CHAPTER 3

MVU ESTIMATION FOR THE GLSD, THE GPD AND THE GNBD AND MODIFIED STIRLING AND C-NUMBERS

3.1 INTRODUCTION

Many authors such as Tate and Goen [61], Ahuja [1, 2], Ahuja and Enneking [3], Patil and Bildikar [53], Gacoullos [7], Gacoullos and Charalambides [9] and Charalambides [12] have worked on MVU estimation for binomial, negative binomial, Poisson, logarithmic series distributions and their truncated forms. In this chapter we will study the problem of MVU estimation for the GLSD, the GPD, the GNBD and their truncated forms. MVU estimation for a PSD involves numbers, sometimes of very interesting nature. For example, in MVU estimation for the logarithmic series, decapitated Poisson and decapitated negative binomial distributions the numbers occurring are the first kind Stirling, the second kind Stirling and C-numbers respectively. While the generalized Stirling numbers of the first kind, the generalised Stirling numbers of the second kind and the generalized
G-numbers are the numbers occurring in the MVU estimation for the left-truncated forms of the logarithmic series, of the Poisson and of the negative binomial distributions respectively (Cherianlambides [12]). Here, in this chapter, we define new numbers occurring in the MVU estimation for the left-truncated forms of the GILSD, of the GPD and of the GNBD and call them as modified Stirling numbers of the first kind, modified Stirling numbers of the second kind and the modified C-numbers respectively. The modified Stirling and G-numbers are obtained by further generalizing the generalized Stirling numbers and the generalized C-numbers using a suitable exponential generating function (e.g.f.) in each case as suggested by the corresponding e.g.f. of the generalized Stirling and C-numbers. For computational purpose, the recurrence relations for the modified numbers are given. It is shown that the all modified numbers approach to the same limiting number. In proving the results of this chapter the results of the previous chapter are used.
3.2 NVU Estimation for the G1SD

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from the G1SD (1.4). Then, since the G1SD is a MPFS, using (2.3.2) the distribution of \( z = \sum_{i=1}^{n} X_i \) is given by the probability function

\[
P(z = z) = p_z(\theta) = \begin{cases} 
\frac{L(z,n) (\theta(1-\theta)^{\beta-1})}{\log(1-\theta)^n}, & z = n, n+1, \ldots \\
0, & \text{otherwise}
\end{cases}
\]

where \( 0 < \theta < 1 \), \( \beta < 1 \) and \( L(z,n) \) is the coefficient of \( \theta(1-\theta)^{\beta-1} \) in the expansion of \( \log(1-\theta)^n \).

Explicitly, \( L(z,n) \) can also be obtained, by using (2.3.3), as

\[
(3.2.2) \quad L(z,n) = \frac{n}{z!} \beta^{z-1} \left[ (1-\theta)^{\beta-1}(\log(1-\theta))^{n-1} \right]_{\theta=0}.
\]

\( L(z,n) \) can be computed for different values of \( z \) and \( n \).

Obviously, if \( z = n \), \( L(z,n) = 1 \) and if \( z < n \), \( L(z,n) = 0 \).

Since the G1SD (1.4) is a truncated form, the other results regarding NVU estimators, the recurrence
relation between the numbers etc. are obtained as a particular cases of the results obtained for truncated GLSD in next section.

3.3 MVU ESTIMATION FOR THE LEFT-TRUNCATED GLSD AND THE MODIFIED STIRLING NUMBERS OF THE FIRST KIND

For a random variable $X$, the probability function of the left-truncated form of the GLSD given by (1.4), with truncation point $r$ ($r > 0$) is defined as

$$ P(x=r) = p_x(r, \beta, \theta) = \frac{\Gamma(x\beta) \left(\theta(1-\theta)^{\beta-1}\right)^x}{x! \Gamma(x\beta+1) \Gamma_1(\theta, r; \beta)}, $$

where $x \in S$, $0 < \theta < 1$, $\beta < 1$, $p_x(r, \beta, \theta) = 0$ whenever $(x\beta-x+1) \leq 0$ and

$$ (3.3.2) \quad \Gamma_1(\theta, r; \beta) = -\log(1-\theta) - \sum_{k=1}^{r-1} \frac{\Gamma(k\beta) \left(\theta(1-\theta)^{\beta-1}\right)^k}{k! \Gamma(k\beta-k+1)}.$$

For $\beta = 1$, (3.3.1) reduces to a left-truncated logarithmic series distribution.
Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ drawn from the population (3.3.1) with $\beta$ and $r$ known. Then $Z = \sum_{i=1}^{n} X_i$ is a complete sufficient for the parametric function $h(\theta) = (1-\theta)^{\beta-1}$ and hence, using (2.4.3), it has the probability function

$$P(Z = z) = p_z(\theta) = \begin{cases} \frac{\text{L}(z,n,r)(\theta(1-\theta)^{\beta-1})^z}{\left(\text{P}_1(\theta,r,\beta)\right)^n}, & \text{if } z = m, m+1, \ldots \\ 0, & \text{otherwise} \end{cases}$$

where $\text{P}_1(\theta,r,\beta)$ is as given in (3.3.2) and

$$(3.3.4) \quad \text{L}(z,n,r) = \sum a(x_1)a(x_2) \cdots a(x_n),$$

$$a(x_i) = \frac{\Gamma(\beta x_i)}{(\beta x_i)!} \frac{\Gamma(\beta x_i+x_i+1)}{(\beta x_i+x_i+1)!}$$

and the summation extends over all ordered $n$-tuplets $(x_1, x_2, \ldots, x_n)$ with $\sum x_i = z$ and all $x_i \geq r$. $\text{L}(z,n,r)$ is also the coefficient of $(\theta(1-\theta)^{\beta-1})^z$ in the expansion

$$(3.3.5) \quad \left(\text{P}_1(\theta,r,\beta)\right)^n = \sum_{z=m}^{\infty} \text{L}(z,n,r)(\theta(1-\theta)^{\beta-1})^z.$$
Now, consider the function

$$f_n(\theta, r, \beta) = \frac{1}{n!} \left[ \log(1+\theta) - \sum_{k=1}^{r-1} (-1)^{k-1} (\beta k - 1) \right] \int_0^n \left( \frac{\theta(1+\theta)^{\beta-1}}{k} \right) \, \text{d}x.$$  

Then expanding $f_n(\theta, r, \beta)$ in terms of $\theta/(1+\theta)^{1-\beta}$, by using the Lagrange formula (1.2) with $g(\theta) = (1+\theta)^{1-\beta}$ and $f(\theta) = f_n(\theta, r, \beta)$, we get

$$f_n(\theta, r, \beta) = \sum_{s=r}^{\infty} s(z_n, r, \beta) \frac{(\theta(1+\theta)^{\beta-1})^s}{s!},$$

where

$$s(z_n, r, \beta) = \theta^{z_n-1} \left[ (1+\theta)^{z_n} - z_n \right] f_n(0, r, \beta) \bigg|_{\theta=0},$$

and $f'$ is the derivative of $f$ w.r.t. $\theta$. We will call the numbers $s(z_n, r, \beta)$ as the modified Stirling numbers of the first kind and they are defined by the e.g.f. given by (3.3.7).

Clearly, for $\beta = 1$, $s(z_n, r, 1) = s(z_n, r)$ is a generalized Stirling numbers of the first kind.
defined by Charalambides [11] with e.g.f.

\[ f_n(\theta, r, 1) = f_n(\theta, r) = \frac{1}{n!} \sum_{k=1}^{r-1} \log(1+\theta) \frac{\theta^k}{k} \int^n_0 \]

\[ = \sum_{s=mn}^{\infty} s(z, n, r) \frac{\theta^z}{z!}, \quad r \in \mathbb{N} \]

and for \( \beta = 1, \ r = 1, \ s(z, n, 1, 1) = s(z, n) \) are the usual Stirling numbers of the first kind with e.g.f.

\[ f_n(\theta, 1, 1) = f_n(\theta) = (\log(1+\theta))^n / n! \ . \] So, the modified Stirling numbers of the first kind are the most generalized form of the usual first kind Stirling numbers.

