CHAPTER 2

MVT ESTIMATION FOR THE CLASS OF THE MPSD
AND LAGRANGE NUMBERS

2.1 INTRODUCTION

The problem of MVU estimation of parameters involved in a class of discrete distributions has been studied by many authors. Roy and Mitra [58] have studied the problem for the class of PSD while Joshi [37], Park [48], Joshi and Park [38], Charalambides [12] and Cacoullos [8] have studied the same for truncated PSD. Patil [50] has studied MVU estimation for the class of G PSD. In this chapter we will study the problem of MVU estimation for the class of non-truncated as well as truncated MPSD which includes all above cases. We will obtain the MVU estimators of the parametric function and those of the probability in the cases of non-truncated and r-left truncated MPSD. We will also discuss the MVU estimator of the unknown truncation point $r$. The numbers appearing in the MVU estimation for the class of MPSD are called Lagrange
numbers, since they are derived with the help of 
Lagrange formula given in (1.2). The right truncated 
and the doubly-truncated cases can be studied in 
similar manner and hence are not discussed here.

2.2 NOTATIONS AND TERMINOLOGY

In the next few sections we will use the 
following notations and terminology:

(i) Let \( T = \{ 0, t_1, t_2, \ldots \} \) be a subset of 
the set \( I \) of the non-negative integers

(ii) Let \( T(i) = \{ t_{1(i)}, t_{2(i)}, \ldots \} \), \( i = 1, 2, \ldots, n \) 
be arbitrary subsets of \( I \). Then the sum
\[
T_n = \sum_{i=1}^{n} T(i)
\]

of the \( n \) given subsets is 
defined as the set of all integers of the form
\[
\sum_{i=1}^{n} t(i), \text{ where } t(i) \in T(i).
\]

(iii) If \( T(i) = T \), \( T_n = \Sigma T \) is denoted by \( n \left\lfloor \frac{T}{T} \right\rfloor \)

(iv) A subset of \( I \), \( \{ x : x \geq r, r \in T \} \) is 
denoted by \( S_r \).
2.3 ESTIMATION FOR THE MPSD

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from the MPSD (1.1) and let \( z = \sum_{i=1}^{n} X_i \). Then, the distribution of \( z \) can be obtained as follows:

Expanding \((f(\theta))^n\) in terms of \( h(\theta) = \theta/s(\theta) \), such that \( h(\theta) \) preserves the properties stated in (1.1), by using the Lagrange formula (1.2) we have

\[
(f(\theta))^n = \sum_{z \in \mathbb{N}^T} L(z,n)(h(\theta))^z
\]

from which we get

\[
P(z = z) = \begin{cases} 
\frac{L(z,n)(h(\theta))^z}{(f(\theta))^n}, & z \in \mathbb{N}^T \\
0, & \text{otherwise}
\end{cases}
\]

where \( L(z,n) = \sum a(x_1)a(x_2) \cdots a(x_n) \) and the summation extends over all ordered \( n \)-tuplets \((x_1, x_2, \ldots, x_n)\) such that \( \sum_{i=1}^{n} x_i = z \). \( L(z,n) \) can also be obtained as the coefficient of \((h(\theta))^z\) in the expansion of \((f(\theta))^n\) given in (2.3.1) as
(2.3.3) \[ L(z,n) = \frac{1}{z^k} D^{-1} \left[ (g(\theta))^z f(\theta)^n \right] \bigg|_{\theta = 0} \]

We will call \( L(z,n) \) as Lagrange numbers. These numbers can be obtained in explicit forms and hence easily be computed for particular values of \( g(\theta) \) and \( f(\theta) \). The distribution of \( Z = \sum_{i=1}^{n} X_i \) given by (2.3.2) obviously belongs to an exponential family of distributions and hence \( Z \) is a complete sufficient statistic for \( h(\theta) \), when the parameters other than \( \theta \) involved in the distribution (1.1) are known. If \( \hat{h}_k(z,n) \) is an unbiased estimator of \( (h(\theta))^k \), where \( k \) is any positive integer, then by virtue of Rao-Blackwell and Lehmann-Scheffe' theorems \( \hat{h}_k(z,n) \) is the MVU estimator of \( (h(\theta))^k \). Following Roy and Mitra [58] and using (2.3.2) we have the following theorem:

**Theorem 2.3.1.** For the MP3D given by (1.1), the MVU estimator \( \hat{h}_k(z,n) \) of \( (h(\theta))^k \), \( k \) is a positive integer, whenever it exists, is given by

\[ \hat{h}_k(z,n) = \frac{L(z-k,n)}{L(z,n)} \]
Corollary 2.3.1. For the EPSP given in (1.1), the MVU estimator $\hat{h}_1(z,n)$ of $h(0)$, whenever it exists, is given by

\[(2.3.5) \quad \hat{h}_1(z,n) = \frac{\hat{b}(z-1,n)}{\hat{b}(z,n)}.\]

Corollary 2.3.2. The MVU estimator of the variance of $\hat{h}_1(z,n)$ obtained in (2.3.5), whenever it exists, is given by

\[(2.3.6) \quad \hat{\sigma}_1(z,n) = [\hat{h}_1(z,n) - \hat{h}_1(z-1,n)]^2.\]

Proof: The variance of $\hat{h}_1(z,n)$ is given by

\[\text{Var}[\hat{h}_1(z,n)] = \text{E}[\hat{h}_1(z,n) - \text{E}(\hat{h}_1(z,n))]^2 = \text{E}[\hat{h}_1(z,n)]^2 - \text{E}(h(0))^2.\]

Hence

\[(2.3.7) \quad \text{Var}[\hat{h}_1(z,n)] = [\hat{h}_1(z,n)]^2 - \hat{\sigma}_2(z,n)\]

is the MVU estimator of the variance of the MVU estimator of $h(0)$. 
From (2.3.4), it can be easily deduced that

\[ \hat{h}_k(z, n) = \prod_{j=0}^{k-1} \hat{h}_1(z - j, n). \]

Using the last result in (2.3.7) we will have (2.3.6).

We now consider the MVU estimation of the probability density of the MPSD.

Let \( p_j(\theta) = P(X = j) \) be the probability of \( X \) at \( X = j, j \in T \), for the MPSD given by (1.1) and let \( \hat{p}_j(z, n) \) be an unbiased estimator of \( p_j(\theta) \). Then by the use of Rao-Blackwell and Lehmann-Scheffe\'s theorems, \( \hat{p}_j(z, n) \) is the MVU estimator of \( p_j(\theta) \).

Following Roy and Mitra [58] and using (2.3.2) we have the following theorem:

**Theorem 2.3.2.** For the MPSD given by (1.1), the MVU estimator \( \hat{p}_j(z, n) \) of \( p_j(\theta) \), whenever it exists, is given by

\[ \hat{p}_j(z, n) = \frac{a(z - j, n)}{L(z, n)} \]
Proof: By the condition of unbiasedness, we have

\[ E\left[ \hat{\theta}_j(z,n) \right] = \theta_j \]

which gives

\[ \sum \frac{\hat{\theta}_j(z,n) L(z,n) (h(\theta))^2}{z} = \sum a(j) L(z, n-1) (h(\theta))^{2+j}. \]

On equating the coefficients of \((h(\theta))^2\) on both the sides we get the required result. //

Corollary 2.3.3. The MVU estimator of the variance of the \(\hat{\theta}_j(z,n)\) obtained in (2.3.8), whenever it exists, is given by

\[ (2.3.9) \quad \hat{\theta}_j(z,n) \hat{\theta}_j(z,n) - \hat{\theta}_j(z-j, n-1) \]

