In this chapter we investigate the enveloping pro-$C^*$-algebra of a complete $lmc^*$-algebra. In particular, taking up the problem raised in [36], we characterize and investigate those complete $lmc^*$-algebras whose enveloping pro-$C^*$-algebras are $C^*$-algebras. They are called $\alpha Q$-algebras in [34] and [36]. In the first section, we give the construction, due to M. Fragoulopoulou [34], of the enveloping pro-$C^*$-algebra of a complete $lmc^*$-algebra with a bounded approximate identity. We also show that the construction, which seems to depend upon a bounded approximate identity, yields a pro-$C^*$-algebra independent of such choice. If $A = \varprojlim A_p$ is a complete $lmc^*$-algebra, then its enveloping pro-$C^*$-algebra $\mathcal{E}(A)$ turns out to be the inverse limit of the enveloping $C^*$-algebras $\mathcal{E}(A_p)$ of the Banach $^*$-algebras $A_p$ [28, §2.7.2, p.48]. Also, for a Banach $^*$-algebra $A$, $\mathcal{E}(A)$ turns out to be a $C^*$-algebra obtained by taking the quotient of $A$ by the kernel of the Gelfand-Naimark pseudonorm on $A$.

Section 2 deals with various characterizations of complete $lmc^*$-algebras whose enveloping pro-$C^*$-algebras are $C^*$-algebras (Definition 4.1.8). The complete $lmc^*$-algebras
having $C^*$-enveloping algebras are shown to be precisely those algebras which admit greatest continuous $C^*$-seminorms $p_\infty$ (Theorem 4.2.1). This main result of the section 2 is used to show that a complete $A_i$ me algebra has $C^*$-enveloping algebra (Corollary 4.2.6). As shown in the section 3, the converse of this does not hold; however, we obtain a weaker converse to this in section 4, under the assumption that the underlying algebra is hermitian (Proposition 4.2.10). It is known [61] that an $A_i$ me algebra $A$ is a $Q$-algebra if and only if $B_r = \{x \in A : r(x) \leq 1\}$ is a neighbourhood of zero. Consequently, $A$ is a $Q$-algebra if and only if there exists a continuous (submultiplicative) seminorm $p$ on $A$ and a constant $K > 0$ satisfying $r(x) \leq Kp(x)$ for all $x \in A$. Similar to this, we characterize the complete $A_i$ me algebras with $C^*$-enveloping algebras (Theorem 4.2.8) as those algebras $A$ which admit a continuous submultiplicative *seminorm* $p$ on $A$ satisfying $r^h(x) \leq Kp(x)$ for all $x \in A$ and for some constant $K > 0$, where $r^h(x)$ denotes the hermitian spectral radius of $x$ (Definition 4.2.7). It is shown (Corollary 4.2.13) that a complete $A_i$ me algebra $A$ has $C^*$-enveloping algebra if and only if $B^0(A)$ is equicontinuous (equivalently, $B^0(A)$ is equicontinuous). For a commutative complete $A_i$ me algebra $A$, the equicontinuity of the hermitian Gelfand space $\mathcal{M}^h(A)$ turns out to be equivalent to having $C^*$-enveloping algebra.
Section 3 deals with the two counter examples showing the following.

(i) A complete lmc *algebra with C*-enveloping algebra may be far from being a Q-algebra (even with the finest locally m-convex topology).

(ii) A closed subalgebra of a complete lmc *algebra with C*-enveloping algebra need not have C*-enveloping algebra.

Section 4 is devoted to investigate some spectral functions and their contribution to making a complete lmc *algebra A an algebra with C*-enveloping algebra. Section 5 is concerned with some examples of algebras with C*-enveloping algebras arising in function algebras. These include $L^\infty$-algebras, the Frechet lmc *algebra $\mathcal{O}(\mathbb{R}^n)$ of all rapidly decreasing $C^\infty$-functions on $\mathbb{R}^n$ and some algebras constructed along the line of Arens' algebra $L^\infty[0,1]$. In section 6, we deal with the topological Segal *algebras. It is shown that for a Banach *algebra A, an A-Segal *algebra B (Definition 4.6.2) is a Q-algebra and $\mathcal{S}(B) = \mathcal{S}(A)$. This yields a number of concrete examples of algebras with C*-enveloping algebras. In the final section, we investigate the enveloping C*-algebras of the topological *algebras with the bases and Köthe sequence spaces $A_\infty(Q)$ over a Köthe power set $Q$. 

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4.1. Construction of enveloping pro-C*-algebras

In what follows we construct the enveloping pro-C*-algebra, due to M. Fragoulopoulou [34], of a complete lmc *-algebra and discuss the fundamental results, concerning enveloping pro-C*-algebras, that we need later.

Definition 4.1.1. Let $A$ be a complete lmc *-algebra and $P \in \mathcal{Q}(A)$ be directed. A net $(e_j)_{j \in J}$ in $A$ is called a bounded approximate identity (bai) if

1. for each $x \in A$, $\lim_{j} e_j x = x = \lim_{j} x e_j$; and
2. for each $p \in P$ and each $j \in J$, $p(e_j) \leq 1$.

In this case, $K_{(e_j)}(A)$ (for brevity $K(A)$, if there is no confusion) will denote the collection of all continuous *-preserving algebra seminorms $p$ on $A$ satisfying $p(e_j) \leq 1$.

Throughout this chapter, $A$ stands for a complete lmc *-algebra and $P$, $(e_j)$ and $K(A)$ are fixed on $A$ as above unless stated otherwise. Also, whenever $A$ is unital, we take $(e_j)$ to be the constant net consisting of 1 only; in this case, $K(A)$ turns out to be the collection of all continuous unital *-preserving algebra seminorms on $A$. Thus without referring to the above definition, we shall freely use the notations $(e_j)$, $P$ and $K(A)$. 
Proposition 4.1.2. For \( p \in K(A) \), let

\[
4.1.2(a) \quad c_p(x) = \sup \{ |\pi(x)| : \pi \in R_p(A) \}, \quad (x \in A);
\]

and

\[
4.1.2(b) \quad I_p = \{ x \in A : c_p(x) = 0 \}.
\]

Also, suppose \( I = \bigcap \{ I_p : p \in K(A) \} \). Then the following hold.

1. For each \( p \in K(A) \) and for each \( x \in A \),

\[
c_p(x) = \sup \{ |\pi(x)| : \pi \in R_p(A) \} = \sup \{ f(x^* x)^{1/2} : f \in p^0(A) \}
\]

and \( c_p \) is a continuous \( C^* \)-seminorm on \( A \).

2. \( I = \bigcap \{ I_p : p \in P \} \), and is a closed two sided \( * \)ideal in \( A \).

3. For each \( p \in K(A) \), \( q_p(x + 1) = \inf \{ c_p(x + i) : i \in I \}, \)

\( (x + 1 \in A/I) \), defines a \( C^* \)-seminorm on the quotient algebra \( A/I \). Also, \( q_p(x + 1) = \sup \{ f(x^* x)^{1/2} : f \in p^0(A) \} = c_p(x) \) for all \( x \in A \) and for all \( p \in K(A) \).

4. The \( C^* \)-calibrations \( \{ q_p : p \in P \} \) and \( \{ q_p : p \in K(A) \} \) on \( A/I \) define the same pro-\( C^* \)-topology on \( A/I \).

Proof. (1) For \( \pi \in R_p(A) \), \( \pi_p : A/N_p \to B(H_n) \), defined by \( \pi_p(x_p) = \pi(x), \ (x_p \in A/N_p) \), is a continuous \( * \)representation of the normed \( * \)algebra \( (A/N_p, ||*||_p) \). We denote the extension of \( \pi_p \)
to the whole of A also by π. Then π_p ∈ R(A_p). By [28, 1.3.7, p.9], π_p is a norm decreasing homomorphism. Consequently, for each x ∈ A, ||π(x)|| = ||π_p(x_p)|| ≤ ||x||_p = p(x). It follows now, that

\[ R_p(A) = \{ π ∈ R(A) : ||π(x)|| ≤ p(x) \}
\]

for all x ∈ A.

Now (1) follows from [34, Lemma 4.1].

(2) Since, for each p ∈ P, \( I_p \) is a closed two sided ideal, it follows that \( \bigcap \{ I_p : p ∈ P \} \) is a closed two sided ideal in A. Now, since \( P ⊆ K(A) \), the inclusion, \( I ⊆ \bigcap \{ I_p : p ∈ P \} \) is obvious. For the reverse inclusion, suppose x ∈ A such that \( c_p(x) = 0 \) for all p ∈ P. Let \( p_1 ∈ K(A) \). Then, since P is directed, by the continuity of \( p_1 \) on A, there exist \( p ∈ P \) and a constant \( K > 0 \) such that \( p_1(y) ≤ Kp(y) \) for all \( y ∈ A \). Consequently, \( c_{p_1}(x) ≤ Kc_p(x) = 0 \) and it follows now, since \( p_1 \) is arbitrary, that x ∈ I. Thus (2) follows.

(3) Let us fix \( p ∈ K(A) \). Then for \( i ∈ I \), \( c_p(i) = 0 \); consequently, \( f(i^*i) = 0 \) for all \( f ∈ P^0_p(A) \) by (1). Also, for \( f ∈ P^0_p(A) \), \( i ∈ I \) and \( x ∈ A \), by the Cauchy-Schwarz inequality [28, 2.1.2, p.70], \( f(i^*x) = 0 = f(x^*i) \). Hence the above observation together with (1) gives,

\[
q_p(x + i) = \inf \{ c_p(x + i) : i ∈ I \}
= \inf \{ \sup \{ f((x + i)^*(x + i))^{1/2} : f ∈ P^0_p(A) \} : i ∈ I \}
\]

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This proves (3) because \( c_p \) is a \( C^\infty \)-seminorm on \( A \).

(4) By (3), it is clear that \( \{ q_p : p \in K(A) \} \) is a family of \( C^\infty \)-seminorms on \( A/I \). Also, by the definition of \( I \), it is obvious that \( \{ q_p : p \in K(A) \} \) defines a Hausdorff topology on \( A/I \). So, it remains to show that each \( q_{p_1} \), \( (p_1 \in K(A)) \), is dominated by \( q_p \) for some \( p \in P \). So, let \( p_1 \in K(A) \). Then as in the proof of (2) above, there exist \( p \in P \) and a constant \( K > 0 \) such that \( p_1(x) \leq Kp(x) \) for all \( x \in A \). Let \( \pi \in R_{p_1}(A) \). Then

\[
||\pi(x)|| \leq p_1(x) \leq Kp(x) \quad \text{for all } x \in A.
\]

Thus \( \pi \in R_p(A) \). Consequently, \( R_{p_1}(A) \subset R_p(A) \). Hence for \( x + 1 \in A/I \),

\[
q_{p_1}(x + 1) = c_{p_1}(x) = \sup \{ ||\pi(x)|| : \pi \in R_{p_1}(A) \}
\]

\[
\leq \sup \{ ||\pi(x)|| : \pi \in R_p(A) \}
\]

\[
= c_p(x)
\]

\[
= q_p(x + 1).
\]
This shows that $p_4$ is continuous in the topology determined by $\{q_p : p \in P\}$ and hence (4) follows. This completes the proof.

Definition 4.1.3. Let $A$ be a complete $lmc$ *algebra with a bai $(e_j)_{j \in J}$. The closed two sided *ideal $I$ of $A$, defined in Proposition 4.1.2, is called the *radical of $A$ and is denoted by $\text{srad}(A)$. The completion of $A/\text{srad}(A)$ with the pro-$C^*$-topology determined by the family $\{q_p : p \in P\}$ of $C^*$-seminorms on $A/\text{srad}(A)$ is called the enveloping pro-$C^*$-algebra of $A$ and is denoted by $(\mathcal{S}(A), \tau)$ (or for brevity, by $\mathcal{S}(A)$), where $\tau$ denotes the topology of $\mathcal{S}(A)$.

As claimed, we first show that the enveloping pro-$C^*$-algebra of a complete $lmc$ *algebra is independent of the choice of a particular directed $m^*$-calibration and a bai.

Theorem 4.1.4. Let $A$ be a complete $lmc$ *algebra. Let $\Gamma$ be the collection of all continuous *preserving algebra seminorms on $A$. Then the following hold.

1. For $p \in \Gamma$, $c_p(x) = \sup \{||\pi(x)|| : \pi \in R^*(A)\}$, $(x \in A)$, defines a continuous $C^*$-seminorm on $A$. Further, $p \in \mathcal{S}(A)$ if and only if $p(x) = c_p(x)$ for all $x \in A$.

2. $\text{srad}(A) = \bigcap_{p \in \Gamma} \{x \in A : c_p(x) = 0\} = \bigcap_{p \in \mathcal{S}(A)} \{N_p : p \in \mathcal{S}(A)\}$. 140
Consequently, for each $p \in \Gamma$, $q_p(x + l) = \inf \{c_p(x + i) : i \in \mathbb{I}\} = c_p(x)$, ($x \in A$), defines a
\(C^n\)-seminorm on $A/\mathbb{I}$, which is continuous in the pro-\(C^n\)-topology on $A/\mathbb{I}$, determined by \(\{q_p : p \in \Gamma\}\). Also, $q_p$ can be
uniquely extended to an element of $S(S(A))$ (the extension will also be denoted by $q_p$).

(3) $p \in S(A) \iff q_p \in S(S(A))$ is a bijective correspondence.

(4) The canonical map $\phi_p : A \rightarrow S(A)$ defined by $\phi_p(x) = x + \text{srad}(A)$, ($x \in A$), is continuous.

