CHAPTER 1

CHARACTERIZATIONS OF PRO-C*-ALGEBRAS

The present chapter is devoted to obtaining several characterizations of pro-C*-algebras. Theorem 1.1.9 characterizes the pro-C*-algebras intrinsically, to which the first section is allotted. Section 2 contains a non-involutive characterization of pro-C*-algebras. In section 3, we deal with those characterizations which are linked to the topological duals. Section 4 is dealt with the investigation of the local structure of the pro-C*-algebras. In section 5, we present an example showing that there exists a topological *-algebra A, in which, for each selfadjoint element h, the *-algebra generated by h is a C*-algebra, but A is not a pro-C*-algebra. In passing, we examine the bounded part b(A) of a pro-C*-algebra A in section 6. Most of the results cited in this chapter have appeared in [15].

1.1. An intrinsic characterization

We begin with certain preliminaries concerning the numerical range theory for Banach algebras and lmc algebras. We recall the following definitions from [18] and [19].
Definitions 1.1.1. Let \((A, || \cdot ||)\) be a Banach algebra. The state space of \((A, || \cdot ||)\) is the set \(D(A, || \cdot ||, 1) = \{ f \in A' : || f || = f(1) = 1 \} = \{ f \in A' : f(1) = 1, || f(x) || \leq || x || \ for \ all \ x \in A \}. For any \(x \in A\), the numerical range of \(x\) (with respect to the norm \(|| \cdot ||\)) is the set \(V(A, || \cdot ||, x) = \{ f(x) : f \in D(A, || \cdot ||, 1) \}. An element of the set \(H(A, || \cdot ||) = \{ x \in A : V(A, || \cdot ||, x) \subset \mathbb{R} \}\) is called a hermitian element of \(A\). A continuous linear functional \(f\) on \(A\) is said to be a hermitian linear functional on \(A\) if \(f(x) \in \mathbb{R}\) for all \(x \in H(A, || \cdot ||)\). We denote the set of all such functionals by \(H(A, || \cdot ||')\) or by \(H(A')\). For \(x \in A\), \(v(x) = \sup \{ \lambda : \lambda \in V(A, || \cdot ||, x) \}\), is called the numerical radius of \(x\).

The following definitions are exactly the analogues of the above definitions. All, but the definition of \(P\)-hermitian linear functionals, can be found in [45].

Definitions 1.1.2. Let \(A\) be a complete \(lmc\) algebra and \(P \in \mathcal{P}(A)\) be directed. For \(p \in P\), let \(D_p = \{ f \in A' : f(1) = 1, f(x) \leq p(x) \ for \ all \ x \in A \}\) and \(D(A, P, 1) = \bigcup_{p \in P} D_p\). For \(x \in A\) and \(p \in P\), let \(V(A, p, x) = \{ f(x) : f \in D_p \}\). Also, the set \(V(A, P, x) = \{ f(x) : f \in D(A, P, 1) \}\) is called the numerical range of \(x\) (or better \(P\)-numerical range of \(x\)). \(v(A, P, x) = \sup \{ |\lambda| : \lambda \in V(A, P, x) \}\) is called the numerical radius of \(x\). Let \(H(A, P) = \{ x \in A : V(A, P, x) \subset \mathbb{R} \}\).
Then each \( x \in H(A, P) \) is called a hermitian element (or a \( P \)-hermitian element) of \( A \). Similarly, an \( f \in H((A, P)' \) = \( \{ f \in A' : f(x) \in \mathbb{R} \text{ for all } x \in H(A, P) \} \) is called a hermitian linear functional (or a \( P \)-hermitian linear functional) on \( A \).

For \( p \in P \), we write \( H^p = \{ sf - tg : s, t > 0 \text{ in } \mathbb{R} \text{ and } f, g \in D_p \} \).

Also, \( B_p = \{ x \in A : v(A, P, x) < \infty \} \) is the collection of all elements of \( A \) with bounded numerical range.

Remark 1.1.3. The one to one correspondence between \( A'(p) \) and \( A_p' \) (Theorem 0.1.31(i)) induces the one to one correspondence between:

(i) \( D_p \) and the state space \( D(A_p, ||\cdot||_p^p) \) of the Banach algebra \( (A_p, ||\cdot||_p^p) \).

(ii) \( H^p \) and \( H((A_p') = H((A_p, ||\cdot||_p^p)^\prime) \), the collection of all hermitian linear functionals on \( (A_p, ||\cdot||_p^p) \).

The same correspondence also gives

\[ V(A, p, x) = V(A_p, ||\cdot||_p^p, x_p). \]

In what follows, we write \( S_p = \{ x \in A : p(x) \leq 1 \text{ for all } p \in P \} \). Throughout this section, we assume a calibration \( P \) on an lmc algebra \( A \) to be directed.
Lemma 1.1.4 (50). Let $A$ be a complete lmc algebra.

1. If $P \in \mathcal{P}(A)$, then $S_P \in \mathcal{B}(\tau)$.

2. Given $B \in \mathcal{B}(\tau)$, there exists $P \in \mathcal{P}(A)$ such that $B \subseteq S_P$.

3. If $A$ is also an lmc $^*$-algebra, then for each $B \in \mathcal{B}(\tau)$, there exists $P \in \mathcal{P}(A)$ such that $B \subseteq S_P$.