From (3.3.6) and (3.3.7) and changing \( \theta \) into 

\(-\theta\), we have

\[(3.3.9) \quad \sum_{s=mn}^{\infty} (-1)^{3-n} s(z, n, r, \beta) \frac{(\theta(1-\theta)^{\beta-1})^z}{z!} \]

\[ = \frac{1}{n!} \left[ -\log(1-\theta) - \sum_{k=1}^{r-1} (\beta k - 1) \frac{(\theta(1-\theta)^{\beta-1})^k}{k} \right]^n \]

\[ = f_n(0, r, \beta), \ say. \]
Now, let us write \((-1)^{n-z} s(z,n,r,\beta) = \left| s(z,n,r,\beta) \right|\)
and call these numbers as the signless modified Stirling numbers of the first kind. Then, using the last relation, they are defined by their e.g.f.

\[
(3.3.10) \quad F_n(\theta,r,\beta) = \sum_{z=0}^{\infty} s(z,n,r,\beta) \frac{(\theta(1-\theta)^{\beta-1})^z}{z!}
\]

Comparing (3.3.2) with (3.3.9) and (3.3.5) with (3.3.10), we get

\[
(3.3.11) \quad (F_1(\theta,r,\beta))^n = n! F_n(\theta,r,\beta)
\]

\[
(3.3.12) \quad B(z,n,r) = \frac{n! s(z,n,r,\beta)}{z!}
\]

Using (3.3.11) and (3.3.12) in (3.3.3) we get

\[
(3.3.13) \quad P(z=z) = p_z(\theta) = \begin{cases} 
\left| s(z,n,r,\beta) \right| \frac{(\theta(1-\theta)^{\beta-1})^z}{z! F_n(\theta,r,\beta)}, & z = mn, mn + 1, \ldots \\
0 & \text{otherwise}
\end{cases}
\]

where \(0 < \theta < 1\), \(\beta > 1\) and \(F_n(\theta,r,\beta)\) is as given in (3.3.9).
Remark 3.3.1. Following Patil and Wani [52] we may call the distribution with the probability function (3.3.13) as the first type modified Stirling distribution.

Remark 3.3.2. For \( r = 1 \), (3.3.13) gives the distribution of \( z \) with probability function

\[
P(z=n) = \begin{cases} \frac{n! \cdot s(z,n,1,\beta) \cdot (\theta(1-\theta)^{\beta-1})^z}{\left(\frac{z!}{-\log(1-\theta)}\right)^n} \\ \text{if } z = n, n+1, \ldots \end{cases}
\]

(3.3.14)

which is the distribution of the sum of independent generalized logarithmic series variables. Comparing (3.3.14) with (3.2.1) we get

\[
H(z,n) = \frac{n! \cdot s(z,n,1,\beta)}{z!}
\]

(3.3.15)

If \( \hat{h}(r,z,n) \) is the MVE estimator of \( h(0) = \theta(1-\theta)^{\beta-1} \), then using (3.3.12) in (2.4.9) it, whenever exists, is given by
and using (3.3.16) in (2.4.10), the MVU estimator of the variance of $\hat{h}(r,z,n)$ is given by

$$(3.3.17) \quad \frac{z \cdot s(z-1,n,r,\beta)}{s(z,n,r,\beta)} \left[ \frac{s(z-1,n,r,\beta)}{s(z,n,r,\beta)} \right]^{-1}$$

Let $p_j(0) = P(X=j)$ be the probability of random variable $X$ at $X=j$, $j \in S_r$, for the distribution defined by (3.3.1). Then using (3.3.12) in (2.4.11) we get the MVU estimator $\hat{p}_j(z,n)$ of $p_j(0)$ as

$$(3.3.18) \quad \hat{p}_j(z,n) = \frac{\Gamma(j+1)(z)}{n! \Gamma(j+1)} \frac{s(z-n+1,r,\beta)}{s(z,n,r,\beta)}$$

where $(z)_j = z(z-1) \ldots (z-j+1)$.

For the case of unknown truncation point $r$ with known $\beta$, let $Z = \sum_{i=1}^{n} X_i$ and $Y = \min(X_1, X_2, \ldots, X_n)$,
where $X_1, X_2, \ldots, X_n$ be a random sample from (3.3.1). Then using (3.3.12) in (2.5.1) we have the probability function of $(Y, Z)$ as

\[
P(Y=y, Z=z) = p_{y, z}(r, \theta) = \begin{cases} 
\frac{|s(z, n, y, \beta)| - |s(z, n, y+1, \beta)|}{P_n(\theta, r, \beta)} \\
\frac{(\theta(1-\theta)^{\beta-1})^z}{z!} \cdot y = r, r+1, \ldots, \left\lfloor \frac{z}{n} \right\rfloor \text{ and} \\
\quad z = nr, nr+1, \ldots \\
0 \quad \text{otherwise}
\end{cases}
\]

where $P_n(\theta, r, \beta)$ as given in (3.3.9). In this case, using the results (2.5.7) to (2.5.14), we have the following results:

1. The MVU estimator $\hat{h}(y, z, n)$ of $h(\theta) = \theta(1-\theta)^{\beta-1}$, whenever it exists, is given by

\[
(3.3.20) \quad \hat{h}(y, z, n) = \frac{s(\beta-1, n, y, \beta) - |s(z-1, n, y+1, \beta)|}{|s(z, n, y, \beta)| - |s(z, n, y+1, \beta)|}
\]
(ii) The MVU estimator of the variance of \( \hat{\sigma}(y,z,n) \)

is given by

\[
(3.3.21) \quad \hat{\sigma}^2(y,z,n) \left[ \hat{\sigma}^2(y,z,n) - \hat{\sigma}^2(y,z-1,n) \right],
\]

where \( \hat{\sigma}^2(y,z,n) \) is as given in (3.3.20).

(iii) The MVU estimator \( \hat{r}(y,z,n) \) of the truncation point \( r \), whenever it exists, is given by

\[
(3.3.22) \quad \hat{r}(y,z,n) = y - \frac{|s(z,n,y+1,\beta)|}{|s(z,n,y,\beta)| - |s(z,n,y+1,\beta)|}
\]

(iv) The MVU estimator of the variance of the MVU estimator of \( r \) is given by

\[
(3.3.23) \quad \frac{|s(z,n,y,\beta)| - |s(z,n,y+1,\beta)|}{(|s(z,n,y,\beta)| - |s(z,n,y+1,\beta)|)^2}
\]

(v) The MVU estimator \( \hat{p}_j(y,z,n) \) of \( p_j(r,\theta) \), the probability of \( X = j \), \( j \in S_r \), for the distribution (3.3.1) with unknown \( r \), whenever it exists, is given by

\[
(3.3.24) \quad \hat{p}_j(y,z,n) = \frac{\Gamma(j\beta) (z)_j}{n \cdot j! \Gamma(j\beta-j+1)} \left[ |s(z-1,n-1,y,\beta)| - |s(z-1,n-1,y+1,\beta)| \right]
\]
3.4 RECURRENCE RELATION BETWEEN THE MODIFIED STIRLING NUMBERS OF THE FIRST KIND AND THAT BETWEEN THE KVK ESTIMATORS

For computational purpose the recurrence relations satisfied by the modified Stirling numbers of the first kind \( s(z,n,r,\beta) \) are given in the following theorem:

**Theorem 3.4.1.** The modified Stirling numbers of the first kind satisfy the following recurrence relations:

\[
(3.4.1) \quad 2^\beta - 1 \ s(z+1,n,r,\beta) + \beta z \ s(z,n,r,\beta) \\
= (1 - 2^\beta - 1) \ s(z, n - 1, r, \beta) \\
+ (-1)^{r-1} \ 2^\beta - 1 \ (z) \ _{r-1}^C \ _{(\beta r - 1)}^C \ s(z-r+1,n-1,r,\beta) \\
+ \sum_{k=1}^{r-1} (-1)^{k-1} \ (z) \ _{k}^C \ [2^\beta - 1 \ (\beta k + \beta - 1) - \beta \ (\beta k - 1)] \\
\cdot s(z-k,n-1,r,\beta)
\]

\[
(3.4.2) \quad s(z,n,r+1,\beta) = \sum_{k=0}^{n} (-1)^r \ (z)_{r}^C \ (r-1)_{k}^C \ s(z-rk,n-k,r,\beta)
\]
(3.4.3) \( s(z,n,r,\beta) = \sum_{k=0}^{n} (-1)^{r+k} \frac{(z)^{r+k}}{r^k k!} (\beta r-1)^k \)

\[ s(z-rk,n-k,r+1,\beta) \]

with initial conditions

\[ s(0,0,r,\beta) = 1, \quad s(0,n,r,\beta) = 0 \quad \text{if} \quad mn > 0 \quad \text{and} \]
\[ s(z,n,r,\beta) = 0, \quad \text{if} \quad z < mn. \]

Proof: Differentiating (3.3.6) w.r.t. \( \theta \) and then multiplying both the sides by \((1+\theta)\) we get

\[(1+\theta) \frac{d}{d\theta} (f_n(\theta,r,\beta))\]

\[ = \left[ 1 - \sum_{k=1}^{r-1} (-1)^{k-1} (\beta k-1) (1+\theta)^{k-1} (1+\theta) \theta^{k-1} (1+\theta) \beta k-k \right] \]

\[ f_{n-1}(\theta,r,\beta). \]

