APPENDIX A

AUTOMATIC CONTINUITY OF *HOMOMORPHISMS

The following theorem has been established by Fragoulopoulou [43, Proposition 3.9]. In this regard also see [41].

Theorem A.1. The image, under a *homomorphism \( \phi \), of a Pták Q lmc * algebra \( E \) onto a barrelled pre-pro-C* algebra \( F \) is, upto a topological algebraic *isomorphism, a C*-algebra.

In this appendix we show that this result holds if (i) the assumption that \( E \) is a Pták space is omitted; (ii) the topological algebraic assumption that \( E \) is a Q locally m-convex * algebra is replaced by the weaker and only algebraic condition that \( E \) is a *sb * algebra (without any topology); and (iii) \( F \) is assumed to be pseudocomplete. On the other hand, as shown in the examples following Theorem A.5, the assumptions that \( F \) is barrelled and that \( F \) carries the pro-C*-topology cannot be omitted.

We begin with the following lemma.
Lemma A.2. Let $I$ be a closed two sided ideal in a topological algebra $A$. If $A$ is a $Q$-algebra, then the quotient algebra $A/I$ is also a $Q$-algebra.

Proof. Let $\pi: A \to A/I = B$ (say), $\pi(x) = x + I$. Then $\pi$ is an open surjective homomorphism satisfying $\pi(A_2) \subseteq B_2$. The conclusion follows from the fact [61, Lemma E.2] that a topological algebra is a $Q$-algebra if and only if the set of all quasiregular elements has nonempty interior.

The following theorem refines [43, Theorem 3.1].

Theorem A.3 [16]. Let $E$ be a $\ast$-sub algebra, $F$ be a pre-pro-$C^\ast$-algebra with $Q \subseteq \mathcal{V}_3(F)$ and $\phi: E \to F$ be a $\ast$-homomorphism. Then $\phi(E) \subseteq b(F)$. Further, if $E$ is a $Q$ locally convex $\ast$-algebra, then $\phi$ is continuous in the norm topology on $b(F)$; and consequently, $\phi$ is continuous in the topology of $F$.

Proof. $F$ can be assumed to be complete without loss of generality. For any $x \in E$, $sp(\phi(x)) \subseteq sp(x)$; consequently, $r(\phi(x)^* \phi(x)) = r(\phi(x^* x)) \leq r(x^* x) < \infty$. Now, for $q \in Q$, $F_q$ is a $C^\ast$-algebra. Hence by [61, Corollary 5.3], $r(\phi(x)^* \phi(x)) = \sup_{q \in Q} q(\phi(x)^* \phi(x))$ for all $x \in E$. Thus, for all $x \in E$, $q \in Q$. 

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\[
\sup_{q \in \mathcal{Q}} q(\phi(x))^2 = \sup_{q \in \mathcal{Q}} q(\phi(x)^* \phi(x)) \\
= r(\phi(x)^* \phi(x)) \\
= r(\phi(x^* x)) \\
\leq r(x^* x) \\
< \infty.
\]
This shows that \( \phi(E) \subseteq b(F) \). Now, let us assume that \( E \) is a \( K \)-locally convex \*algebra. By [73], there exist a continuous seminorm \( q \) on \( E \) and a constant \( K > 0 \) such that \( r(x) \leq Kp(x) \) for all \( x \in E \). Thus,

\[
\|\phi(x)\|_\infty = \sup_{q \in \mathcal{Q}} q(\phi(x))^2 \\
= \sup_{q \in \mathcal{Q}} q(\phi(x)^* \phi(x)) \\
\leq r(x^* x) \\
\leq Kp(x^* x),
\]
for all \( x \in E \). Now, the desired continuity of \( \phi \) follows from the continuity of the involution and the joint continuity of multiplication on \( E \).