Proof. (1) Let us note that for $x \in A$,

$$c_p(x)^2 = \left[ \sup \{||\pi(x)|| : \pi \in R_p'(A)\} \right]^2$$

$$= \sup \{||\pi(x)||^2 : \pi \in R_p'(A)\}$$

$$= \sup \{||\pi(x^*x)|| : \pi \in R_p'(A)\}$$

$$= c_p(x^*x).$$

This, in conjunction with [71, Theorem 2, p.2], proves that $c_p$ is a \(C^n\)-seminorm on $A$. Also, $R_p'(A) \subseteq R_p(A)$ and so, by
4.1.2(c), $||\pi(x)|| \leq p(x)$, ($x \in A, \pi \in R_p'(A)$). Thus the continuity of $c_p$ follows by taking suprema over all $\pi \in R_p'(A)$ on
the left hand side of the above relation. The conclusion that $p \in S(A)$ if and only if $p(x) = c_p(x)$ for all $x \in A$ is a
simple verification.
(2) Let \( y \in \text{srad}(A) \). So, \( c_p(y) = 0 \) for all \( p \in K(A) \). Let \( p_1 \in \Gamma \) and \( \pi \in R'(\pi p_1) \). Then \( \pi \in R'(A) \). Consequently,

\[
0 \leq |\pi(y)| \leq c_p(y) = 0,
\]

for some \( p \in K(A) \). It follows now, that \( c_p(y) = 0 \) and hence \( y \in \bigcap \{ x \in A : c_p(x) = 0 \} \) for all \( p \in \Gamma \} \), giving \( \text{srad}(A) \subseteq \bigcap \{ x \in A : c_p(x) = 0 \} \) for all \( p \in \Gamma \}. \)

The reverse inclusion is obvious since \( K(A) \subseteq \Gamma \). The last equality is the consequence of the fact that \( S(A) = \{ c_p : p \in \Gamma \} \). This proves (2).

(3) The proof is a simple verification.

(4) Let \( \{ x_i \} \) be a net in \( A \) such that \( x_i \to 0 \) in \( A \). Then for \( p \in K(A) \), \( q_p(\phi(x)) = q_p(x_i + \text{srad}(A)) = c_p(x_i) \leq p(x_i) \to 0 \), giving the desired continuity of \( \phi \). This completes the proof.

Enveloping \( C^* \)-algebras of Banach \( ^* \) algebras have been introduced in [28, §2.7.2, p.48] as follows:

Let \( (A, p) \) be a Banach \( ^* \) algebra. Then \( A \) admits the greatest continuous \( C^* \)-seminorm \( |\cdot|_p \) called the Gelfand-Naimark pseudo-norm on \( A \). Also, as in Proposition 4.1.2(1) and the relation 4.1.2(a),

\[
|x|_p = \sup \{ f(x^* x)^{1/2} : f \in P(A), ||f|| \leq 1 \}
= \sup \{ ||\pi(x)|| : \pi \in R(A) \}, (x \in A);
\]

and hence \( c_p(x) = |x|_p \) for all \( x \in A \). This gives,
\[ \text{srad}(A) = \{ x \in A : |x|_p = 0 \} = \{ x \in A : f(x^*x) = 0 \text{ for all } f \in P(A) \text{ with } ||f|| \leq 1 \}. \]

The completion \( \mathcal{E}(A) \) of \( (A/\text{srad}(A), \pi_p) \) is a \( \text{C}^* \)-algebra, called the enveloping \( \text{C}^* \)-algebra of \( A \). Thus the following proposition holds.

**Proposition 4.1.5.** Let \((A, \pi)\) be a Banach \( \ast \) algebra. Then the enveloping pro-\( \text{C}^* \)-algebra \( \mathcal{E}(A, \pi) \) is the \( \text{C}^* \)-algebra obtained by completing \((A/\text{srad}(A), \pi_p)\) and \( \text{srad}(A) = \{ \pi_n \} \).

We also note the following theorem due to M. Fragoulopoulos [34, Theorem 4.3].

**Theorem 4.1.6.** \( (\mathcal{E}(A), \tau) = \lim_{p \in P} \mathcal{E}(A)_p = \lim_{p \in P} \mathcal{E}(A)_p \).

The following theorem, due to M. Fragoulopoulos [34, Theorem 4.1, p. 69], reduces the study of the bounded representations of a (complete) \( \text{lmc}^* \)-algebra to that of its enveloping pro-\( \text{C}^* \)-algebra.

**Theorem 4.1.7.** Let \( A \) be a complete \( \text{lmc}^* \)-algebra. Given \( \pi \in \text{R}(A) \) (respectively, \( \text{R}'(A) \)), there exists a unique (up to unitarily equivalence) \( \sigma \in \text{R}(\mathcal{E}(A)) \) (respectively, \( \text{R}'(\mathcal{E}(A)) \)) such that \( \pi = \sigma \circ \phi \). Further, \( (\mathcal{E}(A), \tau) \) is the unique pro-\( \text{C}^* \)-
algebra (up to a homeomorphic *isomorphism) satisfying this universal property for continuous representations of A.

Definition 4.1.8. A complete lmc *algebra A is said to have C*-enveloping algebra (or A is said to be an algebra with C*-enveloping algebra) if (\mathcal{E}(A), \tau) is a C*-algebra.

In the representation theory of a complete lmc *algebra A with a bai, the enveloping pro-C*-algebra \mathcal{E}(A) of A has been introduced in [22], [34] and [49], which provides a solution to the universal problem for continuous *representations of A into the C*-algebras of all bounded linear operators on Hilbert spaces (Theorem 4.1.7). These correspond to the construction of the enveloping C*-algebras of Banach *algebras [28, §2.7.2, p.48]. In [34] and [36], a complete lmc *algebra A is called a bQ-algebra if (\mathcal{E}(A), \tau) is a barrelled space, which is also a Q-algebra; however, as we have seen in Proposition 0.1.39 (in this regard we also refer to [39, Corollary 2.2] and [62, Proposition 1.14]), a Q pro-C*-algebra is a C*-algebra and hence (\mathcal{E}(A), \tau) becomes C*-algebra on putting the Q-condition only. Thus the barrelledness becomes redundant. Also, the name bQ-algebra loses its meaning (p for barrelledness and Q for the Q-algebra); as a result, we prefer to call a bQ-algebra of [34], [36] and

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other papers of M. Fragoulopoulou, unambiguously, an algebra having $C^*$-enveloping algebra (or algebra with $C^*$-enveloping algebra).

Topological algebras with $C^*$-enveloping algebras seem to be of importance for a number of reasons. Though non-normed (in general), they are well-behaved. In the literature, the $bQ$-condition on $A$ is assumed in several aspects, like tensor products [36], hermitian $K$-theory [59]; and representation theory [34]. In fact, the representation theory of such algebras is quite similar to that of Banach $^*$ algebras (for instance, [34]). Also, as exhibited in [14] and the latter sections of the present chapter, there are several classes of examples of such algebras arising naturally in function theory, Fourier series, abstract harmonic analysis, complex analysis and nuclear spaces—in particular, sequence spaces.

4.2. Characterizations

The following main theorem corresponds to the fact that a Banach $^*$ algebra admits the greatest continuous $C^*$-semi-norm. It also answers a question posed in [36], to completely characterize the class of $bQ$-algebras. This result is applied
later to obtain more information regarding algebras with $C^*$-enveloping algebras and obtain more characterizations of the same algebras.

**Theorem 4.2.1** [14]. Let $A$ be a complete $lmc^*$ algebra with a directed $m^*$-calibration $P$ and a bai $(e_j)$ as in Definition 4.1.1. Then $A$ has $C^*$-enveloping algebra if and only if $A$ admits greatest continuous $C^*$-seminorm. In this case, if $p_\infty$ denotes the greatest continuous $C^*$-seminorm, then for $x \in A$,

$$p_\infty(x) = \sup \{ c_p(x) : p \in P \}$$

$$= \sup \{ ||\pi(x)|| : \pi \in R(A) \}$$

$$= \sup \{ ||\pi(x)|| : \pi \in R'(A) \};$$

and $(\mathcal{D}(A), \tau)$ is the $C^*$-algebra isometrically isomorphic to $A_{p_\infty}$.

**Proof.** By Proposition 4.1.2(3), $q_p(x + srad(A)) = c_p(x)$ for all $x \in A$ and $p \in P$. Now, suppose that $A$ has $C^*$-enveloping algebra. Thus $(\mathcal{D}(A), \tau)$ is a $C^*$-algebra and the topology $\tau$ of $\mathcal{D}(A)$ is determined by some $C^*$-norm $||\cdot||$. By [62, p.185], for $z \in D(A)$, $\sup \{ q_p(z) : p \in P \} < \infty$ and $||z|| = \sup \{ q_p(z) : p \in P \}$. Thus $p_\infty(x) = ||x + srad(A)|| = \sup \{ c_p(x) : p \in P \}$, $(x \in A)$, defines a $C^*$-seminorm $p_\infty$ on $A$; and there exist a constant $K > 0$ and $p_1 \in P$ such that for all $x \in A$, $p_\infty(x) = ||x + srad(A)|| \leq q_{p_1}(x + srad(A)) \leq Kc_{p_1}(x) \leq Kp_1(x)$, using [34, p.69]. This shows that $p_\infty$ is continuous on $A$. Now, let
p be any continuous $C^*$-seminorm on $A$, so that, for some constant $M > 0$ and for some $p_1 \in P$, $p(x) \leq M p_1(x)$ for all $x$ in $A$. Then $R_p(A) \subseteq R_{p_1}(A)$ and for all $x \in A$, $c_p(x) \leq c_{p_1}(x)$. Identifying $R_p(A)$ and $R(A_p)$ canonically by [34, Proposition 3.5] and using the fact that $A_p$ is a $C^*$-algebra, it follows that for each $x \in A$,

$$p(x) = ||x + N_p||_p$$

$$= \sup \{||\pi(x + N_p)|| : \pi \in R(A_p)\}$$

$$= \sup \{||\pi(x)|| : \pi \in R_p(A)\}$$

$$= c_p(x)$$

$$\leq c_{p_1}(x)$$

$$\leq p_\infty(x).$$

Thus $p_\infty$ is the greatest continuous $C^*$-seminorm on $A$.

Conversely, suppose that $A$ admits greatest continuous $C^*$-seminorm denoted by $p_\infty$. By the continuity of $p_\infty$, there exist $p_1 \in P$ and a constant $K > 0$ such that $p_\infty(x) \leq K p_1(x)$ for all $x \in A$. Thus $R_{p_\infty}(A) \subseteq R_{p_1}(A)$, giving $p_\infty(x) = c_{p_\infty}(x) \leq c_{p_1}(x)$ for all $x \in A$. Thus $p_\infty(x) = c_{p_\infty}(x)$ for all $x \in A$. Now, it follows that for $x \in A$,

$$\sup \{c_p(x) : p \in P\} \leq p_\infty(x) = c_{p_1}(x) \leq \sup \{c_p(x) : p \in P\}$$

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and hence $p_\infty(x) = \sup \{c_p(x) : p \in P\}$. This gives $\text{srad}(A) = N$ and for any $x \in A$,

$$||x + \text{srad}(A)||_{p_\infty} = p_\infty(x) = \sup \{c_p(x) : p \in P\} = \sup \{q_p(x + \text{srad}(A)) : p \in P\} = c_{p_\infty}(x) = q_{p_\infty}(x + \text{srad}(A))$$

It follows now, that the topology $\tau$ on $A$ is determined by $||\cdot||_{p_\infty}$ and hence $(A, \tau)$ is a $C^*$-algebra. This completes the proof.

A complete lmc $^*$ algebra having $C^*$-enveloping algebra behaves, in certain respects, like a Q-algebra (even though, the class of these algebras is quite larger than Q lmc $^*$ algebras). Before we compare these classes, we note certain known facts about Q lmc algebras. The following proposition is from [61, Proposition 13.5].

Proposition 4.2.2. An lmc algebra $A$ is a Q-algebra if and only if the set $B = \{x \in A : r(x) \leq 1\}$ is a neighbourhood of 0 in $A$. 

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Corollary 4.2.3. Let $A$ be an lmc algebra and $P$ be a directed $m$-calibration on $A$. Then $A$ is a $Q$-algebra if and only if there exist $p \in P$ and a constant $K > 0$ such that

$$4.2.3 \chi \quad r(x) \leq Kp(x) \text{ for all } x \in A.$$ 

Proof. From Proposition 4.2.2, $A$ is a $Q$-algebra if and only if $B_r$ is a neighbourhood of 0 in $A$. Thus $A$ is a $Q$-algebra if and only if there exist $p \in P$ and a constant $K > 0$ such that $KU \subseteq B_r$, which is equivalent to the inequality, $r(x) \leq Kp(x)$ for all $x \in A$. This completes the proof.

The following corollary is a consequence of [61, Lemma E.3].

Corollary 4.2.4. Let $A$ be a complete $Q$ lmc algebra. Then each element of $A$ has compact spectrum. Consequently, $A$ is sb.

Definition 4.2.5. A complete lmc algebra is said to be strongly spectrally bounded (ssb) if there exists $Q \subseteq \mathcal{Z}(A)$ satisfying $\sup \{q(x) : q \in Q\} < \infty$ for all $x \in A$.

In general, a $Q$-algebra is ssb and an ssb algebra is sb.
The following is a corollary to Theorem 4.2.1 and is noted in [14, Corollary 2.2].

Corollary 4.2.6. A complete $Q$-algebra has $C^\ast$-enveloping algebra.

Proof. Let $A$ be a $Q$-algebra and $P$ be as in Definition 4.1.1. Now, by Corollary 4.2.3, there exist $p_1 \in P$ and a constant $K > 0$ such that $\tau(x) \leq Kp_1(x)$ for all $x \in A$. Let $q \in S(A)$. Then by the continuity of $q$, there exist $p \in P$ and a constant $M > 0$ such that $q(x) \leq Mp(x)$ for all $x \in A$. Now, for $h \in A^h$ and for all $n \in \mathbb{N}$, $q(h) = q(h^n)^{1/2^n} \leq M^{1/2^n}p(h^n)$. Hence by the spectral radius formula [61, p. 22],

$$q(h) = q(h^n)^{1/2^n} \leq \limsup_{n \to \infty} M^{1/2^n}p(h^n)^{1/2^n} \leq \sup_{p \in P} \limsup_{n \to \infty} p(h^n)^{1/2^n} = r(h) \leq Kp_1(h).$$

Hence, for any $x \in A$, $q(x) = q(x^n)^{1/2} \leq Kp_1(x^n)^{1/2} \leq Kp_1(x)$. Thus $\sup \{q(x) : q \in S(A)\} \leq Kp_1(x)$, establishing that $p_\omega(x) = \sup \{q(x) : q \in S(A)\}$ is the greatest continuous $C^\ast$-seminorm on $A$. It follows now, by Theorem 4.2.1, that $A$ has $C^\ast$-enveloping algebra.
Definition 4.2.7 [14]. Let $A$ be an $lmc$ $^*$-algebra. Then for $x \in A$,

$$r^h(x) = \sup \{ r(\pi(x)) : \pi \in R'(A) \}$$

is called the hermitian spectral radius of $x$ in $A$, where, for a continuous (topologically irreducible) $^*$-representation $\pi : A \to B(H_\pi)$, $r(\pi(x))$ is the spectral radius of the operator $\pi(x)$ in the $C^*$-algebra $B(H_\pi)$.