2.4 MVU ESTIMATION FOR THE LEFT-TRUNCATED HPSD

THE TRUNCATION POINT KNOWN

Let the probability function of a random variable \(X\) be
(2.4.1) \[ P(X = x) = p_x(r, \theta) = \frac{a(x)}{f(\theta, r)}(h(\theta))^x, \quad x \in S_r \cap T \]

where \[ f(\theta, r) = \sum_{x \in S_r \cap T} a(x)(h(\theta))^x, \quad r \in T, \quad S_r \text{ as defined in Section 2.2} \]

and \( a(x), h(\theta) \) and \( f(\theta) \) are same as defined in (1.1). Then \( X \) is said to have the left-truncated MPSD with truncation point \( r \).

We will have two cases: (i) the truncation point \( r \) is known and (ii) the truncation point \( r \) is unknown.

In this section we will consider the first case. The second case will be considered in the next section.

If the truncation point \( r \) is known, then the problem of estimation is the same as that for one parameter family of distributions given by (1.1).

That is, expanding \( (f(\theta, r))^n \) in terms of \( h(\theta) = \theta/s(\theta) \), by using Lagrange formula, such that \( h(\theta) \) preserves the properties stated in (1.1), we have

(2.4.2) \[ (f(\theta, r))^n = \sum_{z \in \mathbb{N}} \mathbb{I}(z, n, r)(h(\theta))^z, \quad z \in \mathbb{N} \cap [S_r \cap T] \]

from which we get
\[ P(Z = z) = \begin{cases} \frac{L(z, n, r) \cdot (h(\theta))^z}{(f(\theta, r))^n} \cdot \sum_{S_T \cap S_r} & \text{for } z \leq \sum_{\{x_i : x_i > r\}} \sum_{i=1}^{n} x_i = z \text{ and all } x_i > r. \\ 0 & \text{otherwise} \end{cases} \]

where \( L(z, n, r) = \sum a(x_1) a(x_2) \cdots a(x_n) \), \( Z = \sum_{i=1}^{n} x_i \), \( x_i (i = 1, 2, \ldots, n) \) is a random sample of size \( n \) drawn from the population (2.4.1) and the summation extends over all ordered \( n \)-tuplets \( \{x_1, x_2, \ldots, x_n\} \) with \( \sum_{i=1}^{n} x_i = z \) and all \( x_i > r \).

Consider the left-truncated form of the Lagrange distribution defined in (1.3) with truncation point \( r \) as given by

\[ P(X = x) = p_x(r, \theta) = \frac{\frac{1}{x!} \theta^{x-1} \sum (g(\theta))^x \cdot p(\theta) \cdot (h(\theta))^x}{f(\theta, r)} \left|_{x = 0}^{(h(\theta))^x} \right| \quad x \in S_r \cap T \]

where

\[ f(\theta, r) = \sum_{x \in S_r \cap T} \frac{1}{x!} \theta^{x-1} \cdot (g(\theta))^x \cdot p(\theta) \cdot (h(\theta))^x. \]
Expanding \((f(\theta, r))^n\) again by using Lagrange formula in terms of \(h(\theta) = \theta/g(\theta)\), where \(f(\theta)\) and \(h(\theta)\) are same as defined in (1.1), we get

\[
(2.4.6) \quad (f(\theta, r))^n = \sum_{2 \in n}^E \left[ (g(\theta))^z \frac{d^z}{d\theta^z} (\theta/g(\theta))^n \right]_{\theta=0} \cdot \left( \frac{h(\theta))^z}{z!} \right).
\]

Equating (2.4.2) and (2.4.6) we have

\[
(2.4.7) \quad L(z, n, r) = \frac{1}{z!} \sum_{2 \in n}^E \left[ (g(\theta))^z \frac{d^z}{d\theta^z} (\theta/g(\theta))^n \right]_{\theta=0} \cdot \left( \frac{h(\theta))^z}{z!} \right).
\]