This is the difference-differential equation satisfied by the numbers \( s(z,n,r,\beta) \), which by the virtue of (3.3.7) gives
Expanding \((1+\theta)^{\beta z-z}\) and \((1+\theta)^{-(\beta-1)(z+k)}\) in a power series expansion and then after some adjustment we get

\[
\sum z s(z, n, r, \beta) \frac{\theta^{z+1} (1+\theta)^{\beta z-z}}{(z-1)!} + \beta \sum z s(z, n, r, \beta) \frac{\theta^{z} (1+\theta)^{\beta z-z}}{(z-1)!} = \sum z s(z, n-1, r, \beta) \frac{\theta^{z} (1+\theta)^{\beta z-z}}{z!}
\]

\[
- \sum z s(z, n-1, r, \beta) \frac{\theta^{z+k-1} (1+\theta)^{-(\beta-1)(z+k)}}{z!} = \sum z s(z, n-1, r, \beta) \frac{\theta^{z+k} (1+\theta)^{-(\beta-1)(z+k)}}{z!}
\]

\[
- \sum z s(z, n-1, r, \beta) \frac{\theta^{z+k-1} (1+\theta)^{-(\beta-1)(z+k)}}{z!} = \sum z s(z, n-1, r, \beta) \frac{\theta^{z+k} (1+\theta)^{-(\beta-1)(z+k)}}{z!}
\]

\[
\sum z s(z+1, n, r, \beta) \frac{\theta^{z+j}}{z!} = \sum z s(z, n, r, \beta) \frac{\theta^{z+j}}{z!}
\]

\[
+ \beta \sum z s(z, n, r, \beta) \frac{\theta^{z+j}}{z!}
\]

\[
= \sum z s(z, n-1, r, \beta) \frac{\theta^{z+j}}{z!}
\]

\[
- \sum z s(z, n-1, r, \beta) \frac{\theta^{z+j}}{z!}
\]

\[
= \sum z s(z, n-1, r, \beta) \frac{\theta^{z+j}}{z!}
\]
Equating the coefficients of $\theta^{z+j} / z!$ in both the sides of the last relation we get (3.4.1).

To prove (3.4.2), using (3.3.6) we have

$$f_n(0, r+1, \beta) = \frac{1}{n!} \left[ (\log(1+\theta) - \sum_{k=1}^{r-1} (-1)^{k-1} \left( \frac{\theta^{k-1}(1+\theta)^{\beta-1}}{k} \right) \right]$$

$$+ (-1)^r \left( \frac{\beta r-1}{r-1} \right) \left( \frac{\theta(1+\theta)^{\beta-1}}{r} \right)^m$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k)! (-1)^{rk} (\beta r-1)^k$$

$$\frac{(\theta(1+\theta)^{\beta-1})^r}{r^k} f_{n-k}(\theta, r, \beta),$$

which by the virtue of (3.3.7) gives

$$\sum_{z=rn}^{\infty} s(z, n, r+1, \beta) \frac{(\theta(1+\theta)^{\beta-1})^z}{z!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^{rk}}{k!} \sum_{z=r(n-k)}^{\infty} s(z, n-k, r, \beta) (\beta r-1)^k$$

$$\frac{\theta^{z+rk}(1+\theta)(\beta-1)(z+rk)}{z! r^k}.$$
In right hand side series replacing $z$ by $z - rz$
and then equating the coefficients of $(0(1+0)^{\beta-1})^z/z!$
in both the sides of the last relation we get (3.4.2).

The result (3.4.3) can be obtained in
similar manner as in (3.4.2).

Remark 3.4.1. Putting $\beta = 1$ and writing
$s(z,n,r,1) = s(z,n,r)$ in (3.4.1) we get

$$s(z+1,n,r) + z s(z,n,r) = (-1)^{x-1}(z)_{x-1} s(z-r+1,n-1,r)$$

which is the recurrence relation for the generalized
Stirling numbers of the first kind, the numbers
appearing in the MVU estimation for the $r$ left-
truncated logarithmic series distribution, and agrees
with Charalambides [11].

Putting $\beta = 1$ and $r = 1$ and writing
$s(z,n,1,1) = s(z,n)$ in (3.4.1) we get

$$s(z+1,n) + z s(z,n) = z s(z,n-1),$$

which is the recurrence relation for the usual Stirling
numbers of the first kind and agrees with Patil and
Bildikar [53].
Remark 3.4.2. Multiplying (3.4.1) by \((-1)^{n-n}\) and writing \((-1)^{n-n} s(z,n,r,\beta) = | s(z,n,r,\beta) |\), we get

\[(3.4.4) \quad 2^{\beta-1} | s(z+1,n,r,\beta) | - \beta z | s(z,n,r,\beta) | = (1 - 2^{\beta-1}) | s(z,n-1,r,\beta) | + 2^{\beta-1} (z)_{r-1} (\beta r-1) | s(z-r+1,n-1,r,\beta) | \]

\[= \sum_{k=1}^{r-1} \binom{z}{k} \left[ 2^{\beta-1} \binom{\beta k+\beta-1}{k} - \beta \binom{\beta k-1}{k-1} \right] \cdot | s(z-k,n-1,r,\beta) |, \]

which is a recurrence relation satisfied by the signless modified Stirling numbers of the first kind defined by (3.3.10).

Remark 3.4.3. For \(r = 1\), the recurrence relation (3.4.4) reduces to

\[(3.4.5) \quad 2^{\beta-1} | s(z+1,n,1,\beta) | - \beta z | s(z,n,1,\beta) | + | s(z,n,1,\beta) | \]
which is the recurrence relation between the numbers appearing in the MVU estimation for the GLSD defined in (1.4). Using (3.3.15), (3.4.5) can be written in the form, which is used in the Section 3.2, as

\[ 2^{\beta-1} (s + 1) L(s+1,n) = \beta z L(z,n) + n L(z,n-1). \]

Using the recurrence relation (3.4.5) we can establish the recurrence relation for the MVU estimator of the parametric function \( h(\theta) = \theta (1-\theta)^{\beta-1} \) in the GLSD as follows:

**Theorem 3.4.2.** The MVU estimator \( \hat{h}(s,n) \) of the parametric function \( h(\theta) = \theta (1-\theta)^{\beta-1} \) in the GLSD defined in (1.4) satisfies the recurrence relation

\[ (3.4.6) \quad \hat{h}(s+1,n) = \frac{2^{\beta-1} (s+1) \hat{h}(s,n-1)}{\beta z \hat{h}(z,n-1) + \beta (s-1) \hat{h}(z,n) + z \cdot 2^{\beta-1}}. \]

**Proof:** From (3.3.16) we have

\[ (3.4.7) \quad \hat{h}(z+1,n) = \frac{(z+1) | s(z,n,1,\beta) |}{| s(z+1,n,1,\beta) |}, \]

Proofs From (3.3.16) we have
and

\[(3.4.8) \quad \hat{h}(s,n-1) = \frac{s | s(z_{-1},n-1,1,\beta) |}{| s(z,n-1,1,\beta) |} \cdot \]  

Using \((3.4.5)\) in \((3.4.7)\) we obtain

\[
\hat{h}(s+1,n) = \frac{\beta^{-1} (s+1) | s(z,n,1,\beta) |}{\beta s | s(z,n,1,\beta) | + | s(z,n-1,1,\beta) |} 
\]

Using \((3.4.5)\), replacing \(s\) by \(s-1\), and \((3.4.8)\) in the last relation we get \((3.4.6)\). //

3.5 \textbf{LU-ESTIMATION FOR THE GPD}

Let \(X_1, X_2, \ldots, X_n\) be a random sample of size \(n\) taken from the GPD defined in \((1.7)\) and let \(z = \sum_{i=1}^{n} X_i\). Then \(z\) is a complete sufficient statistic for the parametric function \((\theta, e^{-\theta z})\), when \(\beta\) is known. Since the GPD is a HPSD, distribution of \(z\) is given by

\[
(3.5.1) \quad P(z=z) = p_z(\theta) = \begin{cases} 
\frac{L(z,n) \left(0, e^{-\theta z}\right)}{e^{n\theta}}, & z \in I \\
0, & \text{otherwise}
\end{cases}
\]
where \(0 > 0, \ |\beta\theta| < 1, \ p_2(\theta) = 0\) whenever \(n + \beta z < 0\) and \(L(z,n)\) is the coefficient of \((\theta e^{-\theta})^z\) in the expansion of \(e^{n\theta}\), which by using (2.3.3) can be obtained as

\[
L(z,n) = \frac{n}{z!} \left[ \sum_{\theta} (n+\beta z)^0 \right] |_{\theta=0}^{z=1} = \frac{n(n+\beta z)^{z-1}}{z!}.
\]

The numbers \(L(z,n)\) can easily be computed for different values of \(z\), by using (3.5.2). Using (3.5.2) in (3.5.1) we get

\[
P(z=n) = p_2(\theta) = \begin{cases} \frac{n(n+\beta z)^{z-1}}{z!} (\theta e^{-\theta})^z, & z \in I \\ 0, & \text{otherwise} \end{cases}\]

**Remark 3.5.1.** For \(\beta = 0\), (3.5.2) and (3.5.3) give

\[
L(z,n) = n^z / z! \quad \text{and} \quad P(z=n) = p_2(\theta) = (n\theta)^z / (z! e^{n\theta}), \quad z \in I
\]
which are respectively the numbers appearing in the MVU estimation and the probability function of $Z$ for the usual Poisson distribution.