Now, we obtain a characterization of a complete $lmc$ $^*$-algebra with $C^*$-enveloping algebra in terms of the hermitian spectral radius, parallel to that of a $Q$ $lmc$ $^*$-algebra involving the spectral radius. This behavior of a complete $lmc$ $^*$-algebra with $C^*$-enveloping algebra is quite similar to that of a $Q$-algebra.

Theorem 4.2.8 [14]. $A$ has $C^*$-enveloping algebra if and only if there exist $p \in K(A)$ and a constant $K > 0$ such that

$$r^h(x) \leq Kp(x)$$

for all $x \in A$.

Proof. Suppose that $A$ has $C^*$-enveloping algebra. By the continuity of the greatest continuous $C^*$-seminorm $p_\infty$ on $A$, there exist $p \in K(A)$ and a constant $K > 0$ such that $p_\infty(x) \leq Kp(x)$ for all $x \in A$. It follows now, from the definition of $r^h(\cdot)$ that $r^h(x) \leq p_\infty(x) \leq Kp(x)$ for all $x \in A$. 

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Conversely, suppose there exist \( p \in \mathcal{K}(A) \) and a constant \( K > 0 \) such that \( r^h(x) \leq Kp(x) \) for all \( x \in A \). Let \( q \in \mathcal{S}(A) \) and \( \sigma \in R'_q(A) \). Then for \( x \in A \),

\[
||\sigma(x)||^2 = ||\sigma(x^* x)|| \leq r(\sigma(x^* x)) \leq r^h(x^* x) \leq Kp(x^* x) \leq Kp(x)^2.
\]

Thus \( \sigma \in R'_p(A) \), giving \( R'_q(A) \subset R'_p(A) \). Hence, for each \( x \in A \),

\[
q(x) = c_q(x) \leq c_p(x). \quad \text{Since} \quad q \in \mathcal{S}(A) \quad \text{is arbitrary,} \quad c_p \quad \text{is the greatest continuous \( C^* \)-seminorm on} \ A. \quad \text{Hence, by Theorem 4.2.1,} \ A \quad \text{has \( C^* \)-enveloping algebra. This completes the proof.}
\]

**Corollary 4.2.9.** A has \( C^* \)-enveloping algebra if and only if there exists \( p \in \mathcal{P} \) such that \( \mathcal{R}(A) = R_p(A) \) (equivalently, if and only if there exists \( p \in \mathcal{P} \) such that \( R'(A) = R'_p(A) \)).

**Proof.** Suppose that \( A \) has \( C^* \)-enveloping algebra. So, by the continuity of \( p_\omega \) on \( A \), there exist \( p \in \mathcal{P} \) and a constant \( K > 0 \) such that \( p_\omega(x) \leq Kp(x) \) for all \( x \in A \). Let \( \pi \in \mathcal{R}(A) \). Then \( \pi \in R_{p_\omega}(A) \) for some \( p_\omega \in \mathcal{P} \). Hence \( ||\pi(x)|| \leq c_{p_\omega}(x) \leq p_\omega(x) \leq Kp(x) \) for all \( x \in A \), giving \( \pi \in R_p(A) \). Thus \( \mathcal{R}(A) \subset R_p(A) \).

The reverse inclusions are obvious in the light of [34, p.67]. Similarly it can be proved that \( R'(A) = R'_p(A) \).

Conversely, suppose that \( \mathcal{R}(A) = R_p(A) \) (or \( R'(A) = R'_p(A) \)) for some \( p \in \mathcal{P} \). Then
\[ p_\infty(x) = \sup \{ ||\pi(x)|| : \pi \in \mathbb{R}(A) \} \]
\[ = \sup \{ ||\pi(x)|| : \pi \in \mathbb{R}_p(A) \} \]
\[ = c_p(x), \]

which makes \( c \) the greatest continuous \( C^\# \)-seminorm on \( A \). So, by Theorem 4.2.1, \( A \) has \( C^\# \)-enveloping algebra. This completes the proof.

In general, the converse of the Corollary 4.2.6 does not hold, as we shall see in the next section, even if the underlying algebra \( A \) is assumed to be a unital commutative Fréchet \( lmc^\# \) algebra. However, in the presence of the hermiticity we have the converse as the following proposition shows.

Proposition 4.2.10 [14]. A hermitian complete \( lmc^\# \) algebra \( A \) is a \( \mathbb{Q} \)-algebra if and only if \( A \) has \( C^\# \)-enveloping algebra.

Proof. One way implication is just Corollary 4.2.6 (even in the absence of the hermiticity). So, to prove the remaining implication, let us assume that \( A \) has \( C^\# \)-enveloping algebra. By [61, Theorem 5.2], the hermiticity of \( A \) implies that, for each \( q \in K(A) \), the Banach \( \# \) algebra \( (A_q, ||\cdot||_q) \) is hermitian. Hence, by [20, Lemma 41.2], for each \( z \in A_q \),

\[ r_{A_q}(z) \leq r_{A_q}(z^\# z)^{1/2} = ||z||_q, \]
where \( |\cdot|_q \) denotes the Gelfand-Naimark pseudonorm on \( A_q \). Let us define \( m_q(z) = |z|_q \), \((z \in A)\). Then \( m_q \) is a continuous \( C^\ast \)-seminorm on \( A \). By Theorem 4.2.1, \( A \) admits the greatest \( C^\ast \)-seminorm \( p_\infty \) on \( A \). Hence, by [51, Corollary 5.3], for each \( x \in A \), the spectral radius in \( A \),

\[
r(x) = \sup \{ r_{A_q}(x) : q \in K(A) \} = \sup \{ m_q(x) : q \in K(A) \} \leq p_\infty(x).
\]

Now, the continuity of \( p_\infty \) on \( A \) applied to the above inequalities gives, \( r(x) \leq p_\infty(x) \leq Kp(x) \), for some \( p \in K(A) \), for all \( x \in A \) and for some constant \( K \) independent of \( x \). By Corollary 4.2.3, \( A \) is a \( \mathcal{Q} \)-algebra. This completes the proof.

Let \( B^0(A) \) and \( B^0(A) \) denote the sets of all extreme points of \( P^0(A) \) and \( P^0(A) \) respectively, where \( p \) is any continuous submultiplicative \( \ast \)-seminorm on \( A \). Then

\[
B^0(A) = \bigcup_{p \in K(A)} B^0(p) = \bigcup_{p \in P} B^0(p) \quad \text{and} \quad P^0(A) = \bigcup_{p \in K(A)} P^0(p) = \bigcup_{p \in P} P^0(p).
\]

Let us recall that the bijective correspondence between \( A'(p) \) and \( A' \), \( f \in A'(p) \leftrightarrow f_p \in A' \), as in Theorem 0.1.31, identifies \( P^0(p) \) with \( P^0(A_p) \) and \( B^0(p) \) with \( B^0(A_p) \).

Our first step now, is to characterize complete \( \mathrm{Im} \ast \)-algebras with \( C^\ast \)-enveloping algebras in terms of the equicontinuity of \( P^0(A) \) and \( B^0(A) \). In this sequel we shall need the following lemma.
Lemma 4.2.11 [14]. Let $A$ be a complete $\text{lmc}^*$ algebra with a directed $P \in \mathbb{P}(A)$ and a bai as in Definition 4.1.1. For $f \in P(A)$, the following are equivalent.

(1) $f \in P^0(A)$.

(2) $|f(x)|^2 \leq f(x^*x)$ for all $x \in A$.

Proof. (1) $\Rightarrow$ (2). Suppose $f \in P^0(A)$. So, as noted above, there exists $p \in K(A)$ such that $f \in P^0_P(A)$. Hence, $|f(x)| \leq p(x)$ for all $x \in A$. Now, by the continuity of $f$ and the Cauchy-Schwarz inequality, for any $x \in A$,

$$|f(x)|^2 = \lim_{j} |f(e_j x)|^2 \leq \lim_{j} f(e_j e_j)e^*e f(x^*x) \leq (\lim_{j} p(e_j e_j)e^*e) f(x^*x) \leq (\lim_{j} p(e_j))^2 f(x^*x) \leq p(x),$$

since $p(e_j) \leq 1$ for all $j$. Thus (2) follows.

(2) $\Rightarrow$ (1). Suppose (2) holds. Since $f$ is continuous and since $K(A)$ is directed, there exist $p \in K(A)$ and a constant $K > 0$ such that $|f(x)| \leq Kp(x)$ for all $x \in A$. If $K \leq 1$, then $|f(x)| \leq p(x)$. If $K > 1$, then $|f(x)|^2 \leq f(x^*x) \leq Kp(x^*x) \leq Kp(x)^2$, giving $|f(x)| \leq K^{1/2}p(x)$. Now, by iterations, it
follows that $|f(x)| \leq K^{2^n} p(x)$ for all $x \in A$ and for all $n \in \mathbb{N}$. Consequently, $|f(x)| \leq p(x)$ for all $x \in A$, giving $f \in P^0(A) \subseteq P^0(A)$. So, (1) follows. This completes the proof.

The following corollary is immediate in view of [34, Lemma 4.1] and Theorem 4.2.1.

Corollary 4.2.12. Suppose $A$ has $C^*$-enveloping algebra. Then for each $x \in A$,

$$p_\infty(x) = \sup \{ f(x^n x)^{1/2} : f \in P^0(A) \}$$

$$= \sup \{ f(x^n x)^{1/2} : f \in B^0(A) \}$$

$$= \sup \{ p(x) : p \in S(A) \}.$$

The following corollary characterizes a complete $\text{lmc}^*$ algebra having $C^*$-enveloping algebra in terms of the equicontinuity of $P^0(A)$ and $B^0(A)$.

Corollary 4.2.13. For a complete $\text{lmc}^*$ algebra $A$, the following are equivalent.

(1) $A$ has $C^*$-enveloping algebra.

(2) $P^0(A)$ is equicontinuous.

(3) $B^0(A)$ is equicontinuous.
Proof. (1) $\Rightarrow$ (2). Suppose (1) holds, so that, by Theorem 4.2.1, the topology of $A/srad(A)$ is determined by the $C^*$-norm $\|x + srad(A)\| = p_{\infty}(x)$, $(x \in A)$. Also, by the continuity of the $C^*$-seminorm $p_{\infty}$, there exist $p \in K(A)$ and a constant $K > 0$ such that $p_{\infty}(x) \leq Kp(x)$ for all $x \in A$. Since the quotient topology $\tau_q$ on $A/srad(A)$, induced by the topology of $A$, is finer than the $C^*$-topology $\tau$ on $A/srad(A)$, it follows that for the given $\varepsilon$, there exists a constant $M > 0$ such that $p_{\infty}(e_j) = \|e_j + srad(A)\| = M$ for all $j$. Let $f \in P^0(A)$. By Corollary 4.2.12 and the Cauchy-Schwarz inequality, for $x \in A$,

$$|f(x)| = \lim_{j} |f(e_j x)|$$

$$\leq \left[ \lim_{j} \sup f(e_j^* e_j) \right]^{1/2} f(x^* x)^{1/2}$$

$$\leq \lim_{j} \sup p_{\infty}(e_j^* e_j)^{1/2} p_{\infty}(x)$$

$$\leq M p_{\infty}(x)$$

$$\leq MKp(x).$$

Since the choice of $p$ does not depend upon $f \in P^0(A)$, the above inequality establishes (2).

(2) $\Rightarrow$ (3). The proof is obvious since $B^0(A) \subset P^0(A)$.

(3) $\Rightarrow$ (1). Suppose (3) holds. So, there exist $p \in K(A)$ and a constant $M > 0$ such that $|f(x)| \leq M p(x)$ for all $x \in A$ and $f \in B^0(A)$. Then $q(x) = \sup \{f(x^* x)^{1/2} : f \in B^0(A)\} \leq M p(x)$,
(x ∈ A), defines the greatest continuous $C^*$-seminorm on A, giving (1). This completes the proof.

Let us recall [60, Lemma 2.26], that a commutative complete lmc algebra $A$ is a Q-algebra if and only if the Gelfand space $\mathcal{M}(A)$ is equicontinuous. Analogous to this, we have the following corollary. Again, this points out that the complete lmc $^*$ algebra having $C^*$-enveloping algebras are similar to Q-algebras.

Corollary 4.2.14. Suppose $A$ is a commutative complete lmc $^*$ algebra. Then the following are equivalent.

1) $A$ has $C^*$-enveloping algebra.

2) The hermitian Gelfand space $\mathcal{M}^h(A)$ is equicontinuous.

Further, in this case,

$$p_\infty(x) = \sup \{|f(x)| : f \in \mathcal{M}^h(A)\} = r^h(x)$$

for all $x \in A$.

4.3. Two counter examples

Theorem 4.3.1. (1) There exists a unital Frechet lmc $^*$ algebra $A$ with $C^*$-enveloping algebra such that $A$ is not sb; and hence $A$ is not a Q-algebra under any topology.

(2) There exist a commutative Frechet lmc $^*$ algebra $B$ and a closed $^*$ subalgebra $D$ of $B$ such $B$ has $C^*$-enveloping algebra but $D$ fails to have $C^*$-enveloping algebra.
(3) There exists a commutative non-$Q$ lmc $^{*}$ algebra $B$ having $C^{*}$-enveloping algebra such that $B$ is ssb.

To see this, we have a number of examples; however, before that we shall need the following lemma.

Lemma 4.3.2. Let $A$ be a commutative unital lmc $^{*}$ algebra. Let $n \in R'(A)$. Then $n$ is one dimensional and there exists $\phi \in M_{n}(A)$ such that $n(x) = \phi(x)1$ for all $x \in A$.