We will call these numbers \(L(z, n, r)\) as modified Lagrange numbers. These numbers are most generalized numbers appearing in MVU estimation for PSDs, GPSDs and MPSDs. Modified Stirling numbers of the first and the second kind, modified C-numbers and \(S^*\)-numbers defined in Chapter 3, the generalized Stirling numbers of the first and the second kind and generalized C-numbers defined by Charalambides in \([11]\), C-numbers defined by Cacoullos and Charalambides in \([9]\) and the usual Stirling numbers
of the first and the second kind are the particular cases of these modified Lagrange numbers $L(z,n,r)$ and obtained by considering the different sets of values for $f(\theta)$ and $g(\theta)$ as follows:

(i) For $f(\theta) = \log(1+\theta)$ and $g(\theta) = (1+\theta)^{-\beta}$,

$L(z,n,r) = s(z,n,r,\beta)$, the modified Stirling numbers of the first kind defined in (3.3.8).

(ii) For $f(\theta) = e^\theta$ and $g(\theta) = e^{\theta\theta}$, $L(z,n,r) = s(z,n,r,\beta)$, the modified Stirling numbers of the second kind defined in (3.6.8).

(iii) For $f(\theta) = (1+\theta)^{k}$ and $g(\theta) = (1+\theta)^{-\beta}$,

$L(z,n,r) = C(z,n,k,r,\beta)$, the modified $C$-numbers defined in (3.9.7).

(iv) For $f(\theta) = \theta$ and $g(\theta) = e^\theta$, $L(z,n,r) = S^*(z,n,r)$, the $S^*$-numbers defined in (3.11.4).

(v) For $\beta = 1$ in (i), $L(z,n,r) = s(z,n,r)$, the generalized Stirling numbers of the first kind defined by Charalambides in [11].

(vi) For $\beta = 1$ and $r = 1$ in (i), $L(z,n,r) = s(z,n)$, the usual Stirling numbers of the first kind.
(vii) For $\beta = 0$ in (ii), $L(z,n,r) = S(z,n,r)$, the generalized Stirling numbers of the second kind defined by Charalambides in [11].

(viii) For $\beta = 0$ and $r = 1$ in (ii), $L(z,n,r) = S(z,n)$, the usual Stirling numbers of the second kind.

(ix) For $\beta = 1$ in (iii), $L(z,n,r) = C(z,n,k,r)$, the generalized C-numbers defined by Charalambides in [11].

(x) For $\beta = 1$ and $r = 1$ in (iii), $L(z,n,r) = C(z,n,k)$, the C-numbers defined by Cacoullos and Charalambides in [9].

The distribution of $z = \sum_{i=1}^{n} x_i$, given by (2.4.3), obviously belongs to an exponential family of distributions and hence $z$ is a complete sufficient statistic for $h(\theta)$, when the parameters other than $\theta$ and the truncation point $r$ involved in the distribution (2.4.1) are known. Let $\hat{h}_k(r,z,n)$ be an unbiased estimator of $(h(\theta))^k$, where $k$ is any positive integer. Then by virtue of Rao-Blackwell and Lehmann-Scheffe' theorems, $\hat{h}_k(r,z,n)$ is the MVU estimator of $(h(\theta))^k$. 
Following Roy and Mitra \cite{56} and using (2.4.3) we will have the following theorems analogous to those given in Section 2.3.

**Theorem 2.4.1.** For the left-truncated MPSE given by (2.4.1), truncated at a known point \( r \), the MVU estimator \( \hat{h}_{k}(r,z,n) \) of \( (h(\theta))^k \), \( k \) being positive integer, whenever it exists, is given by

\[
(2.4.8) \quad \hat{h}_{k}(r,z,n) = \frac{L(z-k, n, r)}{L(z, n, r)}.
\]

**Corollary 2.4.1.** For the left-truncated MPSE given by (2.4.1), truncated at a known point \( r \), the MVU estimator \( \hat{h}_{1}(r,z,n) \) of \( (h(\theta)) \), whenever it exists, is given by

\[
(2.4.9) \quad \hat{h}_{1}(r,z,n) = \frac{L(z-1, n, r)}{L(z, n, r)}.
\]

