CHAPTER I

INTRODUCTION

This thesis is devoted to a study of two major classes of non-normal operators, both of them being extensions of the class of hyponormal operators. The first of these classes pertains to generalization related to certain growth conditions satisfied by the resolvent operator, while the second consists of generalization suggested by norm conditions satisfied by the operator itself. It has been known for a long time now that if the resolvent of an operator has a first order rate of growth with respect to its spectrum then certain conditions on its spectrum ensure normality (even selfadjointness and unitariness) of the operator; see Nieminen [46] and Donoghue [14]. Stampfli [70] first observed that the resolvent of a hyponormal
operator satisfies first order growth condition \( (G_1) \) relative to its spectrum and investigated implications of this condition. Later, this and more general growth conditions an operator may satisfy received considerable attention; see for instance, Stampfli [70,72,73,74,75], Orland [47], Putnam [55,56], Luecke [41,42,43], Radjabanipour [61,62,63,64], Wadhwa [78,79], Gohberg and Krein [22], Saito [65], and Fuji [16]. Holbrook [30] introduced and studied the concept of operator radius \( w_\rho(\cdot) \) based on the theory of unitary \( \rho \)-dilation of Nagy and Foias [44,45]. Recently, Patel [48,49] introduced class \( M_\rho \) of operators, the operator radius of whose resolvent satisfies a growth condition relative to its spectrum, generalizing the condition \( (G_1) \). We take up here a detailed study of operators of class \( M_\rho \). Next, results are obtained regarding the relations between the spectra of certain convexoid operators and those of their cartesian and polar components, similar to recent results of Putnam [58] on hyponormal operators. Finally, we introduce and study a new class of operators, namely, the class \( Q(k) \) of \( k \)-quasihyponormal operators defined by certain norm condition. Various generalizations of
hyponormal operators suggested by norm conditions have been studied; see, Istrătescu [33,34,35], Shila devi [63], Patel [50,51] and Wadhwa [77]. Our interest in these classes $Q(k)$ of $k$-quasihyponormal operators is mainly due to the fact that these are independent of normaloid and convexoid operators (two major generalizations of hyponormal operators widely studied) and contain nilpotent operators.

In this chapter, we set up the notations and terminology, and present a chapterwise résumé of the results contained in this thesis.

By an operator we shall mean a bounded linear operator on a Complex Hilbert space $H$ with inner product $(\ ,\ )$ and norm $||.||$. $B(H)$ will denote the Banach algebra of all operators on $H$. For $T \in B(H)$, $T^*$ is the Hilbert space adjoint of $T$, $R(T)$ is the range and $N(T)$ the null space of $T$.

For $M \subset H$, let $M^\perp$ denote the orthogonal complement of $M$ in $H$. If a (closed linear) subspace $M$ is invariant under $T$, $T/M$ will denote the restriction of $T$ to $M$. 
For basic properties of Hilbert spaces and operators we refer to \[2,27,78\].

Let \( P \) be a property that an operator may possess. We say that \( T \) is \textit{restriction}-\( P \) if the restriction of \( T \) to each of its invariant subspaces possesses the property \( P \). \( T \) is said to be \textit{reduction}-\( P \) if the restriction of \( T \) to each of its reducing subspaces possesses the property \( P \).

An operator \( T \) is said to be \textit{selfadjoint} if \( T^* = T \), \textit{positive} \((T \geq 0)\) if \((T_x, x) \geq 0\) for all \( x \in H \), \textit{idempotent} if \( T^2 = T \), \textit{projection} if \( T^* = T = T^2 \), \textit{unitary} if \( T^* T = TT^* = I \), \textit{normal} if \( T^* T = TT^* \) (equivalently, if \(|T^* x| = |T x|\) for all \( x \in H \)), \textit{hyponormal} if \( T^* T - TT^* \geq 0 \) (equivalently, if \(|T^* x| \leq |T x|\) for \( x \in H \)), \textit{quasihyponormal} if \( T^*(T^* T - TT^*)T \geq 0 \) (equivalently, if \(|T^* x| \leq |T^2 x|\) for \( x \in H \)), \textit{paranormal} if \(|T x|^2 \leq |T^2 x|\) for all unit vectors \( x \) in \( H \).