Proof. Since $A$ is commutative, $\pi(A)$ is commutative. Hence $\pi(A) \subseteq \pi(A)'$, the commutant of $\pi(A)$ in $B(H_{n})$. Thus we have $\pi(A) \subseteq \pi(A)' \approx C$; and $\pi(A)$ is an algebra. This gives $\pi(A) \approx C$.

Thus $\pi(A)$ is one dimensional and $\pi(A)$ consists of the scalar multiples of identity. But then all closed subspaces of $H_{n}$ are invariant under $\pi(A)$. So, $H_{n}$ must be one dimensional.

Thus summarizing all the facts, $n$ is one dimensional and $H_{n}$ are invariant under $\pi(A)$. So, $\pi(x) = \phi(x)1$ for some $\phi \in M_{n}(A)$. This completes the proof.

Example 4.3.3 (14). Let $U$ be the open unit strip $\{z \in C : -1 < \text{Re} z < 1\}$ in $C$ and $C(\overline{U})$ be the algebra of all complex-valued continuous functions on the closure $\overline{U}$ of $U$, with the pointwise operations and the compact open topology. Let us consider the strip algebra $A = \{f \in C(\overline{U}) : f$ is analytic on $U\}$.  

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We define, for \( f \in A \), \( f^*(z) = f(\bar{z}), \) \( (z \in \overline{U}) \). Then \( ^* \) is an involution on \( A \) and \( A^* \), with this involution and relative topology, is a Frechet \( lmc \) algebra. The topology of \( A \) is determined by the \( m \)-calibration consisting of the seminorms \( p_n, \) \( (n \in \mathbb{N}) \), defined by \( p_n(f) = \sup \{|f(z)| : z \in K_n\} \), where for each \( n \in \mathbb{N}, \) \( K_n = \{z \in \overline{U} : n \leq \text{Im}z \leq n + 1\} \). It is easily seen that the Gelfand space consists of the point evaluations \( \phi_z, \) \( (z \in \overline{U}) \), defined as \( \phi_z(f) = f(z), \) \( (f \in A) \). Also, the hermitian Gelfand space \( M^h(A) = \{\phi_z : z = x + i0, -1 \leq x \leq 1\} \). In view of Lemma 4.3.2 and Theorem 4.2.1,

\[
p_\infty(f) = \sup \{||p(z)| : p \in R'(A)\}
\]

\[
= \sup \{|f(z)| : z = x + i0, -1 \leq x \leq 1\}
\]

< \infty, \( (f \in A), \)

defines a continuous \( C^* \)-seminorm on \( A \). Consequently, \( \text{srad}(A) = \{f \in A : f([-1,1]) \neq \{0\}\} \) and \( \mathfrak{B}(A) \) is the sup norm \( C^* \)-algebra \( C[-1,1] \). Now, by [61, Corollary 5.6], \( \text{sp}(f) = \{f(z) : z \in \overline{U}\} \) for each \( f \in A \) and so, it is obvious that \( A \) is not sb. In view of Corollary 4.2.4, \( A \) fails to be a \( Q \)-algebra under any \( lmc \) topology on \( A \). This establishes Theorem 4.3.1(1).

In what follows, we slightly modify [20, Example 16, p.202].

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Example 4.3.4 [14]. Let $C$ be a complete unital $\text{imc }^*$ algebra. Suppose $B = C \otimes C$ with the product topology defined by the seminorms $p((x, y)) = \max \{p(x), p(y)\}$, $((x, y) \in B)$, where $p$ runs over all unital continuous submultiplicative seminorms on $C$. The operations on $B$ are the pointwise operations and involution on $B$ is defined by $(x, y)^* = (y^*, x^*)$, $((x, y) \in B)$. All this makes $B$ into a complete $\text{imc }^*$ algebra with unit, on which, for any positive linear functional $f$, $f(z^* z) = 0$ for all $z \in B$. Hence the only positive linear functional on $B$ is the zero functional, giving $\mathbb{R}(B) = \mathbb{R}^*(B) = \{0\}$, $B = \text{srad}(B)$ and $\mathfrak{z}(B) = \{0\}$, the trivial $C^*$-algebra.

(i) To establish Theorem 4.3.1(2), we take $D = \{(x, x) : x \in C\}$, a closed $^*$subalgebra of $B$. It is obvious that $D$ is homeomorphically $^*$isomorphic to $C$. In particular, taking $C$ to be Frechet $\text{imc }^*$ algebra $C(\mathbb{R})$ with compact open topology, pointwise operations and complex conjugation as the involution, the resulting $^*$algebra $D$ does not have $C^*$-enveloping algebra.

(ii) In order to establish Theorem 4.3.1(3), we note that $C$ is ssb (respectively, a $Q$-algebra) if and only if so is $B$. Now, take $C$ to be the $^*$algebra $C[0,1]$ of all continuous functions on $[0,1]$ with the topology of uniform convergence on all countable compact subsets of $[0,1]$. Then the algebra $B$ is not a $Q$-algebra but is ssb. Also, $B$ has $C^*$-enveloping algebra.
4.4. Some spectral functions

For a complete $^*$-algebra $A$, $f \in \mathcal{P}(A)$ and $y \in A$, we define the positive linear functional $f_y : A \rightarrow \mathbb{C}$ by $f_y(x) = f(y^*xy)$, $(x \in A)$. Also, for $f \in M^0(A)$, the $C^*$-seminorm $p_f$ on $A$ is defined by $p_f(x) = ||\pi_f(x)||$, $(x \in A)$, where $\pi_f$ is the GNS-representation associated with $f$ (Definition 0.1.33).

Definitions 4.4.1. Let $A$ be a complete $^*$-algebra. We define the following spectral functions for those $x$ for which the respective right hand side is finite.

$l(x) = \sup \{p(x) : p \in \mathcal{P}(A)\}$,

$u(x) = \sup \{p_f(x) : f \in M^0(A)\}$,

$m(x) = \sup \{f(x^*x)^{1/2} : f \in M^0(A)\}$ and

$s(x) = r(x^*x)^{1/2}$.

Definition 4.4.2. A $^*$-algebra $A$ is called $^*$spectrally bounded ($^*$sb) if $sp(x^*x)$ is bounded for all $x \in A$.

The following proposition is a generalization of Shirali-Ford Theorem. It is previously recorded in the literature (for instance, [42]).
Proposition 4.4.3. A complete lmc *algebra A is hermitian if and only if A is symmetric.

Lemma 4.4.4 [14]. Suppose A is a complete lmc *algebra.

(1) Let \( a \in \mathcal{A}^h \) with \( r(a) < 1 \). Then there exists \( x \in \mathcal{A}^h \) with \( r(x) < 1 \) such that \( 2x - x^2 = a \).

(2) Let \( f \in \mathcal{A}(A) \) and \( b \in A \). Then

\[
\begin{align*}
(\text{i}) \ |f^n(b)h| & \leq r(h)f^n(b)b) \quad \text{for all } h \in \mathcal{A}^h. \\
(\text{ii}) \ |f^n(a)| & \leq r^n(a)f^n(b)b) \quad \text{for all } a \in A.
\end{align*}
\]

(3) Given \( p \in \mathcal{A}(A) \) and \( b \in A \), there exists \( f \in \mathcal{A}(A) \) such that \( |f(x)| \leq p(x) \) for all \( x \in A \) and \( f^n(b)b) = p(b^n b) \).

Proof. (1) Let \( a \in \mathcal{A}^h \) with \( r(a) < 1 \). So, \( \text{sp}_\mathcal{A}(a) \subseteq \{ z \in \mathbb{C} : |z| \leq r(a) \} \). Hence as in the proof of [45, Theorem 4.1], there exists an \( m^\mathcal{A} \)-calibration \( Q \) such that \( \text{sp}_\mathcal{A}(a) \subseteq V(\mathcal{A}, Q, a) \subseteq \{ z \in \mathbb{C} : |z| \leq r(a) \} \). Thus \( a \in B_\mathcal{A} = \{ x \in A : v(\mathcal{A}, Q, a) < \infty \} \).

Now, by [45, Theorem 3.1], \( V(B_\mathcal{A}, \cdot \cdot |_{\mathcal{E}_\mathcal{A}}, a) = V(\mathcal{A}, Q, a) \subseteq \{ z \in \mathbb{C} : |z| \leq r(a) \} \). Also, \( \text{sp}_\mathcal{E}(a) \subseteq V(\mathcal{E}, \cdot \cdot |_{\mathcal{E}_\mathcal{A}}, a) \) is always true.

Hence \( r_\mathcal{E}(a) < 1 \). Now, by Ford's square root lemma [20, Proposition 12.11] for Banach * algebras, there exists \( x = x^* \) in \( B_\mathcal{A} \) with \( r_\mathcal{E}(x) < 1 \) such that \( 2x - x^2 = a \). Since \( r(x) = r_\mathcal{E}(x) \leq r_\mathcal{E}(a) \), (1) follows.

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(2) Let \( f \in \mathcal{B}(A) \), \( b \in A \) and let \( h \in A^h \). If \( r(h) = \infty \), then

(i) is obvious. Let us assume, first, that \( r(h) < 1 \). In this case, by (1) above, there exist \( x, y \in A^h \) with \( r(x) < 1 \) and \( r(y) < 1 \) such that 

\[
h = 2x - x^2 \quad \text{and} \quad -h = 2y - y^2.
\]

Let \( u = b - xb \) and \( v = b - yb \). Then \( u^*u = b^*(1 - x)^2b = b^*(1 - h)b \) and \( v^*v = b^*(1 - y)^2b = b^*(1 + h)b \). This gives 

\[
f(b^*(1 - h)b) = f(u^*u) \geq 0
\]

and 

\[
f(b^*(1 + h)b) = f(v^*v) \geq 0.
\]

Consequently, 

\[
f(b^*b) \geq f(b^*h).
\]

Now, assume that \( 1 \leq r(h) < \infty \). Let \( \varepsilon > 0 \) and 

\[
k = \frac{h}{r(h) + \varepsilon}.
\]

Then 

\[
r(k) = \frac{r(h)}{r(h) + \varepsilon} < 1.
\]

Hence 

\[
|f_b(b_{r(h) + \varepsilon})| \leq f(b^*b). \]

This gives 

\[
|f_b(h)| \leq f(b^*b)(r(h) + \varepsilon).
\]

(i) follows now, by taking \( \varepsilon \to 0 \).

Now, for \( a \in A \), by the Cauchy-Schwarz inequality and above (i),

\[
|f_b(a)|^2 = |f(b^*(ab))|^2
\]

\[
\leq f(b^*b)f((ab)^*ab)
\]

\[
= f(b^*b)f(b^*(a^*a)b)
\]

\[
= f(b^*b)f_b(a^*a)
\]

\[
\leq f(b^*b)f(b^*b)r(a^*a)
\]

\[
= f(b^*b)^2s(a)^*.
\]
Hence, \( |f_b(a)| \leq f(b^*b)s(a) \), which establishes (ii). So (2) follows.

(3) Since \( A \) is a \( C\)-algebra, by \([20, \text{Lemma 39.11}]\), there exists a positive linear functional \( f_p : A \to \mathbb{C} \) such that
\[
|f_p(x)| \leq ||x|| \quad \text{and} \quad f_p(b^*b) = ||b^*b||. 
\]
It follows now, that \( f = f_p \circ \chi_p \) is a positive linear functional on \( A \) with
\[
|f(x)| \leq p(x) \quad \text{for all} \quad x \in A \quad \text{and} \quad f(b^*b) = p(b^*b). 
\]
Also, by \([20, \text{Lemma 37.11(iii)}]\), for \( x, y \in A \),
\[
f(y^*x^*xy) = f_p(y^*x^*x^*y) \leq f_p(y^*x^*x^*y) = f(y^*x^*x^*y) \leq f(y^*x^*x^*y) = f(y^*x^*y). 
\]
Hence, for all \( x \in A \),
\[
\frac{f(y^*x^*xy)}{f(y^*x^*y)} \leq r_{A_p}(x^*x) 
\]
showing that \( f \in \mathcal{M}(A) \). Also, by \([20, \text{Lemma 39.15}]\), \( |f(x)|^2 = |f_p(x)|^2 \leq f_p(x^*x) = f(x^*x) \) for all \( x \in A \). Thus \( f \in \mathcal{M}^0(A) \).

Hence (3) follows. This completes the proof.

**Proposition 4.4.6 [14].** Let \( A \) be a complete \( lmc \) \( * \)-algebra.

(1) If \( A \) is a \( F \)rechet \( lmc \) \( * \)-algebra, then \( \mathcal{J}(A) = S(A) \).

(2) If \( A \) is \( * \)sb, then for each \( p \in \mathcal{J}(A) \), \( p(x) \leq s(x) \), \((x \in A)\).
(3) If $A$ is $\ast$ sb and hermitian, then

(i) $r(x) \leq s(x)$ for all $x \in A$; and

(ii) $x \mapsto s(x)$ defines a $C^*$-seminorm on $A$.

Proof. (1) The proof is a consequence of the automatic continuity of the $\ast$ homomorphism from a Fréchet lmc $\ast$ algebra with a bai to a $C^*$-algebra.

(2) Let $p \in \mathcal{Y}(A)$. Then for $x \in A$,

$$p(x) = ||x_p||_p = \frac{r_A(x \ast x)^{1/2}}{s(x)} \leq \frac{r(x \ast x)^{1/2}}{s(x)} = s(x).$$

(3) Let us note that $A_p$ is a hermitian Banach $\ast$ algebra for each $p \in K(A)$. Hence, by [20, Lemma 41.2], for $x \in A$,

$$r_p(x_p) \leq \frac{r(x \ast x)^{1/2}}{s(x)} \leq \frac{r(x \ast x)^{1/2}}{s(x)} = s(x).$$

Now, taking supremum over all $p \in K(A)$, on the left hand side of the above inequality, we have $r(x) \leq s(x)$ for all $x \in A$. This establishes (i).

Also, for $x \in A$, $s(x \ast x) = r(x \ast x \ast x)^{1/2} = r((x \ast x)^2)^{1/2} = r(x \ast x) = s(x)^2$, since $x \ast x$ is selfadjoint and $A$ is hermitian. Thus $s$ is a $C^*$-seminorm on $A$. This completes the proof of Proposition 4.4.5.

Now, we come to one of the main results of this section.
Theorem 4.4.6 [14]. A complete * sb lmc * algebra $A$ admits greatest $C^*$-seminorm (not necessarily continuous) say $|\cdot|$. In this case, $|x| = l(x) = u(x) = m(x)$ for all $x \in A$.