**Corollary 2.4.2.** The MVU estimator of the variance of \( \hat{h}_{1}(z,n) \) obtained in (2.4.9), whenever it exists, is given by

\[
(2.4.10) \quad \hat{h}_{1}(r,z,n) \left[ \hat{h}_{1}(r,z,n) - \hat{h}_{1}(r,z-1,n) \right]
\]
We now consider the MVU estimation of the probability density of the left-truncated MPSD in this case. The result is given in the following theorem.

**Theorem 2.4.2.** For the left-truncated MPSD given by (2.4.1), truncated at a known point \( r \), the MVU estimator \( \hat{P}_j(z, n) \) of \( p_j(r, \theta) \), the probability of \( X \) at \( X = j, j \in S_r \cap T \), is given by

\[
(2.4.11) \quad \hat{P}_j(z, n) = \frac{g(j) L(z - j, n - 1, r)}{L(z, n, r)}
\]

2.5 **MVU Estimation for the Left-Truncated MPSD, The Truncation Point Unknown**

If the truncation point \( r \) and the parameter \( \theta \) both are unknown, then for the two parameters family of distributions given by (2.4.1), the problem of the estimation for both \( r \) and \( h(\theta) \) arises. Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from the truncated MPSD (2.4.1) and \( S_r \subset T \). Further let

\[
z = \sum_{i=1}^{n} X_i \quad \text{and} \quad Y = \min \left( X_1, X_2, \ldots, X_n \right).
\]

Then, since \( z \) is a complete sufficient statistic for \( h(\theta) \), using the results of Fraser [20] one can say that the
statistic \((Y, Z)\) is jointly complete sufficient for \((r, h(\theta))\). The distribution of \((Y, Z)\) is obtained in the following theorem:

**Theorem 2.5.1.** Let \(X_1, X_2, \ldots, X_n\) be a random sample of size \(n\) from (2.4.1) and \(Z = \sum_{i=1}^{n} X_i\), \(Y = \min(X_1, X_2, \ldots, X_n)\). Then \((Y, Z)\) is jointly complete sufficient statistic for \((r, h(\theta))\) and has probability function

\[
P(Y=y, Z=z) = P_{Y,Z}(r, \theta) = \begin{cases} 
\frac{L(z,n,y)-R(z,n,y+1)(h(\theta))^{z}}{(r(\theta,r))^{n}}, \\
\quad y = r, r+1, \ldots, \left\lceil \frac{z}{n} \right\rceil \\
0 & \text{otherwise}
\end{cases}
\]

where

\[
L(z,n,y) = \sum a(x_1)a(x_2) \ldots a(x_n),
\]

the summation extends over all ordered tuplets \((x_1, x_2, \ldots, x_n)\) of integers \(x_i \geq y\) with

\[
\sum_{i=1}^{n} x_i = z.
\]
Proof: We have

\[(2.5.3) \quad p_{y,z}(x,\theta) = P(Y = y, Z = z)\]

\[= P(Y \geq y, Z = z) - P(Y \geq y + 1, Z = z)\]

\[= P(X_1 \geq y, \ldots, X_n \geq y, Z = z)\]

\[\quad - P(X_1 \geq y + 1, \ldots, X_n \geq y + 1, Z = z).\]

But since \(X_1, X_2, \ldots, X_n\) are independent, using (2.4.1), we have

\[(2.5.4) \quad P(X_1 \geq y, \ldots, X_n \geq y, Z = z)\]

\[= \frac{\sum a(x_1)a(x_2) \ldots a(x_n)}{(f(\theta,r))^n} (h(\theta))^z.\]