We shall denote by \( \sigma(T) \), \( \pi_0(T) \) and \( \pi(T) \) the spectrum, the point spectrum (the set of all eigenvalues) and the approximate point spectrum (the set of all
approximate eigenvalues) of $T$ respectively. An eigenvalue $z$ is said to be of finite geometric multiplicity if its eigenspace $N(T-z)$ is finite dimensional. We shall write $\pi_0(T)$ for the set of all isolated points of $\sigma(T)$ which are eigenvalues of finite geometric multiplicity. An eigenvalue $z$ is said to be reducing eigenvalue if the eigenspace $N(T-z)$ corresponding to $z$ is reducing for $T$. An eigenvalue $z$ is said to be a normal eigenvalue of $T$ if $N(T-z) = N(T^* - \overline{z})$. An approximate eigenvalue $z$ is said to be a normal approximate eigenvalue of $T$ if $E(T-z) = E(T^* - \overline{z})$ where

$$E(T-z) = \left\{ \left\{ x_n \right\} : \|x_n\| = 1 \text{ and } (T-z)x_n \to 0 \right\}.$$ 

The residual spectrum of $T$ is the set of all scalars $z$ such that $N(T-z) = \{0\}$ and $R(T-z)$ is not dense.

Let $z_0$ be an isolated point of $\sigma(T)$. If $\Gamma$ is a simple closed rectifiable curve in the resolvent set of $T$ enclosing only the point $z_0$ of $\sigma(T)$ then the operator

$$P = \frac{1}{2\pi i} \int_{\Gamma} (T-z)^{-1} \, dz$$

is an idempotent (not necessarily selfadjoint) projection. $P$ is called the spectral
projection corresponding to the isolated point \( z_0 \). The range \( R(P) \) is invariant under \( T \) and \( \sigma(T/R(P)) = \{ z_0 \} \) \cite{40}. An isolated eigenvalue \( z_0 \) is said to be of finite algebraic multiplicity if \( \dim R(P) < \infty \) \cite{11,35}.

An operator \( T \) is said to be iseloid if every isolated point of \( \sigma(T) \) is an eigenvalue.

We denote by \( W(T) \) and \( \text{conv } \sigma(T) \) the numerical range of \( T \) and the convex hull of \( \sigma(T) \) respectively. We shall write \( r(T) \) and \( w(T) \) for the spectral radius and the numerical radius of \( T \) defined by

\[
r(T) = \sup \left\{ |z| : z \in \sigma(T) \right\}
\]

\[
w(T) = \sup \left\{ |z| : z \in W(T) \right\}.
\]

An operator \( T \) is convexoid if \( \overline{W(T)} = \text{conv } \sigma(T) \), where \( \overline{W(T)} \) denotes the closure of \( W(T) \).

Given \( \rho > 0 \), an operator \( T \) on \( H \) is said to have a unitary \( \rho \)-dilation if there exists a Hilbert space \( H' \) containing \( H \) as a subspace and a unitary operator \( U \) on \( H' \) such that \( T^n x = \rho^n P_H U^n x \) for all \( x \in H \) (\( n=1,2,3,\ldots \)), where \( P_H \) is the orthogonal projection of \( H' \) onto \( H \);
U is called the unitary \( \rho \)-dilation of \( T \). We denote by \( \mathcal{C}_\rho \) the class of all operators \( T \) which have a unitary \( \rho \)-dilation. It is known that \( \mathcal{C}_1 = \{ T : \|T\| \leq 1 \} \) and \( \mathcal{C}_2 = \{ T : w(T) \leq 1 \} \). To provide a unified framework for these characterizations of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), Holbrook [30] introduced the concept of operator radii \( w_\rho(T) \) (\( 0 < \rho < \infty \)) of \( T \) defined by

\[
w_\rho(T) = \inf \left\{ \alpha : \alpha > 0 \text{ and } \alpha^{-1} T \in \mathcal{C}_\rho \right\}.
\]