The following is the Vidav-Palmer theorem for $C^*$-algebras [18, p.65]. Theorem 1.1.6(2) is pro-$C^*$-analogue of it, characterizing the pro-$C^*$-algebras among the lmc algebras.

Theorem 1.1.5 (Vidav-Palmer) [18]. Let $(A, ||\cdot||)$ be a Banach algebra. Then $(A, ||\cdot||)$ is a $C^*$-algebra with some involution if and only if $A = H(A, ||\cdot||) \oplus IH(A, ||\cdot||)$.

Theorem 1.1.6 [45]. Let $A$ be a complete lmc algebra. Let $P \in \mathcal{P}(A)$ be directed.

1. $B_P = A(S_P)$ and $||x||_{S_P} = \sup \{p(x) : p \in P\}$, $(x \in B_P)$.

Further, for each $x \in B_P$, $V(A, P, x) = V(B_P, ||\cdot||_{S_P}, x)$, where $-$ denotes the closure.

2. $A = H(A, P) \oplus IH(A, P)$ if and only if there exists an involution on $A$, making $A$ a pro-$C^*$-algebra, with $P \subseteq S(A)$. The involution is determined by the above direct sum.
As a corollary to the above theorem, we have the following result.

Corollary 1.1.7. Let $A$ be a complete lmc algebra with a directed $P \in \mathcal{E}_2(A)$. Then $A$ is a pro-$C^*$-algebra with $P \in \mathcal{E}_3(A)$ if and only if $D(A,P,1) \subseteq A^{'h}$.

Proof. Suppose $A$ is a pro-$C^*$-algebra with $P \in \mathcal{E}_3(A)$. Let $f \in D(A,P,1)$. Then by Definitions 1.1.2, there exists $p \in P$ such that $f \in D_p$, and hence $f_p \in D(A_p,||\cdot||_p,1_p)$ by Remark 1.1.3(i). Thus $||f_p|| \leq 1$ and $f_p(1_p) = 1$. So, by $(28, \text{Proposition 2.1.9})$, $f_p \in A^{'h}_p$. Again, an appeal to Theorem 0.1.31(3) gives $f \in A^{'h}$.

Conversely, suppose $D(A,P,1) \subseteq A^{'h}$. Let $h \in A^h$ and $f \in D(A,P,1)$. Then $f \in A^{'h}$. Hence $f(h) = f^*(h) = f(h^*) = f(h)$. Thus $f(h) \in \mathbb{R}$. Since $f \in D(A,P,1)$ is arbitrary, it follows that $A^h \subseteq H(A,P)$, giving $A = A^h + iA^h \subseteq H(A,P) + iH(A,P)$, a direct sum because $H(A,P) \cap iH(A,P) = \{0\}$. So, by Theorem 1.1.6(2), $A$ is a pro-$C^*$-algebra with $P \subseteq S(A)$. This completes the proof.

Before we go to the final and the main result of this section, we note the following lemma (45, Lemma 3, p.88).
Lemma 1.1.8. Let $A$ be a complete lmc algebra and $P \in \mathcal{K}_1(A)$. Then $H(A,P)$ is closed in $A$.

We note that topological *algebras, more general than the $C^*$-algebras, called the Generalized $B^*$-algebras (or in short, $GB^*$-algebras), introduced by Dixon [29], have been studied by Bhatt ([6], [7]). The source of inspiration for our result is his analogous result [7, Theorem 21]. He proves it by using somewhat complicated numerical range theory. Our proof mainly uses the definition and the ideas of the inverse limits and $C^*$-algebra results.

Theorem 1.1.9. Let $A$ be a complete lmc *algebra. Then $A$ is a pro-$C^*$-algebra if and only if $A$ contains a *subalgebra $B$ such that

(i) $B$ is a $C^*$-algebra with some norm $||\cdot||$; and

(ii) the inclusion $(B, ||\cdot||) \to A$ is continuous embedding with the dense range.

Further, if $K = \{x \in B : ||x|| \leq 1\}$ is closed in $A$, then $B = b(A)$.

Proof. Suppose that $A$ is a pro-$C^*$-algebra. Let $x \in A$ and $y = x^* x$. Then $y$ is normal; and so, by functional calculus [62, Proposition 1.8, p.184] (and also Theorem 0.1.37), there exists a unique *homomorphism $\psi : C(sp(y)) \to A$ which sends 1.
to 1 and the identity function to $y$. But by Theorem 0.1.36(2), $\text{sp}(y) \subset [0, \infty)$. Thus the function $g : \text{sp}(y) \to \mathbb{R}$ defined by $g(\lambda) = 1 + \lambda$, $(\lambda \in \text{sp}(y))$, does not vanish on $\text{sp}(y)$ and hence $g$ is invertible in $C(\text{sp}(y))$. Since $1 + y = \Psi(g)$, $1 + y \in A^{-1}$.