Using (2.5.4) with (2.5.2) in (2.5.3) we have the result (2.5.1). //

Theorem 2.5.2. For the left-truncated MPFD given by (2.4.1), the truncation point \(r\) is unknown, the MVU estimator \(\hat{h}_k(y,z,n)\) of \((h(\theta))^k\), \(k\) being a positive integer, whenever it exists, is given by
\( h_k(y, z, n) = \frac{L(z - k, n, y) - L(z - k, n, y + 1)}{L(z, n, y) - L(z, n, y + 1)} \)

**Proof:** Let \( \hat{h}_k(y, z, n) \) be an unbiased estimator of \( (h(\theta))^k \). Then by virtue of Rao-Blackwell and Lehmann-Scheffe theorems \( \hat{h}_k(y, z, n) \) is the MVU estimator of \( (h(\theta))^k \).

Now, since \( \hat{h}_k(y, z, n) \) is an unbiased estimator of \( (h(\theta))^k \), we have

\[
\mathbb{E}[\hat{h}_k(y, z, n)] = (h(\theta))^k
\]

which in virtue of (2.5.1) gives

\[
(2.5.6) \quad \sum_{y} \hat{h}_k(y, z, n) \left[ L(z, n, y) - L(z, n, y+1) \right] (h(\theta))^z
\]

\[
= \sum_{y} \hat{h}_k(y, z, n) \left[ L(z, n, y) - L(z, n, y+1) \right] (h(\theta))^{z+k}.
\]

Equating the coefficients of \( (h(\theta))^z \) in both the sides of (2.5.6) we get (2.5.5). //

**Corollary 2.5.1.** For the left-truncated MPSD given by (2.4.1), truncated at unknown point, the MVU estimator \( \hat{h}_1(y, z, n) \) of \( h(\theta) \), whenever it exists, is given by
\[(2.5.7) \quad \hat{h}_1(y, z, n) = \frac{L(z-1, n, y) - L(z-1, n, y+1)}{L(z, n, y) - L(z, n, y+1)}.
\]

**Corollary 2.5.2.** The MVU estimator of the variance of \(\hat{h}_1(y, z, n)\) obtained in \((2.5.7)\), whenever it exists, is given by

\[(2.5.8) \quad \hat{h}_1(y, z, n) = \hat{h}_1(y, z, n) - \hat{h}_1(y, z-1, n).
\]

**Proof:** The proof is similar to that of Corollary 2.5.2 and hence it is left. //

Since the truncation point \(r\) is unknown, the MVU estimator of \(r\) can be obtained as follows:

**Theorem 2.5.3.** For the left-truncated MPSD given by \((2.4.1)\), truncated at unknown point \(r\), the MVU estimator \(\hat{r}_k(y, z, n)\) of \(r^k\), where \(k\) is any positive integer, whenever it exists, is given by

\[(2.5.9) \quad \hat{r}_k(y, z, n) = y^k - \frac{L(z, n+1)}{L(z, n, y) - L(z, n, y+1)} \sum_{j=0}^{k-1} \binom{k}{j} y^j.
\]

**Proof:** Let \(\hat{r}_k(y, z, n)\) be an unbiased estimator of \(r^k\), where \(k\) is a positive integer. Then by virtue of
Rao-Blackwell and Lehmann-Scheffe theorems, $\hat{r}_k(y,z,n)$ is the MVU estimator of $r^k$ and hence

$$B[\hat{r}_k(y,z,n)] = r^k$$

which in virtue of (2.5.1) gives

$$\sum_{y} \sum_{z} \hat{r}_k(y,z,n) \left[ L(z,n,y) - L(z,n,y+1) \right] (h(\theta))^z = \sum_{y} r^k \left[ L(z,n,y) - L(z,n,y+1) \right] (h(\theta))^z.$$