Holbrook observed that the family of operator radii \( w_\rho(T) \) includes the familiar operator norms \( \|T\| = w_1(T) \) and \( w(T) = w_2(T) \). The function \( w_\rho(T) \) is continuously decreasing for \( \rho \in (0, \infty) \) and \( \lim_{\rho \to \infty} w_\rho(T) = r(T) \). Furthermore, \( w_\rho(T) \leq 1 \) if and only if \( T \in \mathcal{C}_\rho \). For other basic properties of operator radii we refer to [30, 31].

An operator \( T \) is said to be \( \rho \)-loid if \( w_\rho(T) = r(T) \) \([27]\); thus 1-loid and 2-loid operators are precisely the normaloid and spectraloid operators respectively.

Motivated by the growth condition \((G_1)\) and the inequality \( w_\rho(T) \leq \|T\| \), Patel [48, 49] introduced the
classes $M_p$ of operators. An operator $T$ is said to be of class $M_p$ if $w_p[(T-z)^{-1}] < 1/d(z, \sigma(T))$ (equivalently, if $(T-z)^{-1}$ is $\rho$-oid) for all $z \notin \sigma(T)$, where $d(z, X)$ denotes the distance of the closed set $X$ from the point $z \notin X$. For $\rho = 1$, $M_1$ is precisely the class of all operators satisfying the growth condition $(G_1)$ of Stampfli [10].

More generally, an operator $T$ is said to satisfy condition $M_p$ relative to a closed set $X$ containing $\sigma(T)$ if

$$w_p[(T-z)^{-1}] < 1/d(z, X)$$

for all $z \notin X$.

An operator $T$ is said to be of class $R$ (Niecke [35]) if

$$\| (T-z)^{-1} \| = 1/d(z, \overline{W(T)})$$

for all $z \notin \overline{W(T)}$.

Let $K$ denote the closed selfadjoint two sided ideal in $B(H)$ of all compact operators on $H$. Let $\hat{T}$ be the
canonical image of $T$ in the Galkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$. Then the **essential spectrum** of $T$, denoted by $\text{essp}(T)$, is defined by $\text{essp}(T) = \sigma(\hat{T})$. The **left essential spectrum** $\text{essp}_1(T)$ of $T$ is the set of all scalars $z$ such that $\hat{T} - z\hat{1}$ is not left invertible. As shown in [15], $z \in \text{essp}_1(T)$ if and only if there is a sequence of unit vectors $x_n$ such that $x_n \to 0$ weakly and $(T-z)x_n \to 0$.

In [56], Putnam showed that $\text{Bdry} \sigma(T) \subset \text{essp}_1(T) \cup \pi_00(T)$, where $\text{Bdry} \sigma(T)$ is the boundary of $\sigma(T)$.

The **Weyl spectrum** $\omega(T)$ of an operator $T$ is defined by $\omega(T) = \bigcap_{K \in \mathcal{K}} \sigma(T + K)$. It is easily seen that $\text{essp}(T) \subset \omega(T) \subset \sigma(T)$.

For an operator $T$, a point $z \in \sigma(T)$ is said to be a **Riesz point** if $H = N(z;T) \oplus F(z;T)$, where $N(z;T)$ is finite dimensional, $F(z;T)$ is closed, $T(N) \subset N$, $T(F) \subset F$ and $(T-z)/N$ is nilpotent while $(T-z)/F$ is a homeomorphism. If each point of $\sigma(T) - \{0\}$ is a Riesz point, $T$ is called a **Riesz operator** [30]. Every Riesz operator $T$ can be decomposed in the form $T = C + Q$, where $C$ is compact and $Q$ quasinilpotent [31].
The tensor product of $H$ with itself is denoted by $H \otimes H$ and for $x, y \in H$, we write $x \otimes y$ for the tensor product of $x$ and $y$. For operators $A$ and $B$ on $H$, the tensor product $A \otimes B$ is the operator on $H \otimes H$ given by

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$ 

For elementary properties of the tensor product of operators we refer to [21].