Proof. Given $p \in \mathcal{P}(A)$ and $b \in A$, there exists $f \in \mathcal{A}^0(A)$ as in Lemma 4.4.4(3). But then,

$$p_f(b)^2 = p_f(b^* b) = \|p_f(b^* b)\| = \sup \{f(x^* b^* bx) : f(x^* x) = 1\} \geq f(b^* b) = p(b)^2,$$

showing that $l(b) \leq u(b)$. Thus $u(x) = l(x)$ for all $x \in A$. Similarly, $p(b)^2 = f(b^* b) \leq m(b)^2$, so, $l(x) \leq m(x)$ for all $x \in A$. Now, let $f \in \mathcal{A}^0(A)$ and $\xi_f \in H_f$ be a topologically cyclic vector for $\pi_f$ such that $\langle \xi_f, \xi_f \rangle = 1$. So, $f(x) = \langle \pi_f(x)\xi_f, \xi_f \rangle$ for all $x \in A$. Also, $f(x^* x) = \|\pi_f(x)\xi_f\|^2 \leq p_f(x)^2 \leq u(x)$, which implies $m(x) \leq u(x)$. Hence, $l(x) = m(x) = u(x) = |x|$ (say) for all $x \in A$, which is the greatest $C^*$-seminorm if it exists. Now, since $A$ is * sb, we get $|x| \leq s(x)$ for all $x$. Indeed, for $f \in \mathcal{A}^0(A)$ and $x \in A$, the boundedness of $\pi_f(x)$ implies,
\[ p_f(x) = \|\pi_f(x)\| \]

\[ = \sup \left\{ \frac{\|\pi_f(x)\eta\|}{\|\eta\|} : \eta \neq 0 \text{ in } H_f \right\} \]

\[ = \sup \left\{ \frac{\|\pi_f(x)(y + N_f)\|}{\|y + N_f\|} : y \in A, f(y^*y) \neq 0 \right\} \]

\[ = \sup \left\{ \frac{f(y^*x^*xy)^{1/2}}{f(y^*y)^{1/2}} : y \in A, f(y^*y) \neq 0 \right\} \]

\[ \leq r(x^*x)^{1/2} \]

\[ = s(x) \]

by Lemma 4.4.4(2). Since \( s(x) < \infty \) for all \( x \in A \), the proof follows.

The following theorem is known [20, Theorem 41.11].

**Theorem 4.4.7.** For a Banach\* algebra \( A \), the following are equivalent.

1. \( A \) is hermitian.
2. \( x \mapsto s(x) \) is the Gelfand Naimark pseudonorm on \( A \).
3. \( s(\cdot) \) is a \( C^* \)-seminorm on \( A \).

Further, each of the above implies \( r(x) \leq s(x) \) for all \( x \in A \).
Theorem 4.4.8 [14]. Let $A$ be a complete $^*$sb $^*$imc $^*$algebra. Then the following are equivalent.

1. $A$ is hermitian.
2. $s(x) = |x|$ for all $x \in A$, where $|\cdot|$ is defined as in Proposition 4.4.6.
3. $x \mapsto s(x)$ is a $C^*$-seminorm on $A$.

Further, each of the above implies that $A$ is $sb$ and $r(x) \leq s(x)$ for all $x \in A$.

Proof. (1) $\Rightarrow$ (3). Suppose (1) holds. Then by Proposition 4.4.5(3), $x \mapsto s(x)$ is a $C^*$-seminorm on $A$. So, (3) follows.

(3) $\Rightarrow$ (2). Since $A$ is a $^*$sb $^*$algebra, $|\cdot|$ is the greatest $C^*$-seminorm on $A$ and $|x| \leq s(x)$ for all $x \in A$. Also, by assumption (3), $s(x) \leq |x|$ as $s(\cdot)$ is a $C^*$-seminorm.

(2) $\Rightarrow$ (1). Suppose (2) holds. Since, $A$ is $^*$sb, $|\cdot|$ is the greatest $C^*$-seminorm on $A$ and so $x \mapsto s(x)$, $(x \in A)$ is the greatest continuous $C^*$-seminorm on $A$. By Theorem 4.4.6 and Definition 4.4.1, $s(x) = m(x) = \sup \{f(x^*x) : f \in \mathcal{A}^0(A)\}$ for all $x \in A$. Let $x \in A$. Since $A$ is $^*$sb, $r(x^*x) < \omega$. Let $t = r(x^*x)$ and $h = t - x^*x \in A_2$, the unitization of $A$. Let $f \in \mathcal{A}^0(A_2)$. Then $f(h^2) = t^2 - 2tf(x^*x) + f((x^*x)^2)$. Now, by Lemma 4.4.4(2)(i), $f((a^*a)^2) \leq r(a^*a)f(a^*a)$ for all $a \in A$ and
so, \( f(h^2) \leq t^2 - 2tf(x^*x) + tf(x^*x) = t^2 - tf(x^*x) \leq t^2. \) Thus 
\[ r(h)^2 = r(h^2) = s(h)^2 = m(h)^2 < t^2, \] so that, \( r(t - x^*x) \leq \alpha. \)

Now, mimicking the proof from [20, Theorem 41.11], it follows that \( A_\alpha \) is hermitian. Consequently, \( A \) is hermitian. This completes the proof.

**Theorem 4.4.9** [14]. Let \( A \) be a complete \( * \)-algebra having the \( C^* \)-enveloping algebra. Then the following are equivalent.

1. \( A \) is hermitian.
2. \( s(x) = p_\omega(x) \) for all \( x \in A \).
3. \( x \mapsto s(x) \) is a continuous \( C^* \)-seminorm on \( A \).

If further, \( A \) is also commutative, then each of the above implies \( r(x) = p_\omega(x) \) for all \( x \in A \).

**Proof.** \( 1 \Rightarrow 2 \). Let us note that for each \( p \in P \), \( A_p \) is hermitian and hence is symmetric by Shirali-Ford's Theorem. So, by Raikov symmetry criterion [65, Theorem 4.7.213], and by Corollary 4.2.12, for each \( x \in A \),

\[
  r(x^*x) = \sup_{p \in P} r_{A_p}(x^*x) \\
  = \sup_{p \in P} \left\{ \sup_{f \in B_0(A_p)} f(x^*x) \right\} \\
  = \sup_{p \in P} \left\{ \sup_{f \in B_0(A)} f(x^*x) \right\}
\]
This, with Proposition 4.4.5(2), gives \( s(x) = p_\omega(x) \) for all \( x \in A \), which proves (2).

(2) \( \Rightarrow \) (3). Since \( p_\omega \) is a continuous \( C^* \)-seminorm, this implication is obvious.

(3) \( \Rightarrow \) (1). This is just (3) \( \Rightarrow \) (1) of Theorem 4.4.8.

Now, suppose that \( A \) is commutative. Suppose (1) (and hence equivalently, (2) and (3)) holds. By Proposition 4.4.3, \( A \) is hermitian if and only if \( A \) is symmetric. Hence by [36, Lemma 3.31], \( A \) is symmetric if and only if \( A(A) = A^h(A) \).

Since, additionally \( A \) is \( * \)-sb too, the hermiticity turns out to be equivalent to the equality \( r(x^*x) = r(x)^2 \) for all \( x \in A \).

Thus \( x \mapsto r(x) = s(x) \) is a \( C^* \)-seminorm determined by some \( p \in K(A) \) as, by Proposition 4.2.10, \( A \) is also a \( \mathbb{Q} \)-algebra.

Thus \( r(x) = p_\omega(x) \). This completes the proof.

Remarks 4.4.10. (1) Let \( A \) be the pro-\( C^* \)-algebra \( C[0,1] \) of all continuous functions on \( [0,1] \) with the topology of uniform convergence on all countable compact subsets of \( [0,1] \). Then \( A \) admits the greatest \( C^* \)-seminorm \( |x| = \|x\|_\infty = \sup_{0 \leq t \leq 1} |x(t)| \), \( (x \in A) \), the sup norm on \( A \). But \( A \) fails to admit the greatest continuous \( C^* \)-seminorm.
(2) It follows from Theorem 4.4.6 together with Proposition 4.4.5(1), that a sub Frechet $\mathfrak{m}$ algebra has $C^*$-enveloping algebra. This is analogous to the result that an sb Frechet algebra is a $Q$-algebra \cite{8, Theorem 1}.

(3) The Frechet $\mathfrak{m}$ algebra $A$ of Example 4.3.3 has $C^*$-enveloping algebra, it admits greatest $C^*$-seminorm but is not $\text{sb}$ (as is exhibited by the function $f(z) = z, (z \in \mathbb{U})$, in $A$).

4.5. Examples : Function algebras

In this section, we investigate certain function algebras having $C^*$-enveloping algebras. Throughout this section, we consider the $\mathfrak{m}$ algebras of functions. The operations on these algebras are pointwise and the involution is the complex conjugation except in the final example.

Example 4.5.1. Let $X$ be a compact, second countable $C^\infty$-manifold. Let $C^\infty(X)$ be the $\mathfrak{m}$ algebra of all $C^\infty$-functions on $X$ with the topology of uniform convergence on $X$ of functions and all their derivatives. It is a Frechet $\text{sb}$ hermitian $\mathfrak{m}$ algebra and $\mathfrak{m}(C^\infty(X)) = C(X)$, the sup norm $C^*$-algebra of all continuous functions on $X$. If $A$ is a complete $\mathfrak{m}$ algebra with $C^*$-enveloping algebra, then by \cite[Corollary 4.3]{34}, $\mathfrak{m}(C^\infty(X) \otimes A) = \mathfrak{m}(C^\infty(X)) \otimes \mathfrak{m}(A) = C^\infty(X) \otimes \mathfrak{m}(A) = \mathfrak{m}(X, \mathfrak{m}(A))$, the algebra of all $\mathfrak{m}(A)$-valued continuous functions on $X$. 

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The following definition is a modification of [61, Definition 15.2].

Definition 4.5.2. A sequence \( \{A_n : n \in \mathbb{N}\} \) of \( \ell^m \)-algebras is called \( \ell^m \)-compatible if the following conditions are satisfied.

(i) Each \( A_n \) is complete.

(ii) \( A_n \) is a proper two sided \( \ast \) ideal of \( A_{n+1} \) for each \( n \in \mathbb{N} \).

(iii) The inclusion mapping \( \text{id}_n : A_n \to A_{n+1} \) is an into homeomorphism for each \( n \in \mathbb{N} \).

Definitions 4.5.3. Let \( \{A_n : n \in \mathbb{N}\} \) be an \( \ell^m \)-compatible sequence of \( \ell^m \)-algebras. Let the algebra \( A = \bigcup_{n=1}^{\infty} A_n \) be equipped with the finest locally convex topology having the property that the inclusion maps \( \Theta_n : A_n \to A \) are continuous.

If \( A \) with this topology is an \( \ell^m \)-algebra, then \( A \) is called an \( \ell^m \)-algebra. Further, if each \( A_n \) is a Frechet \( \ell^m \)-algebra, then the \( \ell^m \)-algebra so obtained is called an \( \ell^m \)-algebra.

The following are examples of \( \ell^m \)-algebras.

Examples 4.5.4. (i) Let \( \{B_k : k = 1, 2, \ldots\} \) be a sequence of commutative non-trivial Frechet \( \ell^m \)-algebras. Let us define for \( n \in \mathbb{N} \), \( A_n = B_1 \oplus B_2 \oplus \ldots \oplus B_n = \{x = (x_1, x_2, \ldots, x_n) : x_k \in B_k, \ldots \} \).
Then for each $n \in \mathbb{N}$, $A_n$ is a Frechet lmc algebra with the product topology. Consequently, $\{A_n : n \in \mathbb{N}\}$ is an $m^*$-compatible sequence of Frechet lmc algebras. By [61, Example 15.5], $A = \bigcup_{n=1}^{\omega} A_n$ is an $\mathcal{L}^\infty^*$-algebra. Also, by [61, Proposition 15.8], $A$ is a $Q$-algebra if and only if each $A_n$ is a $Q$-algebra. In particular, taking $B_k$ to be the unital Banach$^*$ algebras for all $k \in \mathbb{N}$, $A$ becomes a $Q\mathcal{L}^\infty^*$-algebra. In this case, by Corollary 4.2.6, $A$ has $C^*$-enveloping algebra.

(ii) Let us fix $k \in \mathbb{N}$. Let $C_n = \{t = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k : |t_i| \leq n, \text{ for } i = 1, 2, \ldots, k\}$, a $k$-cell for each $n \in \mathbb{N}$. Let $A_n = \{f \in C(\mathbb{R}^k) : f(t) = 0 \text{ for } t \in \mathbb{R}^k - C_n\}, (n \in \mathbb{N})$, be the sup norm algebra. Then $\{A_n : n \in \mathbb{N}\}$ is an $m^*$-compatible sequence of Frechet lmc algebras and $A = \bigcup_{n=1}^{\omega} A_n = C_c(\mathbb{R}^k)$, the algebra of all continuous functions on $\mathbb{R}^k$ with the compact support, is an $\mathcal{L}^\infty^*$-algebra with the local base at $0$ given by the scalar multiples of the sets of the form $U_m = \{f \in A : |f(t)| \leq m(t) \text{ for all } t \in \mathbb{R}^k\}$, where $m$ run over all real-valued positive continuous functions on $\mathbb{R}^k$. Also, by [61, Example 3.5], $A$ is a $Q\mathcal{L}^\infty^*$-algebra. The Gelfand space of $A$ consists of the point evaluation maps $\phi_t : A \to \mathbb{C}$, $\phi_t(f) = f(t), (f \in A, t \in \mathbb{R}^k)$. Let us note that $\phi_t^*(f) = \phi_t(f^*) = f^*(t) = \overline{f(t)} = f(t)$ for all $f \in A$ and all $t \in \mathbb{R}^k$. Consequently, $\mathcal{M}(A) = \mathcal{M}^h(A) = \{\phi_t : t \in \mathbb{R}^k\}$. Hence, by Corollary 4.2.14,
\[ p_\infty(f) = \sup_{t \in \mathbb{R}^k} |\phi(t)| = \sup_{t \in \mathbb{R}^k} |f(t)| = \|f\|_\infty \] the sup norm over \( \mathbb{R}^k \). This gives, \( A = A/N \) and \( \mathfrak{H}(A) = C_0(\mathbb{R}^k) \), the completion of \( (C_0(\mathbb{R}^k), \|\cdot\|_\infty) \), which is the \( C^* \)-algebra of all continuous functions on \( \mathbb{R}^k \) vanishing at infinity endowed with the sup norm.