**Remarh 3.5.2.** For $\beta = 1$, (3.5.2) and (3.5.3) reduce to

$$L(z,n) = n(n+z)^{z-1}/z! \quad \text{and}$$

$$p(z=x) = p(z; \theta) = n(n+z)^{z-1} \theta^z e^{-(n+z)\theta} / z!, \quad z \in \mathbb{I}$$

which are the results for the displaced Borel distribution with one parameter $\theta$.

Using (3.5.2) in (2.3.5), the MVU estimator $\hat{h}(z,n)$ of $h(\theta) = \theta e^{-\theta}$, whenever it exists, is given by

$$(3.5.4) \quad \hat{h}(z,n) = \frac{z}{n+\beta z} \left[ \frac{n+z-1}{n+\beta z} \right]^{z-2}$$

and using (3.5.4) in (2.3.6), the MVU estimator of the variance of $\hat{h}(z,n)$ is given by
\( (3.5.5) \quad \hat{V}(\hat{h}(z,n)) = \frac{n}{n + \beta z} \left( \frac{n + \beta z - \beta}{n + \beta z} \right)^{z-2} \)

\[
\begin{align*}
& \left[ \frac{n}{n + \beta z} \left( \frac{n + \beta z - \beta}{n + \beta z} \right)^{z-2} \\
& \quad - \frac{z}{n + \beta z} \frac{n + \beta z - 2\beta}{n + \beta z} \left( \frac{n + \beta z - 2\beta}{n + \beta z} \right)^{z-3} \right].
\end{align*}
\]

Let \( p_j(\theta) = P(X = j) \) be the probability of a random variable \( X \) at \( X = j \), \( j \in I \), for the GPD (1.7) and let \( \hat{p}_j(z,n) \) be an unbiased estimator of \( p_j(\theta) \). Then using (1.7) and (3.5.2) in (2.3.8) we have

\( (3.5.6) \quad \hat{p}_j(z,n) = \left( \frac{n - 1}{n} \right) \binom{z}{j} \frac{(1 + \beta j)^{j-1}}{(n + \beta z - 1 - \beta j)^j} \)

\[
\cdot \left[ 1 - \frac{1 + \beta j}{n + \beta z} \right]^{z-1}.
\]

**Remark 5.5.3.** For a special case \( \beta = 0 \) the above results reduce to

\[
\hat{h}(z,n) = \frac{z}{n}, \quad \hat{V}(\hat{h}(z,n)) = \frac{z}{n^2}
\]

and

\[
\hat{p}_j(z,n) = \binom{2}{j} (n-1)^{z-j} / n^3.
\]
which are respectively the MVU estimators of (i) the parameter $\theta$, (ii) the variance of the MVU estimator of $\theta$ and (iii) the probability in usual Poisson distribution and these results agree with those of Patil [50]. The similar results for the displaced Borel distribution can be obtained by putting $\beta = 1$ in (3.5.4) to (3.5.6).

### 3.6 MVU ESTIMATION FOR THE LEFT-TRUNCATED GPD AND THE MODIFIED STIRLING NUMBERS OF THE SECOND KIND

For a random variable $X$, the probability function of the left-truncated form of the GPD given in (1.7) with truncation point $r$ ($r > 0$) is defined as

$$
(3.6.1) \quad F(x-x) = p_x(r, \beta, \theta) = \frac{(1 + \beta x)^{x-1} \theta e^{-\theta} x}{x! \varphi_1(\theta, r, \beta)},
$$

where $x \in S_r$, $\theta > 0$, $|\beta\theta| < 1$, $p_x = 0$ whenever $1 + \beta x < 0$ and

$$
(3.6.2) \quad \varphi_1(\theta, r, \beta) = e^{\theta - \sum_{k=0}^{r-1} (1 + \beta k)^{k-1} \frac{(\theta e^{-\theta})^k}{k!}}.
$$
For $\beta = 0$ (3.6.1) reduces to a left-truncated Poisson distribution and for $\beta = 1$ to a left-truncated displaced Borel distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ taken from (3.6.1) with $\beta$ and $r$ known.

Then $Z = \sum_{i=1}^{n} X_i$ is a complete sufficient for the parametric function $h(\theta) = \theta e^{-\theta}$ and hence, using (2.4.3), it has the probability function

\[
(3.6.3) \quad P(Z=z) = p_z(\theta) = \frac{\mathbb{L}(z,n,r) (\theta e^{-\theta})^z}{(\mathcal{O}_1(0,r,\beta))^n},
\]

\[
z = rm, rm + 1, \ldots
\]

\[
= 0 \quad \text{otherwise}
\]

where $\mathcal{O}_1(0,r,\beta)$ is as given in (3.6.2) and

\[
(3.6.4) \quad \mathbb{L}(z,n,r) = \sum a(x_1)a(x_2) \ldots a(x_n),
\]

\[
a(x_i) = (1 + \beta x_i)^{x_i-1} / (x_i)! \quad \text{and the summation is as explained in (3.3.4).} \quad \mathbb{L}(z,n,r) \text{ is also the coefficient of } (\theta e^{-\theta})^z \text{ in the expansion}
\]

\[
(3.6.5) \quad (\mathcal{O}_1(0,r,\beta))^n = \sum_{z=rm}^{\infty} \mathbb{L}(z,n,r) (\theta e^{-\theta})^z.
\]
Now consider the function

\[ \phi_n(\theta, r, \beta) = \frac{1}{n!} \left[ e^{\theta} - \sum_{k=0}^{r-1} \frac{(1 + \beta k)^{k-1}}{k!} \right]^n \]

and let \( \phi_n(\theta, r, \beta) \) be expanded in terms of \( (\theta e^{-\theta}) \)
by using the Lagrange formula (1.2). Then using

(2.4.6) we get

\[ \phi_n(\theta, r, \beta) = \sum_{z=0}^{\infty} S(z, n, r, \beta) (\theta e^{-\theta})^z / z! \]

where

\[ S(z, n, r, \beta) = \left[ e^{\beta z} \phi_n'(\theta, r, \beta) \right]_{\theta=0}, \]

and \( \phi' \) is the derivative of \( \phi \) w.r.t. \( \theta \).

We will call the numbers \( S(z, n, r, \beta) \) as the modified Stirling numbers of the second kind and they are defined by e.g.f. given in (3.6.7).

Clearly, for \( \beta = 0 \) \( S(z, n, r, 0) = S(z, n, r) \)
is a generalized Stirling numbers of the second kind defined by Charalambides [11] with e.g.f.
- 61 -

\[ \phi_n(0, r, 0) = \phi_n(0, r) = \frac{1}{n!} \left[ e^{\theta} - \sum_{k=0}^{r-1} \frac{\theta^k}{k!} \right]^n \]

\[ = \sum_{\varepsilon=mn}^{\infty} S(z, n, r) \theta^\varepsilon / \varepsilon! , \ r \in N \]

and for \( \beta = 0 \) and \( r = 1 \), \( S(z, n, 1, 0) = S(z, n, 1) = S(z, n) \) is the usual Stirling number of the second kind with e.g.f.

\[ \phi_n(\theta) = \frac{1}{n!} (e^{\theta} - 1)^n = \sum_{\varepsilon=mn}^{\infty} S(z, n) \theta^\varepsilon / \varepsilon! . \]

So, the modified Stirling numbers of the second kind are the most generalized form of the usual second kind Stirling numbers.

Comparing (3.6.2) with (3.6.6) and (3.6.5) with (3.6.7) we get

(3.6.9) \( (\phi_1(\theta, r, \beta))^n = n! \phi_n(\theta, r, \beta) \)

and

(3.6.10) \( L(z, n, r) = n! S(z, n, r, \beta) / \varepsilon! . \)

Using (3.6.9) and (3.6.10) in (3.6.3) we get

\[
\begin{align*}
S(z,n,r,\theta) (e^{-\theta})^z \\
\frac{z! \, \mathcal{A}_n(\theta, r, \beta)}{z! \, \mathcal{A}_n(\theta, r, \beta)} \\
z = rn, \, rn+1, \, \ldots
\end{align*}
\]

(3.6.11) \( P(z=z) = p_z(\theta) = \begin{cases} 
S(z, n, r, \theta) (e^{-\theta})^z \\
\frac{z! \, \mathcal{A}_n(\theta, r, \beta)}{z! \, \mathcal{A}_n(\theta, r, \beta)} \\
z = rn, \, rn+1, \, \ldots \\
0 \text{ otherwise}
\end{cases} \)

\( \theta > 0, \ | \beta \theta | < 1 \) and \( \mathcal{A}_n(\theta, r, \beta) \) is as given in (3.6.6).

Remark 3.6.1. Following Patil and Wani [32], we may call the distribution with the probability function (3.6.11) as the second type modified Stirling distribution.