Using the concept of 'generalized limit' Berberian [3] has constructed an extension $H^0$ of a given Hilbert space $H$ such that every operator $T$ on $H$ extends to an operator $T^0$ on $H^0$ and the mapping $T \mapsto T^0$ is a faithful $\ast$-representation of $B(H)$ into $B(H^0)$; thus $(S + T)^0 = S^0 + T^0$, $(zT)^0 = zT^0$, $(ST)^0 = S^0T^0$, $(T^*)^0 = (T^0)^*$, $1^0 = 1$, $\|T^0\| = \|T\|$ and $\sigma(T^0) = \sigma(T)$. Further (i) $T \geq 0$ if and only if $T^0 \geq 0$ and (ii) for any $T \in B(H)$,

$\pi(T) = \pi(T^0) = \pi_0(T^0).$

We now summarize chapterwise, the contents of the thesis.

In Chapter II, we study the classes $M_\rho$ of operators.
Since there is no non-zero $\rho$-loid operator if $0 < \rho < 1$ [17], it follows that $M_\rho$ is empty for $0 < \rho < 1$ and we shall therefore confine our attention to the case when $\rho \geq 1$. We note that for any complex numbers $\alpha$ and $\beta$, $\alpha T + \beta$ and $T^*$ belong to $M_\rho$ whenever $T \in M_\rho$. The classes $M_\rho$ are contained properly in the class of convexoid operators [48,49]. Since $w_\rho(\cdot)$ is a non-increasing function of $\rho$, it is easily seen that $M_\rho \subset M_{\rho'}$ whenever $\rho' > \rho$. In particular, $M_1$, the class of all operators which satisfy condition $(G_1)$ is contained in the class $M_\rho(\rho > 1)$. We start by giving an example to show that there exists an operator in $M_\rho(\rho > 1)$ which is not in $M_1$, and list some of the elementary properties of operators in $M_\rho$.

In general, $\sigma(T)$ is not a continuous function of $T$. Luecke has proved in [41] that $\sigma(T)$ is continuous if $T$ is restricted to the class $M_1$. We show here that the spectrum $\sigma(T)$ is continuous in the Hausdorff metric even if $T$ is restricted to the wider class $M_\rho$. As a consequence of this we obtain that the class $M_\rho$ is a uniformly closed subset of $B(H)$. 
Hildebrandt [39], Putman [55, 56], Stampfli [73], Saito [65] and others have investigated conditions for certain boundary points of the spectrum of an operator to be normal approximate eigenvalues. We prove here that if \( T \in \mathcal{M}_\rho \), \( z_0 \in \sigma(T) \) and if there exist sequences \( \{z_n\} \) and \( \{r_n\} \) such that \( D_n = \{ z : |z-z_n| < r_n \} \) is contained in the complement of \( \sigma(T) \), \( z_n \to z_0 \) and \( |z_n-z_0|/r_n \to 1 \), then \( z_0 \) is a normal approximate eigenvalue of \( T \). In fact, we prove this result with the weaker assumption that \( T \) satisfies the condition

\[
\rho \frac{1}{\| (T-z)^{-1} \|} \leq 1/d(z, X)
\]

for all \( z \notin X \), where \( X \) is a closed set containing \( \sigma(T) \), with the sequence \( D_n \) contained now in the complement of \( X \). This includes the result for the case \( \rho = 1 \) of Saito [65]. We deduce that if \( T \) satisfies condition \( M_\rho \) relative to a closed superset \( X \) of \( \sigma(T) \), then every semi-bare point of \( X \) in \( \sigma(T) \) is a normal approximate eigenvalue.