(iii) For a fix \( k \in \mathbb{N} \), let \( A \) be the test functions algebra \( \mathcal{C}^n(\mathbb{R}^k) \) of all \( C^n \)-functions with compact support and with the Schwartz topology, \( (1 \leq n \leq \omega) \). Then \( A \) is a \( \mathcal{L}^S \)-algebra. It is easily seen that \( \mathfrak{H}(\mathcal{C}^n(\mathbb{R}^k)) = \mathfrak{H}(\mathcal{C}_c^\infty(\mathbb{R}^k)) = (C_0(\mathbb{R}^k), \|\cdot\|_\infty) \).

(iv) For a fixed \( k \in \mathbb{N} \) and for \( n \in \mathbb{N} \), let \( \mathcal{C}_n \) be the \( k \)-cell as defined in (i) above. Let \( A_n \) be the \( \text{lim} \)-algebra of all rapidly decreasing \( \mathcal{C}^\infty \)-functions on \( \mathbb{R}^k \) with support in \( \mathcal{C}_n \).

For a given multi-index \( l = (l_1, l_2, \ldots, l_k) \) in \( \mathbb{N}^k \), let us define \( |l| = l_1 + l_2 + \ldots + l_k \). Also, for a multi-index \( l = (l_1, l_2, \ldots, l_k) \) and for \( f \in A_n \), we define \( D^l(f) : \mathbb{R}^k \rightarrow \mathbb{R} \), by

\[ D^l(f)(t) = \frac{\partial^{|l|}}{\partial s_1^{l_1} \partial s_2^{l_2} \ldots \partial s_k^{l_k}} f(t), \quad (t \in \mathbb{R}^k). \]

For \( m \in \mathbb{N} \), let

\[ p_{m,n}(f) = \sup_{|l| \leq m} \sup_{t \in \mathcal{C}_n} |D^l(f)(t)|, \quad (f \in A_n). \]

Then the family \( \{p_{m,n} : m = 1, 2, \ldots\} \) of seminorms defines the Schwartz topology on \( A_n \). Now, by [60, p.133], \( A_n \) is a Fréchet \( \text{lim} \)-algebra. Also, \( \{A_n : n \in \mathbb{N}\} \) is an \( * \)-compatible sequence and the \( \mathcal{L}^S \)-algebra \( A \) determined by \( \{A_n : n \in \mathbb{N}\} \) is the Schwarz.
space $s(\mathbb{R}^k)$ of all rapidly decreasing $C^\infty$-functions on $\mathbb{R}^k$. We also note that $s(\mathbb{R}^k)$ is an AE-algebra in the sense of [27] and hence is an Ime algebra [27, Proposition 1]. Also, as noted in [60, p.133], $A$ is not a $Q$-algebra. Again, it is easily seen that $\mathcal{S}(A) = C_0(\mathbb{R}^k)$.

Now, we come to the examples of algebras whose construction is similar to that of Arens' algebra $L^\omega[0,1]$, ([3], [4]); however, before that we recall the construction of Arens' algebra $L^\omega[0,1]$. $L^\omega[0,1] = \bigcap_{1 \leq p < \infty} L^p[0,1]$ is the Arens' algebra of all complex-valued functions $f$ on $[0,1]$ such that $\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p} < \infty$ for $1 \leq p < \infty$ with the topology determined by the family $\{\| \cdot \|_p : p \in [1,\infty)\}$ of norms, with the pointwise operations and the complex conjugation as the involution. Since the Lebesgue measure of $[0,1]$ is finite, we have $L^p[0,1] \subseteq L^r[0,1]$ whenever $r \leq p$. Thus the topology of $L^\omega[0,1]$ is determined by the family $\{| \cdot |_n : n \in \mathbb{N}\}$ and hence $L^\omega[0,1]$ is metrizable. Also, for $p, q, r \in [1,\infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, $\|fg\|_q \leq \|f\|_q \|g\|_r$ for all $f, g \in L^\omega[0,1]$. This gives the joint continuity of the multiplication on $L^\omega[0,1]$. However, $L^\omega[0,1]$ does not contain any absolutely convex idempotent neighbourhood of $0$ except $L^\omega[0,1]$ itself. Thus $L^\omega[0,1]$ cannot be an Ime algebra. How-
ever, parallel to the construction of Arens' algebra, one can construct a number of complete lmc *algebras. We now, investigate certain complete lmc *algebras constructed along this line.

**Example 4.5.5.** For $1 < p < \infty$, let $AC^p[0,1] = \{ f \in C[0,1] : \text{the derivative } f' \text{ of } f \text{ exists a.e. and } f' \in L^p[0,1] \}$. Let us define $||f||_p = ||f||_{\infty} + \left( \int_0^1 |f'(t)|^p dt \right)^{1/p}$, $(f \in AC^p[0,1])$. Then $AC^p[0,1]$ is a Banach *algebra with pointwise operations and complex conjugation as the involution. Let $AC^\omega[0,1] = \bigcap_{1 \leq p < \infty} AC^p[0,1]$, a Frechet lmc *algebra whose topology is determined by the $m$-calibration $\{ || \cdot ||_p : 1 \leq p < \infty \}$. Also, $AC^\omega[0,1] = \lim_{1 \leq p < \infty} AC^p[0,1]$. Now, by [65, p.303], $\mathcal{M}(AC^p[0,1]) = [0,1]$ for all $p \in [1,\infty)$. Hence, applying [61, Proposition 7.5], $\mathcal{M}(AC^\omega[0,1]) = [0,1]$. The algebra $AC^\omega[0,1]$ is a hermitian Q-algebra and $\mathfrak{R}(AC^\omega[0,1]) = C[0,1]$.

**Example 4.5.6.** Similar to the above example, one can also consider the Sobolev spaces $W^{k}_{p}[0,1] = \{ f \in C^{k-1}[0,1] : f^{(k-1)} \in AC[0,1] \text{ and } f^{(k)} \in L^p[0,1] \}$, where $k = 1,2,\ldots$, and $p \in [1,\infty)$. Let us define
\[ ||f||_{p,k} = \sup_{0 \leq t \leq 1} \left( \sum_{r=0}^{k-1} \left| \frac{f^{(r)}(t)}{r!} \right| + \left( \int_0^1 |f^{(k)}(t)|^p dt \right)^{1/p} \right). \]

Then \( W_{p,k}[0,1], ||\cdot||_{p,k} \) is a Banach algebra with pointwise operations and complex conjugation as the involution.

Now, analogous to the Arens' algebra \( L^0[0,1] \), we get \( W_{\omega,k}[0,1] = \bigcap_{1 \leq p < \omega} W_{p,k}[0,1] \), a complete imprimitivity algebra with \( C^* \)-enveloping algebra.

Example 4.5.7. Let \( C_b(\mathbb{R}) \) denote the \( C^* \)-algebra of all bounded continuous functions on \( \mathbb{R} \). Let

\[ BV_{loc}(C_b(\mathbb{R})) = \{ f \in C_b(\mathbb{R}) : f \text{ is of bounded variation on } [-n,n] \text{ for each } n \in \mathbb{N} \}. \]

For \( f \in BV_{loc}(C_b(\mathbb{R})) \), if \( V_n(f) \) denotes the total variation of \( f \) on \([-n,n]\), then the seminorms \( p_n(f) = ||f||_\infty + V_n(f) \) (\( f \in BV_{loc}(C_b(\mathbb{R})), n \in \mathbb{N} \)) determine a topology making \( BV_{loc}(C_b(\mathbb{R})) \) a Fréchet imprimitivity algebra. Also, \( BV_{loc}(C_b(\mathbb{R})) \) has \( C^* \)-enveloping algebra. In fact, \( \mathcal{E}(BV_{loc}(C_b(\mathbb{R}))) = C_b(\mathbb{R}) \) with the sup norm.

Example 4.5.8. Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), the open unit disc and \( H(D) \) denote the algebra of all holomorphic functions on \( D \) with pointwise operations. For \( f \in H(D) \), let \( \overline{f}(z) = f(\overline{z}) \), \( (z \in D) \). Then \( f \mapsto \overline{f} \) defines an involution on \( H(D) \). For
n ∈ ℕ, let \( A^n(D) = \{ f ∈ H(D) : f^{(k)} \text{ has the continuous extension on } \overline{D} \text{ for } 0 ≤ k ≤ n \} \). Then \( A^n(D) \) is a Banach * algebra with the norm defined by

\[
\| f \|_n = \sup \left\{ \frac{1}{k!} \left| f^{(k)}(z) \right| : z ∈ \overline{D} \right\}.
\]

The Frechet lmc * algebra \( A^Ω(D) = \bigcap_{n=0}^{∞} A^n(D) \), with the topology determined by the \( m^* \)-calibration \( \{ \| · \|_n : n = 0, 1, 2, \ldots \} \) is a non-hermitian Q lmc * algebra with \( ℳ(A^Ω(D)) = C[-1,1] \).

### 4.6. Topological Segal * algebras

Let us recall the following definition from [23, Definition 12].

**Definition 4.6.1.** Let \( (A, \| · \|) \) be a Banach algebra. The sub-algebra \( B \) of \( A \) is an *A-Segal algebra* if the following conditions are satisfied.

1. **(i)*** \( B \) is a dense left ideal of \( A \).

2. **(ii)*** \( B \) is a Banach space with some norm \( | · | \).

3. **(iii)*** \( | x | ≤ K | x | \) for all \( x ∈ B \) and for some constant \( K \).

4. **(iv)*** \( | xy | ≤ M | x | | y | \) for all \( x, y ∈ B \) and for some constant \( M \) independent of \( x \) and \( y \).

The following is a modification of the above definition, tailored for the present set up.
Definition 4.6.2 [14]. Let \((A, \| \cdot \|)\) be a Banach algebra with a bai. A subalgebra \(B\) of \(A\) is said to be an \(A\)-Segal algebra if there exists a topology \(\tau\) on \(B\) satisfying the following conditions.

(i) \(B\) is a dense ideal in \(A\).

(ii) \((B, \tau)\) is a complete metric algebra with a bai.

(iii) The inclusion mapping \(\text{id} : (B, \tau) \to (A, \| \cdot \|)\) is continuous.

(iv) The multiplication \((A, \| \cdot \|) \times (B, \tau) \to (B, \tau)\) is continuous.

Theorem 4.6.3 [14]. If \(B\) is an \(A\)-Segal algebra, then \(B\) is a \(Q\)-algebra. Also, \(\mathcal{S}(B) = \mathcal{S}(A)\), the enveloping \(C^\infty\)-algebra of \(A\).

Proof. Let \(x \in A_{-1} \cap B\) and \(y = x_{-1}\). So, \(x + y - xy = x \circ y = 0 \in B\). Since \(B\) is an ideal, \(y = xy - x \in B\). Thus \(A_{-1} \cap B = B_{-1}\). Also, since \(A_{-1}\) is open in \(A\), by the continuity of the inclusion map \(\text{id} : (B, \tau) \to (A, \| \cdot \|)\), \(B_{-1} = A_{-1} \cap B\) is open in \((B, \tau)\). Hence \(B\) is a \(Q\)-algebra.

Now, let \(P\) be a directed \(\mathcal{M}\)-calibration on \(B\) and \((e_j)_{j \in J}\) be a bai of \(B\) satisfying \(p(e_j) \leq 1\) for all \(p \in P\) and for all \(j \in J\). Let \(K(B)\) be the collection of all continuous submultiplicative \(\mathcal{M}\)-seminorms \(p\) on \(B\) such that \(p(e_j) \leq 1\) for
all \( j \in J \). Again, using the continuity of the inclusion map, we fix \( p_0 \) in \( K(B) \) satisfying \( \|x\| \leq kp_0(x) \) for all \( x \in B \) and for some constant \( k \) independent of \( x \). We show that \( R'(A) = R'(B) \) by restricting each \( \pi \in R'(A) \) to \( B \). Let \( \pi \in R'(A) \) and \( \xi \in H_\pi \) be a topologically cyclic vector for \( \pi \), that is, \( H_\pi \) is the closed linear span of \( \{\pi(x) : x \in B\} \). Let us fix \( y \in B \). For \( x \in A \), \( xy \in B \) and \( e_j \to y \) in \((B, \tau)\). By Definition 4.6.2(iv), \( xe_j \to xy \) in \((B, \tau)\); hence \( \|\pi(xe_j) - \pi(xy)\| \to 0 \). Since the net \( (e_j) \) is \( \tau \)-bounded, there exists \( M_x > 0 \) such that \( \|\pi(xe_j)\| < M_x < \infty \). Thus,

\[
\|\pi(xy)\| = \lim_{j} \|\pi(xe_j)\pi(y)\| \\
\leq \lim_{j} \|\pi(xe_j)\| \|\pi(y)\| \\
\leq M_x \|\pi(y)\|.
\]