Remark 3.6.2. For \( r = 1 \), (3.6.11) reduces to

(3.6.12) \( P(z=z) = p_z(\theta) = \begin{cases} 
\frac{n! \, S(z, n, 1, \beta) (e^{-\theta})^z}{z! \, (e^\theta - 1)^n} \\
z = n, \, n+1, \, \ldots
\end{cases} \)

which is the distribution of the sum of independent decapitated generalized Poisson variables. Gupta [22] has obtained the distribution (3.6.12) independently.
Comparing (3.6.12) with that of Gupta [22] we get

\[(3.6.13) \quad S(z,n,1,\beta) = \sum_{m=1}^{z-1} (\frac{z-1}{m-1}) (\beta z)^{z-m} S(m,n)\]

where \(S(m,n)\) are usual Stirling numbers of the second kind.

For particular case \(\beta = 0\), the distribution (3.6.11) was derived by Charalambides [12] and the distribution (3.6.12) by Tate and Goon [61], Cacoullos [7] and Douglas [18].

Using (3.6.10) in (2.4.9), the MVU estimator \(\hat{h}(r,z,n)\) of \(h(\theta) = 0.e^{-\theta}\), whenever it exists, is given by

\[(3.6.14) \quad \hat{h}(r,z,n) = \frac{z S(z-1,n,r,\beta)}{S(z,n,r,\beta)}\]

and using (3.6.14) in (2.4.10), the MVU estimator of the variance of \(\hat{h}(r,z,n)\) is given by

\[(3.6.15) \quad \frac{z S(z-1,n,r,\beta)}{S(z,n,r,\beta)} \left[ \frac{z S(z-1,n,r,\beta)}{S(z,n,r,\beta)} - \frac{(z-1)S(z-2,n,r,\beta)}{S(z-1,n,r,\beta)} \right] \]
If \( p_j(\theta) = P(X = j) \) is the probability of a random variable \( X \) at \( X = j, j \in S_r \), for the distribution defined by (3.6.1), then using (3.6.10) in (2.4.11) we get the MVU estimator \( \hat{p}_j(z,n) \) of \( p_j(\theta) \) as

\[
(3.6.16) \quad \hat{p}_j(z,n) = \left( \frac{(1+\beta)j^{-1}}{n} \right) S(z-j,n-1,r,\beta) / S(z,n,r,\beta).
\]

For the case of unknown truncation point \( r \), the distribution of \((Y, Z)\), where \( Y = \min(X_1, X_2, \ldots, X_n) \), \( Z = \sum_{i=1}^{n} X_i \) and \( X_i (i = 1, 2, \ldots, n) \) is a random sample from (3.6.1), can be obtained by using (3.3.10) in (2.5.1) as

\[
(3.6.17) \quad P(Y=y, Z=z) = \begin{cases} 
\frac{S(z,n,y,\beta) - S(z,n,y+1,\beta)}{\phi_n(\theta,r,\beta)} & \text{if } y = r, r+1, \ldots, \lfloor z/n \rfloor, \\
(y \cdot e^{-\beta \theta})^y / z! & \text{if } z = nr, nr+1, \ldots, \\
0 & \text{otherwise}
\end{cases}
\]

where \( \phi_n(\theta,r,\beta) \) is as given in (3.6.6). For this case, the results regarding MVU estimator of \( h(\theta) = 0 \cdot e^{-\theta} \),
the MVU estimator of the variance of the MVU estimator of \( h(\theta) \), the MVU estimator of the truncation point \( r \),
the MVU estimator of the variance of the MVU estimator of \( r \) and the MVU estimator of the probability of a
random variable \( X \) for the distribution (3.6.1) can
be obtained by using (3.6.10) and (3.6.17) in the
results (2.5.5) to (2.5.14) of the previous chapter.

3.7 RECURRENT RELATION BETWEEN THE MODIFIED
STIRLING NUMBERS OF THE SECOND KIND AND
THAT BETWEEN THE MVU ESTIMATORS

The recurrence relations between the
modified Stirling numbers of the second kind
\( S(z,n,r,p) \), useful for computational purpose, are
given in the following theorem:

Theorem 3.7.1. The modified Stirling numbers of
the second kind \( S(z,n,r,p) \) satisfy the following
recurrence relations:
\begin{align*}
(3.7.1) \quad & S(z+1,n,r,\beta) = e^\beta \cdot (\beta z + n) \cdot S(z,n,r,\beta) \\
& \quad + (\beta r + 1)^{r-1} \binom{z}{r-1} \cdot S(z-r+1,n-1,r,\beta) \\
& \quad + \sum_{k=0}^{r-1} e^\beta (1+\beta k)^k - (\beta k + 1)^k \\
& \quad \cdot \binom{z}{k} S(z-k,n-1,r,\beta) \\
(3.7.2) \quad & S(z,n,r+1,\beta) = \sum_{k=0}^{n} \frac{(-1)^k (1+\beta r)^{rk-k} (z)^{rk}}{k! (r!)^k} \\
& \quad \cdot S(z-rk,n-k,r,\beta) \\
(3.7.3) \quad & S(z,n,r,\beta) = \sum_{k=0}^{n} \frac{(1+\beta r)^{rk-k} (z)^{rk}}{k! (r!)^k} \\
& \quad \cdot S(z-rk,n-k,r+1,\beta)
\end{align*}

with initial conditions

\begin{align*}
(3.7.4) \quad & S(0,0,r,\beta) = 1; \quad S(0,n,r,\beta) = 0, \text{ if } mn > 0 \text{ and } \\
& \quad S(z,n,r,\beta) = 0, \text{ if } z < mn.
\end{align*}

\textbf{Proof:} Differentiating \((3.6.6)\) w.r.t. \(\theta\) we get the difference-differential equation
\[
\frac{d}{d\theta} (g_n(\theta, r, \beta)) \\
= n \cdot g_n(\theta, r, \beta) + \left[ \sum_{k=0}^{r-1} (1+\beta k)^{k-1} \cdot \frac{(\theta \cdot e^{-\theta})^k}{k!} \right] \\
- \sum_{k=1}^{r-1} \frac{(1-\beta \theta)^k}{\theta} (1+\beta k)^{k-1} \frac{(\theta \cdot e^{-\theta})^k}{(k-1)!} \\
\cdot g_{n-1}(\theta, r, \beta)
\]

which by virtue of (3.6.7) gives

\[
\sum_{z} S(z, n, r, \beta) \frac{\theta^{z-1} \cdot e^{-\beta z \theta}}{(z-1)!} - \beta \sum_{z} S(z, n, r, \beta) \frac{\theta^{z} \cdot e^{-\beta z \theta}}{(z-1)!} \\
= n \sum_{z} S(z, n, r, \beta) \frac{\theta^{z} \cdot e^{-\beta z \theta}}{z!} + \sum_{z} \sum_{k=0}^{r-1} (1+\beta k)^{k-1} S(z, n-1, r, \beta) \\
\cdot \frac{(\theta \cdot e^{-\theta})^{z+k}}{z!} \\
- \sum_{z} \sum_{k=0}^{r-1} (1+\beta k)^{k-1} S(z, n-1, r, \beta) \frac{\theta^{z+k-1} \cdot e^{-\theta (z+k)}}{z! (k-1)!} \\
+ \beta \sum_{z} \sum_{k=0}^{r-1} (1+\beta k)^{k-1} S(z, n-1, r, \beta) \frac{(\theta \cdot e^{-\theta})^{z+k}}{z! (k-1)!}
\]
Expanding $e^{-\beta z}$ and $e^{-(\beta + k)\psi}$ by a power series expansion and then adjusting we get

$$\sum_{z} S(z+1,n,r,\beta) \sum_{j=0}^{\infty} (-1)^j (\beta z + \rho)^j \frac{\rho^{2+j}}{z!} j!$$

$$\sum_{z} (\beta z + \rho) S(z,n,r,\beta) \sum_{j=0}^{\infty} (-1)^j (\beta z)^j \frac{\rho^{2+j}}{z!} j!$$

$$+ \sum_{z} \sum_{k=0}^{r-1} (1+\beta k)^k (\frac{z}{k}) S(k,n-1,r,\beta) \sum_{j=0}^{\infty} (-1)^j (\beta z)^j \frac{\rho^{2+j}}{z!} j!$$

$$- \sum_{z} \sum_{k=1}^{r-1} (1+\beta k)^{r-1} (\frac{z}{k-1}) S(k+1,n-1,r,\beta) \sum_{j=0}^{\infty} (-1)^j (\beta z)^j \frac{\rho^{2+j}}{z!} j!$$

Equating the coefficients of $\frac{\rho^{2+j}}{z!}$ in both the sides of the last relation, we get (3.7.1). //
To prove (3.7.2), using (3.6.6) we have

\[ \Phi_n(\theta, r+1, \beta) = \frac{1}{n!} \left[ (e^\theta - \sum_{k=0}^{r-1} (1+\beta r)^k \frac{(\theta e^{-\theta})^k}{k!}) \right] 

- (1+\beta r)^{r-1} \frac{(\theta e^{-\theta} \cdot r)}{r!} \right] \]

\[ = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! (1+\beta r)^{k(r-1)} \]

\[ \cdot \Phi_{n-k}(0, r, \beta) \frac{(\theta e^{-\theta} \cdot rk)}{(r!)^k} \]

which by virtue of (3.6.7) gives

\[ \Sigma_{z=mn}^{\infty} S(z_n, r+1, \beta) \left( \theta e^{-\theta} \right)^z \frac{z!}{z!} \]

\[ = \sum_{z=mn}^{\infty} \sum_{k=0}^{n} (-1)^k (1+\beta r)^{k(r-1)} \frac{(z)^{rk}}{k! (r!)^k} \]

\[ \cdot S(z-rk, n-k, r, \beta) \left( \theta e^{-\theta} \right)^z \frac{z!}{z!} \]

Equating the coefficients of \((\theta e^{-\theta})^z / z!\) in both the sides of the last relation we get (3.7.2).