For an operator satisfying condition \((G_\lambda)\), Stampfli [70, Theorem 4] showed that every isolated point
of its spectrum is an eigenvalue. We prove in Theorem 6 that every operator in \( M_\rho \) is isoloid; in fact, more generally, if for some \( k < \infty \),

\[
w_\rho [ (T-z)^{-1} ] \leq k / d(z, \sigma(T)), \quad z \notin \sigma(T)
\]

then every isolated point of \( \sigma(T) \) is an eigenvalue of \( T \). This leads to several corollaries implying normality for \( T \in M_\rho \). Thus it follows that the class \( M_\rho \) is precisely the class of normal operators if \( \dim \mathcal{H} < \infty \). Further if \( T \) is reduction-\( M_\rho \) and each eigenvalue of finite multiplicity is a semi-bare point of \( \sigma(T) \), then Weyl's Theorem holds for \( T \).

We close this chapter with two independent results. Firstly, we prove that if \( T \in M_\rho \) then \( T + K \in \overline{R_1} \) for every compact operator \( K \), where \( \overline{R_1} \) is the uniform closure in \( \mathcal{B}(\mathcal{H}) \) of the class \( R_1 \) of all operators having one dimensional reducing subspace. This extends a result of Stampfli on hyponormal operators [71]. The second result is an answer to a conjecture made by Istratescu in [37] for operators in \( M_1 \).

In Chapter III, we continue the study of operators
satisfying condition $M_\rho$ relative to a closed set $X$ containing the spectrum, and obtain three main results. The first result is an extension of the well known classical result of Nieminen [46]. We prove here that if $T$ is an operator such that the imaginary axis (with possible exception of zero) is contained in the resolvent set of $T$ and if

$$w_\rho[(T-z)^{-1}] \leq 1/|\alpha|$$

for every real $\alpha \neq 0$, then $T$ is selfadjoint. For $1 \leq \rho \leq 2$, this result has been recently established by Lin [39]. Nieminen had asked in his paper [46] whether $T$ is unitary if $\sigma(T)$ lies in the unit circle $\mathbb{C}$ and

$$|\sigma(T)| \leq \frac{1}{1/(|z|-1)}$$

for $z \not\in \mathbb{C}$, to which Donoghue answered in affirmative [14]. In Theorem 10 we establish a generalization of Donoghue's Theorem. It is proved that if

(i) $\sigma(T) \subseteq \mathbb{C}$

(ii) $w_\alpha(T^{-1}) \leq 1$ and

(iii) $w_\rho[(T-z)^{-1}] \leq 1/(|z|-1), 1 < |z| < \delta (\rho > 2)$

then $T$ is unitary. As a consequence we prove that $T$ is
unitary if (i), (ii) and (iii) with $1 \leq \rho \leq 2$ are satisfied. Further we deduce that if $T \in \mathcal{M}_\rho$ and $\sigma(T) \subseteq \mathbb{C}$ then $T$ is unitary.

Stampfli has very extensively studied implications of growth conditions for operators whose spectra are thin \cite{72,73}. He demonstrates that if the spectrum of $T$ lies on a $C^1$-Jordan curve $\Gamma$ and is countable then certain growth conditions inside and outside $\Gamma$ imply normality of $T$ \cite[Theorem 5]{73}. We prove here in Theorem 11 that if $\sigma(T) \subseteq \Gamma$, $\Gamma$ being a $C^1$-Jordan curve, and if

1. $w_\rho[(T-z)^{-1}] \leq 1/d(z,\Gamma)$ for $z$ outside $\Gamma$
2. $\| (T-z)^{-1} \| \leq k/d(z,\Gamma)^n$ for $z$ inside $\Gamma$,

for some positive integer $n$, then every isolated point of $\sigma(T)$ is a normal eigenvalue of $T$. It follows from this result that if $\sigma(T)$ is a countable subset of $\Gamma$ and $T$ satisfies the above growth conditions (i) and (ii) then $T$ is normal. This improves the result of Stampfli \cite[Theorem 5]{73}. We then consider, in particular, operators whose spectrum lies on the unit circle $\mathbb{C}$ and show that if in this case $T$ satisfies $\mathcal{M}_\rho$ condition in an open
neighbourhood of \( C \) relative to the circle \( C \) then \( T \) is unitary. We conclude with the observation that Theorem 2 of [73] and Corollary 1 to Theorem 1 of [63] can be restated, replacing the norm by the operator radius \( w_\rho(\cdot) \).