Let us define \( \tilde{\pi}(x) : \{\pi(y) : y \in B\} \to \{\pi(y) : y \in B\} \) by \( \tilde{\pi}(x)(\pi(y)) = \pi(xy) \). Then the above relation gives the continuity of \( \tilde{\pi}(x) \) on the dense subspace \( \{\pi(y) : y \in B\} \) of \( H_\pi \). We denote the unique continuous extension of \( \tilde{\pi}(x) \) to \( H_\pi \) also by \( \tilde{\pi}(x) \). Then \( \tilde{\pi}(x) \in B(H_\pi) \). Also, \( \tilde{\pi} \) now becomes a topologically irreducible representation of \( A \) with \( \tilde{\pi}|_B = \pi \). Since \( B \) is dense in \( A \), it is easy to see that \( \tilde{\pi} \) is the unique extension of \( \pi \). Without ambiguity, we denote the extension \( \tilde{\pi} \)
Thus $R'(A) = R'(B)$. Now, for any $y \in B$,

$$p_\infty(y) = \sup_{\pi \in \mathcal{P}} \rho_\pi(y)$$

$$= \sup \{ ||\pi(y)|| : \pi \in R'(B) \}$$

$$= \sup \{ ||\pi(y)|| : \pi \in R'(A) \}$$

$$\leq ||y||$$

$$\leq kp_\infty(y).$$

Thus the greatest continuous $C^*$-seminorm $p_\infty$ on $(B, \tau)$ is the restriction of the Gelfand-Naimark pseudonorm (denoted by $p_\infty$ only) on $A$. Now, we show that $\mathcal{S}(B) = [B/(N_\infty \cap B), ||\cdot||_{p_\infty}]^\sim$

$$= (A/N_\infty, ||\cdot||_{p_\infty})^\sim = A_{p_\infty} = \mathcal{S}(A).$$

For this, let us define

$$\phi : B/(N_\infty \cap B) \to A/N_\infty \text{ by } \phi(x + N_\infty \cap B) = x + N_\infty.$$ Then $\phi$ is a well-defined onto isomorphism. Thus $\mathcal{S}(B)$ is a $C^*$-subalgebra of $\mathcal{S}(A)$. Let us fix $z \in \mathcal{S}(A)$ and $\varepsilon > 0$ and choose $x \in A$ such that $||x + N_\infty - z||_{p_\infty} < \varepsilon/2$. Also, by the denseness of $B$ in $A$, there exists $y \in B$ such that $||x - y|| < \varepsilon/2$. Then

$$\phi(y + (N_\infty \cap B)) = y + N_\infty \in \phi(B/(N_\infty \cap B))$$

and

$$||z - y + N_\infty||_{p_\infty} \leq ||z - x + N_\infty||_{p_\infty} + ||y - x + N_\infty||_{p_\infty}$$

$$\leq ||z - x + N_\infty||_{p_\infty} + ||y - x||$$

$$< \varepsilon.$$
giving the required denseness of \( \mathfrak{H}(B) \) in \( \mathfrak{H}(A) \). So, \( \mathfrak{H}(B) \) is a dense \( C^\ast \)-subalgebra of \( \mathfrak{H}(A) \), giving, \( \mathfrak{H}(B) = \mathfrak{H}(A) \), which completes the proof.

Next, we come to the examples of the A-Segal \( \mathfrak{H} \)-algebras for suitably chosen Banach \( \mathfrak{H} \)-algebras. However, before that we note the basic facts we shall require.

Let \( G \) be a locally compact unimodular group with the Haar measure \( \mu \). Since \( G \) is unimodular, \( \mu \) is left as well as right invariant. For \( 1 \leq p < \infty \), let \( L^p(G) \) denote the Banach space \( \{ f : G \to \mathbb{C} : ||f||_p = \left( \int f(t)^p dt \right)^{1/p} < \infty \} \), with the norm \( ||\cdot||_p \). For \( f \in L^1(G) \) and \( g \in L^p(G) \), let us define the convolution \( f \ast g(t) = \int g(s^{-1} t) g(s) ds, \ (t \in G) \). Then,

\[
4.6.3(a) \quad ||f \ast g||_p = \left( \int \left( \int f(s^{-1} t) g(s) ds \right)^p dt \right)^{1/p} \\
\leq \left( \int \left( \int f(s^{-1} t) g(s) ds \right) dt \right)^{1/p} \\
\leq ||f||_1 \left( \int ||g||_p ds \right)^{1/p} \\
= ||f||_1 ||g||_p .
\]

In particular, for \( f, \ g \in L^1(G) , \)

\[
4.6.3(b) \quad ||f \ast g||_1 \leq ||f||_1 ||g||_1 .
\]
which makes $L^4(G)$ into a Banach algebra. Also, if $G$ is compact, then the relation 4.6.3(a) reduces to

$$
4.6.3c) \quad \|f^\ast g\|_p \leq \|f\|_p \|g\|_p, \quad (f, g \in L^p(G)),
$$

which makes $L^p(G)$ into a Banach algebra. Let us also note that for a locally compact group $G$, $f \in L^p(G)$ and $g \in L^4(G)$, the relation

$$
4.6.3d) \quad \|f^\ast g\|_p \leq \|f\|_p \|g\|_4,
$$

can be proved analogously.

Now, for $f \in L^p(G)$, let

$$
4.6.3e) \quad f^\ast = f(t^{-1}), \quad (t \in G).
$$

It can be seen that the mapping $f \in L^p(G) \mapsto f^\ast \in L^p(G)$ satisfies properties (i), (iii) and (iv) of the involution. Now, by 4.6.3(a) and 4.6.3(d), if, either $f \in L^4(G)$ and $g \in L^p(G)$ or if $f \in L^p(G)$ and $g \in L^4(G)$, then $f^\ast g$ and $g^\ast f^\ast$ are elements of $L^p(G)$. In this case,

$$
4.6.3f) \quad g^\ast f^\ast = \int g(s^{-1})f^\ast (s)ds
\quad = \int g(t^{-1}t) \cdot \frac{f^\ast (s)}{s}ds
\quad = \int g(t^{-1}s)f(s^{-1})ds
$$

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\[
\left( \int g(s)f(s^{-1}t)ds \right)\left( t^{-2} \right)
\]

Thus for such \( f \) and \( g \), \((f \ast g)^{n} = g^{n} \ast f^{n}\). All this implies that \( L^{2}(G) \) is a Banach algebra and if \( G \) is compact, then \( L^{P}(G) \) is also a Banach algebra. The enveloping \( C^{*} \)-algebra of \( L^{2}(G) \) is called the group \( C^{*} \)-algebra of \( G \) and is denoted by \( C^{*}(G) \).

Example 4.6.4 [14]. For a locally compact abelian group \( G \), let \( A = L^{2}(G) \) and \( B = \ell^{\infty}(G) = \{ f \in L^{2}(G) : f \in L^{P}(G) \text{ for all } p \in (1,\infty) \} \). For an integer \( n \geq 2 \), if \( p_{n}(f) = ||f||_{1}^{r} + ||f||_{n}^{r} \) (\( f \in B \)), then \( p_{n} \) is a \( \ast \) seminorm on \( B \). Also, by 4.6.3(b),

\[
p_{n}(f \ast g) = ||f \ast g||_{1}^{r} + ||f \ast g||_{n}^{r}
\]

\[
\leq ||f||_{1} ||g||_{1}^{r} + ||f||_{1} ||g||_{n}^{r}
\]

\[
\leq ||f||_{1} ||g||_{1}^{r} + ||f||_{1} ||g||_{n}^{r} + ||f||_{n} ||g||_{1}^{r} + ||f||_{n} ||g||_{n}^{r}
\]

\[
= (||f||_{1} + ||f||_{n} ||g||_{1}^{r} + ||g||_{n}^{r})
\]

\[
= p_{n}(f)p_{n}(g)
\]

for all \( f, g \in B \). Thus \( \{p_{n} : n = 2,3,\ldots\} \) is an \( m^{\infty} \)-calibration on \( B \).
It follows from the relations 4.6.3(a) and 4.6.3(d) that $B$ is an ideal of $A$. The denseness of $B$ in $A$ is a consequence of the denseness of $C_c^\omega(G)$ (all continuous functions on $G$ with the compact support) in $L^I(G)$. The continuity of the inclusion map $\text{id}: (B, \tau) \to (A, \| \cdot \|_A)$ is obvious from the definition of the seminorms $p^\alpha$. Again, by using 4.6.3(a), the continuity of the multiplication mapping $(A, \| \cdot \|_A) \times (B, \tau) \to (B, \tau)$ follows. Thus $B$ is an $A$-Segal algebra (and also a Frechet imc algebra in its own right). It follows now, from Theorem 4.6.3, that $B$ has $C^\omega$-enveloping algebra and $\mathcal{B}(B) = \mathcal{B}(A) = C^\omega(G)$, the group $C^\omega$-algebra of $G$.

Example 4.6.5 [14]. Let $A = L^I(\mathbb{R})$, $B = \{ f \in A : f \in C^\omega(\mathbb{R}) \}$ and $f^{(n)} \in L^I(\mathbb{R})$ for all $n = 1, 2, \ldots$. Let us consider the topology $\tau$ defined on $B$ by the seminorms $p^\alpha(f) = \| f \|_A + \| f^{(k)} \|_A$, $k = 0, 1, 2, \ldots$. It can be easily verified that $D(f * g) = (Df) * g = f * (Dg)$, for all $f, g \in B$, where $D$ denotes the differential operator $\frac{d}{dt}$. Thus, inductively, we have,

\begin{align*}
4.6.5(c) & \quad D^k(f * g) = (D^k f) * g = f * (D^k g),
\end{align*}

for all $f, g \in B$ and for $k = 0, 1, 2, \ldots$. Now, by 4.6.3(b), we have,

\begin{align*}
4.6.5(b) & \quad p^\alpha(f * g) = \| f * g \|_A + \| D^k(f * g) \|_A \\
& \quad = \| f * g \|_A + \| (D^k f) * g \|_A
\end{align*}
Thus 4.6.5(b) and a few simple verification make \((B, \tau)\) into an \(\mathbb{A}\)-Segal algebra. Thus \((B, \tau)\) has \(\mathbb{C}\)-enveloping algebra.

**Example 4.6.6** [14]. Let \(G\) be a compact group and \((S, |\cdot|)\) be a Banach algebra with a bai. For \(1 \leq p < \infty\), let \(B^p(G, S)\) be the Banach algebra of all functions \(f : G \to S\) satisfying
\[
|f|_p = \left( \int |f(g)|^p d\mu(g) \right)^{1/p} < \infty, \text{ with the norm } |\cdot|_p.
\]
As noted in [57, p. 485], \(B^p(G, S) = L^p(G) \otimes S\), the completion of the normed algebra \((L^p(G) \otimes S, \eta_p)\), where the norm \(\eta_p\) is defined as
\[
\eta_p(\sum_{i=1}^m f_i \otimes x_i) = \left( \int |\sum_{i=1}^m f_i(g) \otimes x_i|^p d\mu(g) \right)^{1/p},
\]
\((\sum_{i=1}^m f_i \otimes x_i \in L^p(G) \otimes S)\). Also, for \(\sum_{i=1}^m f_i \otimes x_i \in L^p(G) \otimes S\),
\[
\eta_p\left(\left(\sum_{i=1}^m f_i \otimes x_i\right)^*\right) = \left( \int |\sum_{i=1}^m f_i^*(g)x_i^*|^p d\mu(g) \right)^{1/p}
\]
\[= \left( \int |\sum_{i=1}^m f_i(g^{-1})x_i|^p d\mu(g) \right)^{1/p},
\]
\[= \left( \int |\sum_{i=1}^m f_i(g^{-1})x_i^*|^p d\mu(g) \right)^{1/p}
\]
\[= \left( \int |\sum_{i=1}^m f_i(g^{-1})x_i|^p d\mu(g) \right)^{1/p}.\]
So, \( L^p(G) \otimes S \) is a Banach * algebra. By [57, Proposition 7.10], for all \( p \), \( \mathfrak{B}(B^p(G, S)) = C^*(G) \otimes_{m_{\text{top}}} S \). Taking \( A = B^*(G, S) \) and \( B = B^{\omega}(G, S) = \bigcap_{1 \leq p < \infty} B^p(G, S) = \lim_{p \to \infty} B^p(G, S) \), with the topology of \( \| \cdot \|_p \)-convergence for each \( p \in [1, \infty) \), \( \mathfrak{B}(B^{\omega}(G, S)) = C^*(G) \otimes_{m_{\text{top}}} S \).

Example 4.6.7 [14]. Let \((A, \| \cdot \|)\) be a commutative hermitian Banach * algebra with a bai. Let \( \mu \) be a positive regular Borel measure on \( M(A) = M^h(A) \). For \( 1 \leq p < \infty \), let \( A^p = \{ x \in A : \hat{x} \in L^p(M(A), \mu) \} \), where \( \hat{x} \) denotes the Gelfand transform of \( x \). Then \( A^p \) is a Banach * algebra with the norm on \( A^p \) defined by \( \| x \|_{A^p} = \| x \| + \| \hat{x} \|_p \), \( x \in A^p \). Also, for \( x \in A \) and \( y \in A^p \),

\[
\| xy \|_p = \| xy \| + \| \hat{y} \|_p
\]

\[
= \| xy \| + \left( \int_{M(A)} |\hat{x}(\phi)\hat{y}(\phi)|^p d\mu(\phi) \right)^{1/p}
\]

\[
\leq \| xy \| + \| \hat{x} \|_\infty \| \hat{y} \|_p
\]

\[
< \infty.
\]
Thus $A_p(\mu)$ is an ideal of $A$. Since $(x^*)^\wedge = (\hat{x})^*$, it follows that $A_p(\mu)$ is an ideal. Now, let us define $B = \bigcap_{1 \leq p < \infty} A_p(\mu)$, an ideal in $A$. Then it is easily seen that $B$ is a complete $\ell^\infty$ algebra with the topology $\tau$, determined by the $\| \cdot \|_{A_p}^\infty$ convergence for $p \in [1, \infty)$. Then $B$ is an $A$-Segal $^*$ algebra and $\mathcal{S}(B) = \mathcal{S}(A)$ by Theorem 4.6.3.

In particular, for a locally compact abelian group $G$, $L^2(G)$ is a hermitian Banach $^*$ algebra. Also, $L^\infty(G)$ always admits a ban [60, p.232] and $\mathcal{M}(L^\infty(G)) = \hat{G}$, the dual group of $G$ consisting of all group homomorphisms $\phi : G \rightarrow \Gamma$, where $\Gamma = \{ z : |z| = 1 \}$. Let $\mu$ be the Haar measure on $\hat{G}$. Then taking $A = L^2(G)$, $B = \{ f \in L^2(G) \colon \text{the Fourier transform } \hat{f} \text{ of } f \text{ is in } L^p(\hat{G}, \mu) \text{ for } 1 \leq p < \infty \}$ becomes an $A$-Segal $^*$ algebra with $\mathcal{S}(B) = \mathcal{S}(A) = C^*(G)$.

4.7. Topological algebra with bases and Köthe sequence algebras

Throughout this section $\omega$ stands for the $^*$ algebra of all scalar sequences with pointwise operations and complex conjugation as the involution. For the details of the sequence spaces we refer to [51], [53] and [66] and in particular, for the ring structure of the Köthe sequence spaces [10].