(3.7.3) can be obtained in similar manner.
Remark 3.7.1. Putting $\beta = 0$ and writing $S(z,n,r,0) = S(z,n,r)$, (3.7.1) reduces to

$$S(z+1,n,r) = n \cdot S(z,n,r) + \binom{z}{r-1} S(z-r+1,n-1,r)$$

which is the recurrence relation for the generalized Stirling numbers of the second kind, the numbers appearing in the MVU estimation for the $r$ left-truncated Poisson distribution, and agrees with Charalambides [11].

Putting $\beta = 0$ and $r = 1$ in (3.7.1) and writing $S(z,n,1,0) = S(z,n)$ we get the recurrence relation for the usual second kind Stirling numbers.

Remark 3.7.2. For $r = 1$, the recurrence relation (3.7.1) reduces to

$$e^{-\beta} \cdot S(z+1,n,1,\beta) = (n+\beta z) \cdot S(z,n,1,\beta) + S(z,n-1,1,\beta)$$

which is a recurrence relation between the numbers appearing in the MVU estimation for the decapitated GPD.
Using the recurrence relation (3.7.5), we can establish the recurrence relation for the 
MVU estimator of the parametric function \( h(0) = 0 \cdot e^{-\beta} \) 
in the decapitated GPD as follows:

**Theorem 3.7.2.** The MVU estimator \( \hat{h}(z,n) \) of 
h(0) = 0 \cdot e^{-\beta} in the decapitated GPD satisfies the 
recurrence relation

\[
(3.7.6) \quad \hat{h}(z+1,n) = \frac{(z+1) e^{-\beta} \hat{h}(z,n-1)}{(n+\beta z) \hat{h}(z,n-1) - (n+\beta z - \beta) \hat{h}(z,n) + z \cdot e^{-\beta}}
\]

**Proof:** From (3.6.14) we have,

\[
\hat{h}(z+1,n) = \frac{(z + 1) S(z,n,1,\beta)}{S(z + 1, n, 1, \beta)}
\]

which after using (3.7.5) gives

\[
\hat{h}(z+1,n) = \frac{e^{-\beta} (z + 1) S(z,n,1,\beta)}{(n+\beta z) S(z,n,1,\beta) + S(z,n-1,1,\beta)}
\]

Using (3.7.5) replacing \( z \) by \( (z - 1) \) and (3.6.14) 
replacing \( n \) by \( (n-1) \) in the last relation, we get 
(3.7.6).  //
3.8 MVU ESTIMATION FOR THE GHBD

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) taken from the GHBD defined in (1.5) and let \( Z = \sum_{i=1}^{n} X_i \). Then \( Z \) is a complete sufficient statistic for the parametric function \( (\theta(1-\theta)^{\beta-1}) \) when \( k \) and \( \beta \) are known. Since the GHBD is a MPBD, using (2.3.2) the distribution of \( Z \) is given by

\[
\begin{cases} 
\frac{L(z,n) (\theta(1-\theta)^{\beta-1})^z}{(1-\theta)^{-nk}}, & z \in \mathbb{I} \\
0, & \text{otherwise}
\end{cases}
\]

(3.8.1) \( P(Z=z) = p_Z(\theta) = \)

where \( 0 < \theta < 1, \quad |\beta\theta| < 1, \quad k > 0, \quad p_Z = 0 \)

whenever \( (k + \beta \theta - z + 1) \leq 0 \) and \( L(z,n) \) is the coefficient of \( (\theta(1-\theta)^{\beta-1})^z \) in the expansion of \( (1-\theta)^{-nk} \), which by using (2.3.3) can be obtained as

(3.8.2) \( L(z,n) = \left. \frac{1}{z!} \left( \frac{\theta(1-\theta)^{\beta-1}}{\theta} \right)^{z-1} \left[ (1-\theta)^{z-\beta} \theta (1-\theta)^{-nk} \right] \right|_{\theta=0} 

= \frac{nk}{nk + \beta z} \left( \frac{nk}{z} + \beta z \right), \quad z \in \mathbb{I}.

The numbers $L(z,n)$ can easily be computed for different values of $k$, $z$ and $n$. Using (3.8.2) in (3.8.1) we get

$$P(Z=z) = p_z(0) = \left\{ \begin{array}{ll}
\frac{nk}{nk + z} (\frac{nk + \beta z}{z} e^z) \\
(1-\alpha)^{nk + \beta z - z}, z \in I \\
0, \text{ otherwise}
\end{array} \right.$$

(3.8.3) 

**Remark 3.8.1.** For $\beta = 1$, (3.8.2) and (3.8.3) reduce to

$$L(z,n) = \frac{nk}{nk + z} (\frac{nk + z}{z})$$

and

$$P(Z=z) = p_z(0) = \frac{nk}{nk + z} (\frac{nk + z}{z}) e^z (1-\alpha)^{nk},$$

which are the results for the negative binomial distribution.

Using (3.8.2) in (2.3.5), the MLV estimator $\hat{h}(z,n)$ of $h(\theta) = \theta (1-\theta)^{\beta-1}$ given in the GNB (1.5), whenever it exists, is given by
\[
(3.8.4) \quad \hat{h}(z,n) = \frac{nk + \beta z}{nk + \beta z - \beta} \left( \frac{nk + \beta z - \beta}{z - 1} \right)
\]

and using (3.8.4) in (2.3.6) the MVU estimator of the variance of \( \hat{h}(z,n) \) is given by

\[
(3.8.5) \quad \hat{V}(\hat{h}(z,n)) = \frac{(nk+\beta z-\beta)}{(nk+\beta z-\beta)} \left( \frac{nk+\beta z-\beta}{z-1} \right) \left( \frac{(nk+\beta z)}{(nk+\beta z-\beta)} \right) \left( \frac{(nk+\beta z)}{z} \right) - \frac{(nk+\beta z)}{(nk+\beta z-\beta)} \left( \frac{(nk+\beta z)}{z-1} \right)
\]

Let \( p_j(\theta) = P(X = j) \) be the probability of a random variable \( X \) at \( X = j, \ j \in I, \) for the GMBD (1.5). Then using (3.8.2) in (2.3.8), the MVU estimator \( \hat{p}_j(z,n) \) of the probability \( p_j(\theta) \), whenever it exists, is given by

\[
(3.8.6) \quad \hat{p}_j(z,n) = \frac{k(n-1) (nk+\beta z) \left( \frac{k+j\beta}{j} \right) \left( \frac{nk+\beta z-k-j}{z-j} \right)}{n(k+j\beta) (nk+\beta z-k-j) \left( \frac{nk+\beta z}{z} \right)} \), \ z \in \mathbb{N}
\]
Remark 3.8.2. For $\beta = 1$, the above results give the MVU estimator of $\theta$ as

$$\hat{h}(z,n) = \frac{z}{nk + z - 1},$$

the MVU estimator of the variance of the MVU estimator of $\theta$ as

$$\hat{V}(\hat{h}(z,n)) = \frac{(nk - 1)z}{(nk + z - 1)^2 (nk + z - 2)},$$

and the MVU estimator of the probability as

$$\hat{p}_j(z,n) = \frac{\binom{k}{j} \frac{nk - k}{z - j} \binom{nk}{z}}{\binom{nk - j}{z}},$$

which are the results for the negative binomial distribution and agree with those in Roy and Mitra [58].

3.9 MVU ESTIMATION FOR THE LEFT-TRUNCATED GNBBD AND THE MODIFIED C-DISTRIBUTION

For a random variable $X$, the probability function of the left-truncated form of the GNBBD given
in (1.5) with truncation point \( r (r > 0) \) is defined as

\[
(3.9.1) \quad P(X = x) = p_x(r, k, \beta, \theta) = \frac{k \Gamma(k+\beta x) (\theta(1-\theta)^{\beta-1})^x}{x! \Gamma(k+\beta x - x + 1) \cdot H_1(\theta, -k, r, \beta)}
\]

where \( x \in S_r \), \( k > 0, \ 0 < \theta < 1, \ |\beta| < 1 \) and

\[
(3.9.2) \quad H_1(\theta, -k, r, \beta) = (1-\theta)^{-k} \sum_{i=0}^{R-1} k^{(i+1)\beta} (\theta(1-\theta)^{\beta-1})^i 
\]

For \( \beta = 1 \), (3.9.1) reduces to a left-truncated negative binomial distribution.