Chapter IV consists of a study of relations between the spectra of certain convexoid operators and those of their real and imaginary parts on the one hand, and those of their unitary and non-negative polar factors on the other hand. As in [58], we consider here only those operators which have polar factorization of the form \( T = UP \), where \( U \) is unitary and \( P \) is non-negative (other factorizations in which \( U \) is not unitary but is only an isometry or a partial isometry [27, Problem 105] will not be considered). We start by establishing that if \( T \in M_\rho \) and \( z_0 \in \text{Bd} \sigma(T) \) then there exists a sequence of unit vectors \( x_n \) in the extended Hilbert space \( H^0 \) such that \( (T^0 - z_0)x_n \to 0 \) and \( (T^{0*} - \overline{z}_0)x_n \to 0 \), where \( T^0 \) is the extension of \( T \) to \( H^0 \) [3]. Using this result we obtain the spectral mapping relation \( \sigma(\text{Re} \ T) = \text{Re} \ \sigma(T) \) if \( T \in M_\rho \) and \( \sigma(T) \) is connected. For \( \rho = 1 \), the relation was proved by Berberian [6] and for hyponormal
operators this result is due to Putnam [53].

Next we discuss the spectra of polar factors of certain types of operators. For operators in $M_p$ we obtain the following results:

(1) If $T = UP$ and $0 \notin \text{Int } \overline{W(T)}$ then $|\sigma(T)| \subseteq \sigma(P)$, where $|\sigma(T)| = \left\{ |z| : z \in \sigma(T) \right\}$.

(2) If $T = UP$ then for $re^{i\theta} \in \sigma(T)$, $r > 0$, $e^{i\theta}$ must belong to $\sigma(U)$. Conversely, if $0 \notin \overline{W(T)}$ and $\sigma(T)$ is connected then $e^{i\theta} \in \sigma(U)$ implies that there exists $r > 0$ such that $re^{i\theta} \in \sigma(T)$.

In fact we show that the first half of (2) is true for any convexoid operator.

For an operator $T$ in Neck's class $R$ we prove:

(1) If $0 \notin \text{Int } \overline{W(T)}$ then $|\sigma(T)| \subseteq \text{essp}(T^*T)^{1/2} \cap \text{essp}(TT^*)^{1/2}$; hence if $T = UP$ then $|\sigma(T)| \subseteq \sigma(P)$. Conversely, if $T = UP$, $0 \notin \overline{W(T)}$ and if $a \in \sigma(P)$ with $a \notin \overline{w(T)}$ then $a \notin |\sigma(T)|$.

(2) Let $T = UP$. (1) If $re^{i\theta} \in \sigma(T)$, $r > 0$ then
e^{i\theta} \in \sigma(U). (ii) If \( \theta \notin \overline{\sigma(T)} \) then for \( e^{i\theta} \in \sigma(U) \) there exists \( r > 0 \) such that \( re^{i\theta} \in \sigma(T) \).

In the fifth and the last chapter, we introduce and study the classes \( Q(k) \) of \( k \)-quasihyponormal operators. \( T \) is called a \( k \)-quasihyponormal operator if

\[
T^k(T^*T - TT^*)T^k \geq 0 \quad \text{(equivalently, if} \quad \| T^k x \| \leq \| T^{k+1} x \| \quad \text{for all} \quad x \in H).
\]

If \( k=1 \), \( T \) is a quasihyponormal operator [67,68]. We are interested in the study of the classes \( Q(k) \), mainly, because of the fact that while they constitute generalizations of the classes of hyponormal and quasihyponormal operators, they also differ from these in some crucial aspects; for instance, classes \( Q(k) \) include nilpotents and are independent of those of normaloid and convexoid operators.