**Corollary A.4.** Let \( E \) be a \( K \)-locally convex \*algebra which is also a Pták space. Let \( F \) be a barrelled pre-pro-C*-algebra. Let \( \phi: E \to F \) be a surjective \*homomorphism. Then the quotient topology on \( E/\ker(\phi) \) is ncrmable, \( E/\ker(\phi) \) is a...
C*-algebra and φ induces a homeomorphic isomorphism between E/ker(φ) and F.

Proof. By Theorem A.3, φ is continuous. Consequently, ker(φ) is a closed two sided ideal; hence by Lemma A.2, E = E/ker(φ) is a Hausdorff Q locally convex algebra. The induced map \( \hat{\phi} : E \to F, \hat{\phi}(x + \text{ker}(\phi)) = \phi(x) \) is a continuous surjective isomorphism. Now, if E is also a Pták space, then by [69, corollary 3, p.165], E is a Pták space. Also, by [69, Corollary 1, p.164], the induced map \( \tilde{\phi} : E \to (F,τ) \) is a homeomorphism. Since a Pták space is complete [69, §8.1, p.162], \( (F,τ) \) is complete. As E is a Q-algebra, \( (F,τ) \) is also a Q-algebra. By [2, Theorem 4.3], \( (F,τ) \) is a C*-algebra. It follows now, that the quotient topology on E is normable and \( \tilde{E} \) is a C*-algebra. This completes the proof.

Theorem A.5 [16]. Let E be a *sb *algebra and F be a barrelled pseudocomplete pre-pro-C*-algebra with Q ∈ \( \mathcal{S}(F) \). Let \( \phi : E \to F \) be a surjective isomorphism. Then the topology of F is normable and F is a C*-algebra.

Proof. Let τ be the topology of F. By Theorem A.3, F = φ(E) ⊆ b(F), hence F = b(F). The set \( B = \{ y \in F = b(F) : ||y||_\infty = \sup_{q≤q} q(y) ≤ 1 \} \) is a τ-closed, τ-bounded, absolutely convex, idempotent subset of F. Also, b(F) = A(B) and \( ||\cdot||_\infty = ||\cdot||_B \). Since F is pseudocomplete, \( (A(B), ||\cdot||_B) = (B, ||\cdot||_\infty) \) is complete.
and hence is a $C^*$-algebra. Now, the open mapping theorem [69, Corollary 1, p.1641], applied to the continuous map $\text{id} : (b(F), \| \cdot \|_{\infty}) \to (F, \tau)$, $\text{id}(x) = x$, $(x \in b(F))$, shows that $\tau$ is normable and $(F, \tau)$ is the $C^*$-algebra $(b(F), \| \cdot \|_{\infty})$. This completes the proof.

Example A.6. In Theorem A.5, the barrelledness of $F$ cannot be omitted. Indeed, let $E$ to be the $C^*$-algebra $C[0,1]$ of all continuous functions on $[0,1]$ with the sup norm and let $F$ also be the same algebra but with the topology of the uniform convergence on all countable compact subsets of $[0,1]$. Also, let $\phi : E \to F$ be $\phi(f) = f$, $(f \in E)$. Then $F$ is a pro-$C^*$-algebra with the barrel $U = \{ f \in F : |\hat{f}(x)| \leq 1 \text{ for all } x \in [0,1] \}$ which is closed but not a neighbourhood of $0$. Let us note that $F$ is not normable.

Example A.7. Let $F = C^0[0,1]$, the Frechet lmc $*$algebra with the topology $\tau$ determined by the $m^*$-calibration $\{ p_n : n \in \mathbb{N} \}$, where $p_n(f) = \sup \{ \sum_{k=0}^{n} |f^{(k)}(x)| : 0 \leq x \leq 1 \}$. $(f \in F, n \in \mathbb{N})$. Then $(F, \tau)$ is not a pro-$C^*$-algebra. Taking $E = F$, $E$ is $*$sb and the map $\phi : E \to F$, $\phi(f) = f$, $(f \in E)$, is a surjective $*$homomorphism. However, $\tau$ is not normable. Thus the assumption that $F$ is a pre-pro-$C^*$-algebra from the hypothesis of Theorem A.5 cannot be omitted.