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from (3.9.1) with \( k, \beta \) and \( r \) known. Then \( \sum_{i=1}^{n} X_i \) is a complete sufficient for the parametric function \( h(\theta) = \theta(1-\theta)^{\beta-1} \) given in (3.9.1) and hence, using (2.4.3), it has the probability function,

\[
(3.9.3) \quad P(Z = z) = p_z(\theta) = \begin{cases} \frac{D(z, n, r) (\theta(1-\theta)^{\beta-1})^z}{(H_1(\theta, -k, r, \beta))^n}, & z = rn, rn + 1, \ldots \\ 0, & \text{otherwise} \end{cases}
\]
where $H_1(\theta,-k,r,\beta)$ is as given in (3.3.2) and

$$L(z,n,r) = \sum a(x_1)a(x_2) \ldots a(x_n),$$

and

$$a(x_i) = \frac{k}{x_i} \left( k + \beta x_i - 1 \right)$$

and the summation is as explained in (3.3.4). $L(z,n,r)$ is also the coefficient of $(\theta(1-\theta)^{\beta-1})^z$ in the expansion

$$(3.9.4) \quad (H_1(\theta,-k,r,\beta))^n = \sum_{z=mn}^{\infty} L(z,n,r) (\theta(1-\theta)^{\beta-1})^z.$$

Now, consider the function

$$(3.9.5) \quad h_n(\theta,k,r,\beta) = \frac{1}{n!} \left[ (1+\theta)^k - \sum_{i=0}^{r-1} k (k-1\beta+i-1) \right] \frac{(\theta(1+\theta)^{\beta-1})^i}{i}$$

and let $h_n(\theta,k,r,\beta)$ be expanded in terms of $\theta/(1+\theta)^{1-\beta}$ by using the Lagrange formula (1.2). Then, using (2.4.6) we get

$$(3.9.6) \quad h_n(\theta,k,r,\beta) = \sum_{z=mn}^{\infty} G(z,n,k,x,\beta) \frac{(\theta(1+\theta)^{\beta-1})^z}{z!}.$$
where

\[(3.9.7) \quad G(z,n,k,r,\beta) = \frac{e^{z-1}}{\theta} \left[ (1 + \theta)^{z-\beta z} h_n(\theta, k, r, \beta) \right]_{\theta = 0}.
\]

We will call the numbers \(G(z,n,k,r,\beta)\) as the modified \(G\)-numbers defined explicitly by (3.9.7) and by o.g.f given in (3.9.6).

Clearly, for \(\beta = 1\), \(G(z,n,k,r,1) = G(z,n,k,r)\) are the generalized \(G\)-numbers defined by Charalambides [11] with o.g.f.

\[h_n(\theta, k, r) = \frac{1}{n!} \sum (1+\theta)^k - \sum_{i=0}^{r-1} \binom{k}{i} \theta^i \]

\[= \sum_{z=rn}^{\infty} G(z,n,k,r) \frac{\theta^z}{z!}
\]

and for \(\beta = 1\) and \(r = 1\), \(G(z,n,k,1,1) = G(z,n,k)\) are the usual \(G\)-Numbers defined by Cacoullos and Charalambides [9]. So, the modified \(G\)-numbers are the most generalized form of the usual \(G\)-numbers.

Comparing (3.9.5) and (3.9.6) and changing \(\theta\) into \(-\theta\) and \(k\) into \(-k\) we get
Now let us write \((-1)^2 C(z, n, -k, r, \beta) = |C(z, n, -k, r, \beta)|\)
and call these numbers as the signless modified \(C\)-numbers.
Then, using (3.9.3), it is defined by the o.g.f.

\[
(3.9.9) \quad H_n(\theta, -k, r, \beta) = \sum_{z=mn}^\infty \frac{C(z, n, -k, r, \beta) (\theta (1-\theta)^{\beta-1})^z}{z!}.
\]

Comparing (3.9.2) with (3.9.8) and (3.9.4) with (3.9.9)
we get
Using (3.9.10) and (3.9.11) in (3.9.3) we get

\[
(3.9.12) \quad P(z) = p_{m}(\theta) = \begin{cases} 
\frac{\left|G(z,n,-k,r,\beta)\right| (\theta(1-\theta)^{\beta-1})^{z}}{z! H_n(\theta,-k,r,\beta)} & \text{if } z = n, n + 1, \ldots, \\
0 & \text{otherwise}
\end{cases}
\]

where \(0 < \theta < 1\), \(k > 0\), \(\beta \theta < 1\) and \(H_n(\theta,-k,r,\beta)\) is given in (3.9.8).

**Remark 3.9.1.** Following Patil and Wani, we may call the distribution (3.9.12) as the modified negative C-distribution.

**Remark 3.9.2.** For \(r = 1\), (3.9.12) reduces to

\[
(3.9.13) \quad P(z) = p_{z}(\theta) = \frac{n! \left|G(z,n,-k,1,\beta)\right| (\theta(1-\theta)^{\beta-1})^{z}}{((1-\theta)^{-k} - 1)^{n} z!}.
\]
which is the distribution of the sum of independent decapitated generalized negative binomial variables. Gupta [23] has obtained the distribution independently. Our result is in terms of the modified C-numbers. Comparing (3.9.13) with that of Gupta [23] we get the explicit value of $|C(z,n,-k,1,\beta)|$ as

$$
(3.9.14) \quad |C(z,n,-k,1,\beta)| = \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} \frac{(\beta z + ki - 1)!}{(\beta z + ki - z)!}.
$$

For particular case $\beta = 1$, the distribution (3.9.13) is derived by Charalambides [12] and the distribution (3.9.13) by Abuja [1] and Cacoullos and Charalambides [9].

Using (3.9.11) in (2.4.9), the MVU estimator $\hat{h}(r,z,n)$ of $h(\theta) = 0(1-\theta)^{\beta-1}$ given in (3.9.1), for the case of known truncation point $r$, whenever it exists, is given by

$$
(3.9.15) \quad \hat{h}(r,z,n) = \frac{2 |C(z-1,n,-k,r,\beta)|}{|C(z,n,-k,r,\beta)|}
$$

and using (3.9.15) in (2.4.10), the MVU estimator of the variance of $\hat{h}(r,z,n)$ is given by
Let \( p_j(\theta) = P(X = j) \) be the probability of a random variable \( X \) at \( X = j, j \in \mathbb{S}_n \), for the distribution defined by (3.9.1) with truncation point \( r \) known. Then using (3.9.11) in (2.4.11) we get the MVU estimator \( \hat{p}_j(z,n) \) of \( p_j(\theta) \) as

\[
(3.9.17) \quad \hat{p}_j(z,n) = \frac{k + j \beta - 1}{j - 1} \frac{k(z_j) \mid C(z-j,n-1,-k,r,\beta) \mid}{n_j \mid C(z,n,-k,r,\beta) \mid}.
\]

For the case of unknown truncation point \( r \), the distribution of \((Y,Z)\), where \( Y = \min(X_1,X_2, \ldots, X_n) \), \( z = \sum_{i=1}^{n} x_i \) and \( X_i (i = 1,2, \ldots, n) \) is a random sample from (3.9.1), can be obtained by using (3.9.11) in (2.5.1) as
where \( H_n(\theta, -k, r, \beta) \) is as given in (3.9.8).

Using (3.9.11) and (3.9.18) in (2.5.7) to (2.5.14) we get the other results for this case.

3.10 **Recurrence Relation Between the Modified C-Numbers and That Between the MVU Estimators**

For computational purpose the recurrence relations satisfied by the modified C-numbers
\( G(z, n, k, r, \beta) \) are given in the following theorem:

**Theorem 3.10.1.** The modified C-numbers \( G(z, n, k, r, \beta) \) satisfy the following recurrence relations:

\[
\left\{ \begin{array}{l}
\frac{|G(z, n, -k, y, \beta)| - |G(z, n, -k, y+1, \beta)|}{H_n(\theta, -k, r, \beta)} \\
\frac{1}{(\theta(1-\theta)^{\beta-1})^z} \\
y = r, r+1, \ldots, \lfloor z/n \rfloor \\
z = nr, nr+1, \ldots,
\end{array} \right.
\]

\[P_{y, z}(r, \theta) = \begin{cases} 
0 & \text{otherwise}
\end{cases}\]
(3.10.1) \( C(x+1,n,k,r,\beta) \)

\[ = 2^{1-\beta} (nk - \beta z) C(z,n,k,r,\beta) + a(r) \binom{z}{r-1} C(z-r+1, n-1, k, r, \beta) \]

\[ + \sum_{j=0}^{r-1} 2^{1-\beta} (k-\beta j) a(j) - a(j+1) \binom{z}{j} \cdot C(z-j, n-1, k, r, \beta) \]

(3.10.2) \( C(z,n,k,r+1,\beta) \)

\[ = \sum_{j=0}^{n} (-1)^{j} (a(r))^j \frac{(z)^{r-j}}{(r!)^{j} \cdot j!} C(z-rj, n-j, k, r, \beta) \]

(3.10.3) \( C(z,n,k,r,\beta) \)

\[ = \sum_{j=0}^{n} (a(r))^j \frac{(z)^{r-j}}{(r!)^{j} \cdot j!} C(z-rj, n-j, k, r+1, \beta) \]

where \( a(x) = k \cdot \frac{\Gamma(k - \beta x + x)}{\Gamma(k - \beta x + 1)} \) and initial conditions are

(3.10.4) \( C(0,0,k,r,\beta) = 1, \ C(0,n,k,r,\beta) = 0 \) if \( m > 0 \)

and \( C(z,n,k,r,\beta) = 0 \) if \( z < rn \).
The proofs are similar to the proofs of the results (3.4.1) to (3.4.3) of Theorem 3.4.1. and hence left.