We begin by proving certain simple facts about the operators in \( Q(k) \). We observe that (i) \( Q(k) \subseteq Q(k+1) \)
(ii) \( Q(k) \) is uniformly closed in \( B(H) \) whereas \( \bigcup_k Q(k) \) is not closed (iii) \( T \in Q(k), \ T(M) \subseteq M \) implies
\( T/M \in Q(k) \) and \( T/R(T^k) \) is hyponormal, in particular, if \( T \) has dense range (and hence, if \( T \) is invertible) then \( T \) is hyponormal (iv) \( T = T_1 \oplus T_2 \in Q(k) \) if and only if
$T_1, T_2 \in Q(k)$ and (v) if $S, T \in Q(k)$, then $S \circ T \in Q(k)$, and if $ST = TS$, $ST^* = T^*S$ then $ST \in Q(k)$.

Next we enquire about quasinilpotent operators in $Q(k)$; it turns out that if $T \in Q(k)$, then $T$ is quasinilpotent if and only if $T$ is nilpotent of index atmost $k$. This is a key result in studying spectral properties and normality conditions for operators in $Q(k)$.

We then show that if $T \in Q(k)$ then $T$ has ascent atmost $k$, and not every operator in $Q(k)$ is of ascent $0$ or $1$, unlike hyponormal or quasihyponormal operators.

A result of Andô [1] asserts that if $T$ and $T^*$ are paranormal and have the same null space then $T$ is normal. Here we prove that if $T \in Q(k)$ ($k > 1$) and $N(T) = N(T^*)$, then $T$ is normal if either $T^* \in Q(k)$ or $T^*$ is paranormal. This result is interesting in view of the fact that paranormals are independent of $k$-quasihyponormal operators for $k > 1$.

Turning to spectral properties of operators in $Q(k)$ we prove that if $T \in Q(k)$ then every non-zero eigenvalue is reducing for $T$ and $T$ is isoloid. Thus if $T \in Q(k)$
and the eigenspaces span $H$ then $T$ is normal. We produce examples to show that for $T \in \mathcal{Q}(k)$ (i) the null space need not be reducing (ii) the null space may reduce $T$ without $T$ being hyponormal.

Spectral properties have been widely used to obtain normality conditions for an operator by several authors; see for instance [7, 8, 13, 32, 69, 70]. We show here that most of such conditions on a $k$-quasihyponormal operator $T$ imply normality up to a nilpotent summand. Any operator which is direct sum of a normal operator and a nilpotent operator of index at most $k$ belongs to $\mathcal{Q}(k)$. Here we prove that each of the following conditions for $T \in \mathcal{Q}(k)$ implies that $T$ is a direct sum of a normal and a nilpotent operator:

(i) $\sigma(T)$ is finite
(ii) $\sigma(T)$ has a finite number of limit points
(iii) $T^m$ is normal for some $m$
(iv) $T^{*m}T^b$ is compact for some $m$ and $b$
(v) $T$ is a Riesz operator
(vi) $T^{*m}T^m$ is a Riesz operator.

As a byproduct of these results it follows that if
dim $H = n$ then every operator $T \in Q(k)$ is a direct sum of a normal and a nilpotent operator, and

$$Q(k) \subset Q(m) \text{ for } k < m \leq n$$

$$= Q(n) \text{ for } k > n.$$  

Several authors have extended the classical theorem of Weyl [22] to various non-normal operators [4, 5, 12, 36]. By using recent results of Bouldin [20] we prove that Weyl's theorem holds for $T \in Q(k)$, that is, $\omega(T) = \sigma(T) - \pi_{oo}(T)$.

If an operator has countable spectrum then a variety of conditions are known to imply normality of the operator; see for instance, [7, 8, 9, 70, 72]. A remarkable recent result of Putnam [67] says that if $T$ is hyponormal and Lebesgue measure of $\sigma(T)$ is zero then $T$ is normal. Stampfli [66] had earlier proved this result under the additional condition that $T$ is similar to a normal operator; see also [54, 57]. It is this latter result that we improve here. In fact, we prove that if $T \in Q(k)$, $T$ is similar to a normal operator, $N(T) \subseteq N(T^*)$ and $\sigma(T)$ has Lebesgue measure zero then $T$ is normal.