The last result holds if and only if

$$(2.5.10) \quad \sum_{y=r}^{z/n} \hat{r}_k(y,z,n) \left[ L(z,n,y) - L(z,n,y+1) \right] = \sum_{y=r}^{z/n} r^k \left[ L(z,n,y) - L(z,n,y+1) \right] = r^k \cdot L(z,n,r),$$

since $L(z,n, \left[ z/n \right] + 1 ) = 0.$
Replacing \( r \) by \((r + 1)\) in (2.5.10) and subtracting the result from (2.5.10) we obtain

\[
\hat{r}_k(x,n) = [L(z,n,r) - L(z,n,r+1)]
\]

\[
= r^k \left[ L(z,n,r) - L(z,n,r+1) \right] - L(z,n,r+1) \sum_{j=0}^{k-1} \binom{k}{j} r^j.
\]

Since the last relation holds for every \( r \in T \), the result (2.5.9) follows. //

**Corollary 2.5.3.** For the left-truncated MESP given by (2.4.1), at unknown truncation point \( r \), the MVU estimator \( \hat{r}_1(y,z,n) \) of the truncation point \( r \), whenever it exists, is given by

\[
(2.5.11) \quad \hat{r}_1(y,z,n) = y - \frac{L(z,n,y+1)}{L(z,n,y) - L(z,n,y+1)}.
\]

**Corollary 2.5.4.** The MVU estimator of the variance of the \( \hat{r}_1(y,z,n) \) obtained in (2.5.11) is given by

\[
(2.5.12) \quad \frac{L(z,n,y) \cdot L(z,n,y+1)}{\left[ L(z,n,y) - L(z,n,y+1) \right]^2}.
\]
Proof: The variance of $\hat{\tau}_1$ is given by

$$\text{Var}[\hat{\tau}_1(y,z,n)] = E[(\hat{\tau}_1(y,z,n) - E(\hat{\tau}_1(y,z,n))]^2$$

$$= E[\hat{\tau}_1(y,z,n)]^2 - \tau^2.$$

Hence

$$(2.5.15) \quad \text{Var}[\hat{\tau}_1(y,z,n)] = [\hat{\tau}_1(y,z,n)]^2 - \hat{\tau}_2(y,z,n)$$

is the MVU estimator of the variance of the MVU estimator of the truncation point $\tau$. By the use of (2.5.9) for $k=1$ and $k=2$ in (2.5.13), the result (2.5.12) is obtained. 

We now consider the MVU estimation of the probability density of the left-truncated MBSD in this case.

Let $p_j(r,\theta) = P(X = j)$ be the probability of $X$ at $X = j$, $j \in S_r \cap T$, of the left-truncated MBSD defined in (2.4.1) for $S_r \subset T$ and truncation point be unknown. Let $\hat{p}_j(y,z,n)$ be an unbiased estimator of $p_j(r,\theta)$. Then by the use of Rao-Blackwell and Lehmann-Scheffe' theorems $\hat{p}_j(y,z,n)$ is the MVU estimator of
$p_j(r, \theta)$. To find $\hat{p}_j(y, z, n)$ we have the following theorem:

**Theorem 2.5.4.** For the left-truncated MFSID given by (2.4.1), the truncation point is unknown, the MVU estimator $\hat{p}_j(y, z, n)$ of $p_j(r, \theta)$, whenever it exists, is given by

$$
(2.5.14) \quad \hat{p}_j(y, z, n) = \frac{a(j) \left[ L(z-j, n-1, y) - L(z-j, n-1, y+1) \right]}{L(z, n, y) - L(z, n, y+1)}
$$

**Proof:** By the condition of unbiasedness we have

$$
E \left[ \hat{p}_j(y, z, n) \right] = p_j(r, \theta)
$$

which after the use of (2.4.1) and (2.5.1) gives

$$
\sum_{z} \sum_{y} \hat{p}_j(y, z, n) \frac{L(z, n, y) - L(z, n, y+1)}{(f(\theta, x))^n} (h(\theta))^z = \frac{a(j) (h(\theta))^d}{f(\theta, x)}
$$

that is
Equating the coefficients of \( (h(\theta))^2 \) in both the sides of the last relation we get (2.5.14).  //