**Remark 3.10.1.** Putting \( \beta = 1 \) and writing \( G(z,n,k,r) = G(z,n,k,r) \) in (3.10.1) we get

\[
G(z+1,n,k,r) = (nk-z) G(z,n,k,r) + (kr)^r \\
\times \binom{z}{r-1} C(z-r+1,n-1,k,r)
\]

which is the recurrence relation for the generalized G-numbers, the numbers appearing in the MVU estimation of the parametric function \( \theta / (1-\theta) \) in the binomial distribution.

Putting \( \beta = 1 \) and \( r = 1 \) and writing

\[
C(z,n,k,1,1) = C(z,n,k) \text{ in (3.10.1) we get the recurrence relation between the usual C-numbers as}
\]

\[
G(z+1,n,k) = (nk-z) G(z,n,k) + k G(z,n-1,k)
\]

**Remark 3.10.2.** Multiplying both the sides of (3.10.1) by \((-1)^{z+1}\) and writing \((-1)^z G(z,n,-k,r,\beta) = |G(z,n,-k,r,\beta)|\) we get
(3.10.5) \[ |C(z+1,n,-k,r,\beta)| \]

\[ = 2^{1-\beta}(nk + \beta z) |C(z,n,-k,r,\beta)| \]

\[ + \frac{k}{k + \beta r} (k + \beta r)^{r-1} \binom{z}{r-1} |C(z-r+1,n-1,-k,r,\beta)| \]

\[ + \sum_{j=0}^{r-1} k \left( 2^{1-\beta}(k + \beta j)^j - (k + \beta j + \beta - 1)^j \right) \binom{z}{j} \]

\[ |C(z-j,n-1,-k,r,\beta)|, \]

(3.10.5) is the recurrence relation between the signless modified G-numbers defined by the e.g.f. given in (3.9.9).

Remark 3.10.3. Putting \( \beta = 1 \) and writing

\[ |C(z,n,-k,r,1)| = |C(z,n,-k,r)| \in (3.10.5) \text{ we get} \]

\[ |C(z+1,n,-k,r)| = (nk + z) |C(z,n,-k,r)| \]

\[ + \binom{z}{r-1} (k + r - 1)^r \]

\[ |C(z-r+1,n-1,-k,r)|, \]

which is the recurrence relation for the signless generalized G-numbers appearing in the MVU estimation of the parameter \( \theta \) in the negative binomial distribution.
Remark 3.10.4. For a particular case $r = 1$, the recurrence relation (3.10.5) reduces to

\[(3.10.6) \quad 2^{\beta-1} \mid C(z+1,n,-k,1,\beta) \mid \]

\[= (nk+\beta z) \mid C(z,n,-k,1,\beta) \mid + k \mid C(z,n-1,-k,1,\beta) \mid \]

which is the recurrence relation between the numbers appearing in the MVU estimation for the decapitated GNBD.

Using the recurrence relation (3.10.6) we can establish the recurrence relation for the MVU estimator $\hat{h}(z,n)$ defined in (3.9.15) as follows:

Theorem 3.10.2. The MVU estimator $\hat{h}(z,n)$ of the parametric function $h(\theta) = \theta (1-\theta)^{\beta-1}$ in the decapitated GNBD satisfies the recurrence relation

\[(3.10.7) \quad \hat{h}(z+1,n) = \]

\[= \frac{2^{\beta-1} (z + 1) \hat{h}(z, n - 1)}{(nk+\beta z) \hat{h}(z,n-1) - (nk+\beta z-\beta) \hat{h}(z,n) + z.2^{\beta-1}} \]

Proof: The proof is similar to that given for Theorem 3.4.2.
3.11 A LIMIT PROPERTY OF THE MODIFIED STIRLING AND C-NUMBERS

In the following we will discuss an interesting limit property of the modified numbers and show that as a limiting case the signless modified Stirling number of the first kind, the modified Stirling number of the second kind and the signless modified C-number approach to a same number, called $S^*$-number.

Since the displaced Borel distribution (DBD) is a particular case of the CFD defined in (1.7), the numbers occurring in the ML estimation for the DBD can easily be derived by putting $\beta = 1$ in (3.6.7). Now, let us define the numbers $S^*(z,n,r)$ by the e.g.f.

$$(3.11.1) \quad \phi_n^*(0,r) = \sum_{z=0}^{\infty} S^*(z,n,r) \frac{(\theta e^{-\theta})^z}{z!}$$

where

$$(3.11.2) \quad \phi_n^*(0,r) = \frac{1}{n!} \left( \theta - \sum_{t=1}^{r-1} (\theta e^{-\theta})^t / t! \right)^n, \theta > 0.$$ |

Then these are the numbers appearing in the ML estimation for the usual Borel distribution. (3.11.2)
can also be written as

\[ \psi_n^*(\theta, r) = \frac{1}{n!} \sum_{t=r}^{\infty} \frac{(\theta e^{-\theta})^t}{t!} \]

which after the use of Cauchy's rule for multiplication yields

\[ (3.11.3) \quad \psi_n^*(\theta, r) = \frac{1}{n!} \sum_{z=n}^{\infty} (\theta e^{-\theta})^z \sum_{t_1, \ldots, t_n} \frac{t_1^{t_1-1} \cdots t_n^{t_n-1}}{t_1! \cdots t_n!} \]

where the second summation in right hand side extends over all ordered \( n \)-tuples of integers \( (t_1, t_2, \ldots, t_n) \) satisfying the conditions \( t_i \geq r, i = 1, 2, \ldots, n \) and \( \sum t_i = z \). Comparing \((3.11.3)\) with \((3.11.1)\) we obtain

\[ (3.11.4) \quad S^*(z, n, r) = \frac{z!}{n!} \sum_{t_1, \ldots, t_n} \frac{t_1^{t_1-1} \cdots t_n^{t_n-1}}{t_1! \cdots t_n!} \]
Now, writing (3.6.6) as

\[ \phi_n(\theta, r, \beta) = \frac{1}{n!} \sum_{t=1}^{\infty} \left(1 + \beta t\right)^{t-1} \frac{\left(\theta e^{-\theta}\right)^t}{t!} \]  

and expanding \((e^\theta - 1)\) by using Lagrange formula (1.2) we get

\[ \phi_n(\theta, r, \beta) = \frac{1}{n!} \sum_{t=r}^{\infty} \left(1 + \beta t\right)^{t-1} \frac{\left(\theta e^{-\theta}\right)^t}{t!} \]  

which after using Cauchy's rule for multiplication and then comparing with (3.6.7) give the modified Stirling number of the second kind as

\[ S(z, n, r, \beta) = \frac{1}{n!} \sum \left(1 + \beta t_1\right)^{t_1-1} \cdots \left(1 + \beta t_n\right)^{t_n-1} \frac{1}{t_1! \cdots t_n!} \]  

where the summation is as explained in (3.11.3).

From (3.11.5), by virtue of (3.11.4), we have

\[ \lim_{\beta \to \infty} \frac{S(z, n, r, \beta)}{\beta z - n} = S^*(z, n, r) \]

Similarly, after expanding \((-\log(1-\theta))\) in terms of \(\theta / (1-\theta)^{1-\beta}\) by Lagrange formula (1.2),

(3.3.9) can be written as
\[ F_n(\theta, r, \beta) = \frac{1}{n!} \left[ \sum_{t=1}^{r-1} \left( \frac{\beta t - 1}{t - 1} \right) (\theta (1-\theta)^{\beta-1})^k \right]_n. \]

Expanding the right hand side of the last result by Cauchy's rule of multiplication and then comparing with (3.3.10) we get the signless modified Stirling numbers of the first kind as

\[ (\beta t_1 - 1) / (t_1 - 1), \ldots, (\beta t_n - 1) / (t_n - 1) \]

where the summation is as explained in (3.11.5).

Using the identity

\[ \lim_{\beta \to \infty} \frac{(\beta x)^x}{\beta^x} = x^x \]

we have

\[ \lim_{\beta \to \infty} \frac{|s(z, n, r, \beta)|}{\beta^{z-n}} = s^*(z, n, r) \]

Similarly, the following result can be proved for the signless modified G-numbers defined in (3.9.9):

\[ \lim_{\beta \to \infty} \frac{|G(z, n, k, r, \beta)|}{\beta^{s-n}} = k^n s^*(z, n, r). \]