = 1 / (|U|.|S|)$$

$$= \frac{1}{(A u_{22} - B^{-1}) (A s_{11} - B)}$$

where $u_{22}$ is as given by (4.4.9).

4.9 COMPARISON OF ASYMPTOTIC EFFICIENCIES

Considering the parametric space

$$(1 - (1 - \theta) / \infty) \leq \theta < 1$$

and other restrictions, as discussed in Section 4.3, the asymptotic efficiencies
of Methods A to D relative to the M.L. method have been computed for different values of $\beta$ and $\theta$ on computer and tabulated in Table 4.9.1. The table shows that:

(i) The methods A and B are equally efficient as the M.L. method except in cases where $\beta \theta$ has large values.

(ii) The method $G$ becomes less efficient, for $\beta > 1$, as $\theta$ increases.

(iii) The method D is as efficient as the M.L. method.
The asymptotic efficiencies $E_A, E_B, E_C, E_D$ of the methods $A, B, C, D$ respectively relative to the M.L. Method.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_A$</td>
<td>1.000</td>
<td>0.996</td>
<td>0.980</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_B$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_C$</td>
<td>0.960</td>
<td>0.936</td>
<td>0.934</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_D$</td>
<td>0.938</td>
<td>0.863</td>
<td>0.767</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_A$</td>
<td>0.992</td>
<td>0.982</td>
<td>0.972</td>
<td>0.960</td>
<td>0.947</td>
<td>0.933</td>
<td>0.918</td>
<td></td>
</tr>
<tr>
<td>$E_B$</td>
<td>0.994</td>
<td>0.991</td>
<td>0.994</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>$E_C$</td>
<td>0.872</td>
<td>0.754</td>
<td>0.645</td>
<td>0.544</td>
<td>0.451</td>
<td>0.366</td>
<td>0.287</td>
<td></td>
</tr>
<tr>
<td>$E_D$</td>
<td>0.965</td>
<td>0.928</td>
<td>0.886</td>
<td>0.840</td>
<td>0.787</td>
<td>0.736</td>
<td>0.655</td>
<td></td>
</tr>
<tr>
<td>$E_A$</td>
<td>0.976</td>
<td>0.947</td>
<td>0.911</td>
<td>0.867</td>
<td>0.812</td>
<td>0.735</td>
<td>0.637</td>
<td></td>
</tr>
<tr>
<td>$E_B$</td>
<td>0.976</td>
<td>0.955</td>
<td>0.932</td>
<td>0.907</td>
<td>0.877</td>
<td>0.834</td>
<td>0.774</td>
<td></td>
</tr>
<tr>
<td>$E_C$</td>
<td>0.902</td>
<td>0.628</td>
<td>0.475</td>
<td>0.344</td>
<td>0.232</td>
<td>0.139</td>
<td>0.067</td>
<td></td>
</tr>
<tr>
<td>$E_D$</td>
<td>0.975</td>
<td>0.947</td>
<td>0.916</td>
<td>0.884</td>
<td>0.849</td>
<td>0.810</td>
<td>0.763</td>
<td></td>
</tr>
<tr>
<td>$E_A$</td>
<td>0.960</td>
<td>0.912</td>
<td>0.854</td>
<td>0.783</td>
<td>0.694</td>
<td>0.590</td>
<td>0.427</td>
<td></td>
</tr>
<tr>
<td>$E_B$</td>
<td>0.963</td>
<td>0.921</td>
<td>0.854</td>
<td>0.819</td>
<td>0.750</td>
<td>0.670</td>
<td>0.519</td>
<td></td>
</tr>
<tr>
<td>$E_C$</td>
<td>0.744</td>
<td>0.533</td>
<td>0.364</td>
<td>0.230</td>
<td>0.129</td>
<td>0.060</td>
<td>0.016</td>
<td></td>
</tr>
<tr>
<td>$E_D$</td>
<td>0.978</td>
<td>0.954</td>
<td>0.929</td>
<td>0.901</td>
<td>0.876</td>
<td>0.871</td>
<td>0.866</td>
<td></td>
</tr>
</tbody>
</table>

(cont'd...)