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Definition 4.7.1. A non-empty subset $Q$ of $\omega$ is said to be a Köthe power set if the following conditions are satisfied.

(i) For each $a = (a_n) \in Q$, $a_n \geq 0$ for all $n \in \mathbb{N}$.

(ii) For given $a, b \in Q$, there exists $c \in Q$ such that $a_n \leq c_n, b_n \leq c_n$ for $n \in \mathbb{N}$.

(iii) For each $n$, there exists $a \in Q$ with $a_n > 0$.

(iv) $a_n > 0$ for each $a \in Q$.

Further, $Q$ is said to satisfy $G_\omega$-property if the following hold.

(v) For each $a \in Q$, $a_n \leq a_{n+1}$ for all $n$.

(vi) For $a \in Q$, there exists $d \in Q$ such that $a_n^2 \leq d_n$.

Definition 4.7.2. Let $Q$ be a Köthe power set satisfying $G_\omega$-property. Then the locally convex space $\Lambda_\omega(Q) = \{x = (a_n)_{n=1}^\infty \in \omega : p_a(x) = \sum |x_n| a_n < \infty$ for all $a \in Q\}$ with the Köthe topology $\tau$ determined by the seminorms $\{p_a : a \in Q\}$ is called the Köthe space of infinite type.

Proposition 4.7.3. $\Lambda_\omega(Q)$ is an $l^1$-Segal * algebra.

Proof. Given $a \in P$, $x \in \Lambda_\omega(Q)$, $\sum |x_n| = \sum \frac{1}{a_n} a_n |x_n| \leq \sum \frac{a_n}{a_1} |x_n| \leq \frac{1}{a_1} p_a(x)$ shows that $||x||_1 \leq \frac{1}{a_1} p_a(x)$. Hence $\Lambda_\omega(Q) \subset l^1$. It is easy to see that $\Lambda_\omega(Q)$ is a vector space; hence a locally
convex space with the Köthe topology. Also, for $x \in \Lambda_\infty(Q)$, $y \in l^1$ and $a \in Q$,

$$\sum_{n=1}^{k} |x_n y_n| a_n \leq \sum_{n=1}^{k} \sum_{m=1}^{k} |x_n y_m| a_n$$

$$= \left( \sum_{n=1}^{k} |y_n| \right) \left( \sum_{n=1}^{k} |x_n| a_n \right)$$

$$\leq \|y\|_1 p_a(x)$$

for all $k \in \mathbb{N}$. Consequently, $p_a(xy) \leq \|y\|_1 p_a(x)$, making $\Lambda_\infty(Q)$ an ideal in $l^1$. Again, it is a matter of simple verification that $\Lambda_\infty(Q)$ is, in fact, an ideal and each $p_a$ is a seminorm.

Let us note that, [10, Theorem 2, p.1761], $\Lambda_\infty(Q)$ is a Schwartz space or $\Lambda_\infty(Q) = l^1$. Also, $\Lambda_\infty(Q)$ is a Schwartz space if and only if there exists $d = (d_n)_{n=1}^{\infty} \in Q$ such that $\frac{1}{d_n} \in c_0$. Now, it is obvious that $l^1$ is an $l^1$-Segal algebra. So, we assume that, there exists $d \in \Lambda_\infty(Q)$ with $\frac{1}{d_n} \to 0$.

Now, let $x = (x_n) \in \Lambda_\infty(Q) \cap (l^1)_{-1}$. Then $y_n = \frac{-x_n}{1 - x_n}$ defines $x_{-1}$ in $l^1$. Thus $x_n \neq 1$ for $n = 1, 2, \ldots$, and $1 \not\in \{x_n\}$ as $x \in l^1$. Thus for some $\delta > 0$, $|1 - x_n| > \delta$ for all $n$. So, $p_a(y) \leq \frac{1}{\delta} p_a(x) < \infty$ for all $a \in Q$, showing that $y \in \Lambda_\infty(Q)$. This shows that $\Lambda_\infty(Q)_{-1} = \Lambda_\infty(Q) \cap (l^1)_{-1}$. Since for $x \in \Lambda_\infty(Q)$
and \( a \in \mathbb{Q}, \ |x|_1 \leq p_a(x), \ \Lambda_\infty(\mathbb{Q}) \) is continuously embedded in \( l^2 \). Consequently, \( \Lambda_\infty(\mathbb{Q}) \) is a \( \mathbb{Q} \)-algebra. The \( \mathbb{Q} \)-property also implies that every multiplicative linear functional on \( \Lambda_\infty(\mathbb{Q}) \) is continuous. Also, let \( x, y \in \Lambda_\infty(\mathbb{Q})^{-1} \). Then we choose \( \delta > 0 \) such that \( |1 - x_n| \geq \delta, \ |1 - y_n| \geq \delta \), for \( n = 1, 2, \ldots \). Then for all \( a \in \mathbb{Q} \),

\[
P_a(x_n - y_n) = \sum \frac{-x_n}{1 - x_n} - \frac{-y_n}{1 - y_n}a_n
\]

\[
= \sum \frac{y_n x_n}{(1 - x_n)(1 - y_n)} a_n < (1/\delta^2)p_a(y - x),
\]

giving the continuity of the inversion map. Now, by [77, p.170], (in this regard we also refer to [74]) it follows that \( \Lambda_\infty(\mathbb{Q}) \) is an \( l^1 \)-Segal *-algebra. This shows that \( \Lambda_\infty(\mathbb{Q}) \) is an \( l^1 \)-Segal *-algebra, which completes the proof.

**Definition 4.7.4.** Let \( A \) be a topological algebra. A sequence \((f_n)\) in \( A \) is called an **orthogonal basis** for \( A \) if

(i) \((f_n)\) is a basis of \( A \); and

(ii) \( f_n f_m = \delta_{nm} f_n \), where \( \delta_{nm} \) is the Kronecker delta.

**Proposition 4.7.5** [14]. Let \( A \) be a hermitian \( lmc \) *-algebra having \( C^* \)-enveloping algebra. Suppose \( A \) admits an orthogonal basis consisting of hermitian elements. Then \( A \) is a \( \mathbb{Q} \)-algebra and \( \mathcal{S}(A) \) is * isomorphic to the \( C^* \)-algebra \( c_0 \) of all sequences converging to zero.
Proof. Let \((f_n)\) be an orthogonal basis for \(A\) satisfying \(f_n^* = f_n\) for all \(n \in \mathbb{N}\). Then \((f_n)\) is a Schauder basis ([11], [48]). Let \(x = \sum_{n=1}^{\infty} x_n f_n\). Let us define the coefficient functionals \(\varphi_n\) for \(n \in \mathbb{N}\), by \(\varphi_n(x) = x_n\). Then the Gelfand space \(\mathcal{M}(A) = \mathcal{M}^b(A) = \{\varphi_n\}\) by the hermiticity of \(A\). Also, the map \(\hat{\varphi} : A \to \omega, \hat{\varphi}(x) = (x_n)_{n=1}^{\infty}, (x \in A)\) is an isomorphism of \(A\) onto a subalgebra \(K\) of \(\omega\), which is continuous for the topology of pointwise convergence on \(\omega\) by [48, Corollary 1.3].

We identify \(A\) with \(K\) algebraically. By [61, Corollary 5.6], for all \(x \in A\), \(sp(x) = \{\varphi_n(x)\} = \{x_n : n = 1, 2, \ldots\}\). Also, by Theorem 4.2.1 and Corollary 4.2.14, \(p_\infty(x) = \sup_{n} |x_n| = |x|_\infty = r(x) < \infty\) for all \(x \in A\). Thus \(A \subseteq l^\infty\). By Corollary 4.2.10, \(A\) is a \(\mathbb{Q}\)-algebra. Further, \(A\) contains (by its identification with \(K\)) the set of all sequences with finitely many non-zero terms. Also, if \(A\) is unital, then by [48, Theorem 2.1], \(A = \omega\). But since \(\omega\) does not have \(C^*\)-enveloping algebra, we conclude that \(A\) cannot have the identity. It follows now, that the completion of \((A, ||\cdot||_\omega)\) contains \(c_0\). On the other hand, \(p_\infty\), being a continuous \(C^*\)-seminorm on \(A\) in the topology of \(A\),

\[ x(n) = \sum_{k=1}^{n} x_k f_k \text{ converges to } \sum_{k=1}^{\infty} x_k f_k = x \text{ in } A \text{ implies that } \]

\[ |x_{n+1}| \leq \sup_{k>n} |x_k| = p_\infty(x - x(n)) \to 0 \text{ as } n \to \infty. \text{ Thus } x \in c_0. \]

This gives \(\mathcal{B}(A) = c_0\), which completes the proof.
Corollary 4.7.6 [14]. Let $Q$ be a Köthe power set satisfying $G_\omega$-property. Then $\mathcal{E}(\Lambda_\omega(Q)) = c_0$.

Example 4.7.7 [14]. Let $(\theta_n)_{n=1}^\infty$ be an increasing sequence of positive real numbers. Let $Q = \{ (k^n)_{n=1}^\infty : k \in \mathbb{N} \}$. Then it is easy to see that the conditions (i), (iii), (iv) and (v) of Definition 4.7.1 are satisfied by $Q$. To see that $Q$ satisfies the condition (ii), let $a = (k^n)_{n=1}^\infty$, $b = (m^n)_{n=1}^\infty \in Q$. Let us assume without loss of generality that $k < m$. Since $\theta_n > 0$ for each $n$, $k^n \leq m^n$. Taking $c = b$, we have $a_n \leq c_n$ and $b_n \leq c_n$ for each $n \in \mathbb{N}$. This establishes (ii). Also, for $a = (k^n)_{n=1}^\infty$ in $Q$, taking $m = k^2$ and $d = (m^n)_{n=1}^\infty$, we have $a_n^2 = d_n$ for all $n$. Thus $Q$ is a Köthe power set satisfying $G_\omega$-property. For such a $Q$, the Köthe space $\Lambda_\omega(Q)$ is investigated in [76], where this is denoted by $\Lambda_\omega[\theta_n]$; and is called the power series space of infinite type [76, p.204]. In particular, taking $\theta_n = \log n$, $(n \in \mathbb{N})$, by [76, p.205], $\Lambda_\omega[\theta_n]$ reduces to $S = \{ x \in \omega : \sum_{n=1}^\infty n^k |x_n| < \infty, \text{ for all } k \in \mathbb{N} \}$, the algebra of all rapidly decreasing sequences.

Example 4.7.8 [14]. Let $A = C^\omega(\Gamma)$, the convolution algebra of all $C^\omega$-functions $u$ on the unit circle $\Gamma$, with the involution $u^* (z) = u(z^{-1})$, $(z \in \Gamma)$. By [32, p.48], for $u \in A$, the Fourier
series expansion $u = \sum_{n=0}^{\infty} \hat{u}(n) e^{i\lambda n}$ gives sequences $(\hat{u}(n)) \in s(\mathbb{Z})$, the algebra of all rapidly decreasing two sided sequences.

The map $\hat{\mathcal{F}} : C^\infty(\Gamma) \to s(\mathbb{Z})$, $\hat{\mathcal{F}}(u) = (\hat{u}(n))_{n=-\infty}^{\infty}$ establishes a *isomorphism of $C^\infty(\Gamma)$ onto $s(\mathbb{Z})$, which is a homeomorphism for the usual Frechet $C^\infty$-topology on $C^\infty(\Gamma)$ and the Frechet-Köthe topology on $s(\mathbb{Z})$ by [66, Theorem 5.1]. Now, $s(\mathbb{Z})$ is a $\mathbb{Q}$-algebra and $s(s(\mathbb{Z})) = c_0$. Thus via $\hat{\mathcal{F}}$, $C^\infty(\Gamma)$ is a complete $\mathbb{Q}$-algebra with $\mathcal{F}(C^\infty(\Gamma)) = \{ \mu \in PM(\Gamma) : \hat{\mu}(n) \in c_0 \}$, where $PM(\Gamma)$ is the convolution algebra of all pseudomeasures on $\Gamma$, isomorphic to $l^\infty$, via the Fourier expansion [32, §12.11].

Example 4.7.9 [14]. Let $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and $H^p(U)$ be the Hardy space for $1 < p < \infty$. Then $H^p(U)$ is a Banach * algebra with $\| \cdot \|_p$, and with the Hadamard product defined by $f \ast g(x) = \frac{1}{2\pi i} \int \frac{f(z)g(xz^{-1})}{z^{-1}} dz$, $|x| < r < 1$. The involution on $H^p(U)$ is defined by $f^*(z) = f(\overline{z})$, $(z \in U)$. Also, the sequence $(e_n^p)_n$, $e_n^p(z) = z^n$, is hermitian orthogonal basis for $H^p(U)$ [48, Example 3]. The Hardy-Arens' algebra $H^\infty(U) = \bigcap_{1 < p < \infty} H^p(U)$ is a Frechet lim * algebra with the $m^\ast$-calibration $\{ \| \cdot \|_p \}$. Also, we note that for $f \in H^\infty(U)$, $f$ has the unique Taylor series expansion $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$. Since the orthogonal basis is a Schauder basis, as in the proof of Proposi-
ition 4.7.5, \( \mathcal{M}(H^\omega(U)) = \mathcal{M}(H^\omega(U)) = \{ \phi_n \} \), where \( \phi_n(f) = \frac{f^{(n)}(0)}{n!} \), (\( f \in H^\omega(U) \)). It is easily seen that \( p_\omega(f) = \sup_n \left| \frac{f^{(n)}(0)}{n!} \right| \leq ||f||_p \) (\( p > 1 \)), defines the greatest continuous \( C^* \)-seminorm on \( H^\omega(U) \), showing that \( H^\omega(U) \) has \( C^* \)-enveloping algebra.

**Example 4.7.10.** Let \( E \) be the Frechet space of all entire functions of one variable with the compact open topology. With the product and involution of the previous example, \( E \) is an \( lmc^* \) algebra with the same hermitian orthogonal basis. The \( * \)-isomorphism \( \mathcal{S}(\sum x_n e_n) = (x_n) \) identifies \( E \) with \( A_\omega[n] \) homeomorphically [76, p.206]. Thus \( \mathcal{S}(E) = c_0 \).