TABLE - 4.9.1
<table>
<thead>
<tr>
<th>$\beta$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_A$</td>
<td>0.946</td>
<td>0.880</td>
<td>0.802</td>
<td>0.715</td>
<td>0.592</td>
<td>0.490</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>$E_B$</td>
<td>0.948</td>
<td>0.888</td>
<td>0.820</td>
<td>0.746</td>
<td>0.640</td>
<td>0.556</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_C$</td>
<td>0.693</td>
<td>0.458</td>
<td>0.283</td>
<td>0.159</td>
<td>0.073</td>
<td>0.023</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_D$</td>
<td>0.980</td>
<td>0.958</td>
<td>0.934</td>
<td>0.920</td>
<td>0.891</td>
<td>0.975</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_A$</td>
<td>0.932</td>
<td>0.851</td>
<td>0.754</td>
<td>0.650</td>
<td>0.588</td>
<td>0.443</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>$E_B$</td>
<td>0.934</td>
<td>0.889</td>
<td>0.771</td>
<td>0.680</td>
<td>0.647</td>
<td>0.593</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_C$</td>
<td>0.648</td>
<td>0.397</td>
<td>0.222</td>
<td>0.110</td>
<td>0.048</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_D$</td>
<td>0.981</td>
<td>0.959</td>
<td>0.938</td>
<td>0.931</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_A$</td>
<td>0.919</td>
<td>0.822</td>
<td>0.715</td>
<td>0.637</td>
<td>0.634</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>$E_B$</td>
<td>0.921</td>
<td>0.830</td>
<td>0.732</td>
<td>0.614</td>
<td>0.685</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_C$</td>
<td>0.608</td>
<td>0.345</td>
<td>0.177</td>
<td>0.075</td>
<td>0.032</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_D$</td>
<td>0.982</td>
<td>0.960</td>
<td>0.946</td>
<td>0.938</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_A$</td>
<td>0.860</td>
<td>0.718</td>
<td>0.756</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>$E_B$</td>
<td>0.862</td>
<td>0.725</td>
<td>0.774</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_C$</td>
<td>0.447</td>
<td>0.179</td>
<td>0.074</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_D$</td>
<td>0.981</td>
<td>0.989</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.10 **EXAMPLE**

In Table 4.10.1 we consider the zero-truncated data of hemacytometer yeast cell counts per square observed by 'Student' [60]. We fit the GLSD by the methods A to D. While using the M.L. method, M.L. equations (4.4.4) and (4.4.5) do not converge for unique solution. We have tried on computer to solve these equations but we failed and could get only an approximate solutions for $\theta$ and $\beta$. The example also shows the fit of the GLSD by these estimates, though it is not a good fit. All other methods give good fits to the data.

For comparison, we give fits by the usual logarithmic series distribution (LSD) and by the zero-truncated negative binomial distribution (Z-NBD).
<table>
<thead>
<tr>
<th>No. of cells per square (x)</th>
<th>No. of squares (N_x)</th>
<th>2-NBD</th>
<th>ISD</th>
<th>M.L. Method</th>
<th>Method A</th>
<th>Method B</th>
<th>Method C</th>
<th>Method D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>126</td>
<td>126.28</td>
<td>133.66</td>
<td>134.25</td>
<td>128.19</td>
<td>128.69</td>
<td>130.00</td>
<td>127.87</td>
</tr>
<tr>
<td>2</td>
<td>37</td>
<td>42.28</td>
<td>34.11</td>
<td>35.08</td>
<td>38.84</td>
<td>38.43</td>
<td>37.92</td>
<td>39.05</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>12.74</td>
<td>11.61</td>
<td>11.44</td>
<td>13.73</td>
<td>13.50</td>
<td>13.04</td>
<td>13.85</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3.64</td>
<td>4.42</td>
<td>3.98</td>
<td>4.75</td>
<td>4.71</td>
<td>4.45</td>
<td>4.77</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1.00</td>
<td>1.81</td>
<td>1.40</td>
<td>1.45</td>
<td>1.43</td>
<td>1.38</td>
<td>1.42</td>
</tr>
<tr>
<td>≥ 6</td>
<td>0</td>
<td>3.36</td>
<td>1.39</td>
<td>0.85</td>
<td>0.04</td>
<td>0.19</td>
<td>0.21</td>
<td>0.04</td>
</tr>
<tr>
<td>Total</td>
<td>187</td>
<td>187.00</td>
<td>187.00</td>
<td>187.00</td>
<td>187.00</td>
<td>187.00</td>
<td>187.00</td>
<td>187.00</td>
</tr>
<tr>
<td>Mean</td>
<td>1.4599</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.6346</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>3.04</td>
<td>5.72</td>
<td>4.89</td>
<td>2.22</td>
<td>2.44</td>
<td>2.62</td>
<td>2.15</td>
<td></td>
</tr>
<tr>
<td>Estimates:</td>
<td>$\theta = 0.2361$</td>
<td>$\phi = 0.5104$</td>
<td>$\phi = 0.5910$</td>
<td>$\theta = 0.7135$</td>
<td>$\phi = 0.7014$</td>
<td>$\phi = 0.6682$</td>
<td>$\theta = 0.7200$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$k = 1.7965$</td>
<td>$\beta = 0.9071$</td>
<td>$\beta = 0.8536$</td>
<td>$\beta = 0.8590$</td>
<td>$\beta = 0.8600$</td>
<td>$\beta = 0.8508$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>