For $p \in S(A)$, let $\Psi_p : C(\text{sp}_p(y)) \to A_p$ be the unique homomorphism and let us consider the mapping $f : \text{sp}(y) \to \mathbb{R}$, defined by $f(\lambda) = \frac{\lambda}{(1 + \lambda)^2}$, $(\lambda \in \text{sp}(y))$. Denoting $f|_{\text{sp}_p(y)}$ also by $f$, we have,

$$
\Psi_p(f) = (1 + y_p)^{-1} y_p (1 + y_p)^{-2} = (1 + x_p x_p)^{-1} x_p x_p (1 + x_p x_p)^{-2}.
$$

Also,

$$
p(x(1 + x^*)^{-1})^2 = \|x_p (1 + x_p x_p)^{-2}\|_p^2
= \|\Psi_p(f)\|_p
\leq \sup \{|f(\lambda)| : \lambda \in \text{sp}_p(y)\}
= \sup \{\frac{\lambda}{(1 + \lambda)^2} : \lambda \in \text{sp}_p(y)\}
\leq 1.
$$

Thus, taking supremum over all $p \in S(A)$, it follows that $x(1 + x^*)^{-1} \in b(A)$ with $\|x(1 + x^*)^{-1}\|_\infty \leq 1$ for all $x \in A$.

For $x \in A$, let $x_n = x(1 + \frac{1}{n} x^*)^{-1} = \sqrt{n} \left(1 + \left(\frac{1}{n}\right)\left(\frac{y}{y_n} \cdot \frac{y}{y_n}\right)^{-1}\right)$. Then $x_n \in b(A)$. Also,
\[ x - x_n = x - x(1 + \frac{1}{n^* x})^{-2} \]
\[ = (x(1 + \frac{1}{n^* x}) - x)(1 + \frac{1}{n^* x})^{-1} \]
\[ = \frac{1}{n^* x} x(1 + \frac{1}{n^* x})^{-1} \]
\[ = \left( \frac{X}{\sqrt{n}} \left(1 + \frac{X}{\sqrt{n}} \right)^{-1} \right). \]

Hence for any \( p \in S(A) \),

\[ p(x - x_n) \leq \frac{1}{\sqrt{n}} p(x x^*) p(1 + \frac{x^* x}{\sqrt{n} \sqrt{n}}) \]
\[ \leq \frac{1}{\sqrt{n}} p(x)^2 \to 0 \text{ as } n \to \infty. \]

Thus \( (b(A), \|\cdot\|_\infty) \) is continuously embedded in \( A \) with the sequentially dense range.

Conversely, let \( A \) be a complete \( \text{lim} \ C^* \) algebra and \( B \) be as given in the hypothesis. Let \( P_1 \in \mathcal{C}_2(A) \) be directed, \( x \in A \) and \( (x_i)_{i \in I} \) be a net in \( B \) satisfying \( x_i \to x \). Then \( x_i^* \to x^* \) and \( x_i^* x_i \to x^* x \). Since \( (B, \|\cdot\|) \) is a \( C^* \)-algebra, \( y_i = (1 + x_i^* x_i)^{-2} \in B \) with \( \|y_i\| \leq 1 \) for each \( i \in I \). Using the boundedness of \( K \) in \( A \), for any \( q \in P_1 \), we get a constant \( M_q > 0 \) satisfying

\[ \text{i.i.} q(x) \leq M_q \text{ for all } x \in K. \]

Thus, for any \( q \in P_1 \),
\[ q(y_i - y_j) = q((1 + x_i^*x_i)^{-1}(x_j^*x_j - x_i^*x_i)(1 + x_j^*x_j)^{-1}) \]
\[ \leq M^2q(x_j^*x_j - x_i^*x_i) \to 0, \]

showing that \((y_i)\) is a Cauchy net in \(A\). Let \(y = \lim_{i} y_i\) in \(A\).

Since \(1 + x_i^*x_i \to 1 + x^*x\) and since \((1 + x_i^*x_i)^{-1} = y_i \to y\)
in \(A\), by the continuity of the inversion map in complete IUC algebras, \(y = (1 + x^*x)^{-1}\). Thus for any \(x \in A\), \(1 + x^*x \in A^{-1}\) and hence \(A\) is symmetric.

Let us note that if \(K\) is closed, then since
\[ ||x_i(1 + x_i^*x_i)^{-2}||^2 = ||x_i^*x_i(1 + x_i^*x_i)^{-2}|| \leq 1, \]
(by the standard functional calculus in \(C^*\)-algebras), \(x_i(1 + x_i^*x_i)^{-2} \in K\), giving \(x(1 + x^*x)^{-1} \in \overline{K} = K\).

Now, since the closure \(\overline{K}\) of \(K\) is in \(\mathcal{B}(A)\), by Lemma 1.1.4(3), there exists \(P \in \mathcal{S}_2(A)\) such that \(\overline{K} \subseteq S_P\). Also, replacing \(P\) by \(\hat{P}\) as defined in Proposition 0.1.15, we assume \(P\) to be directed. This gives \(B \subseteq B_P = \{x \in A : \nu(A,P,x) < \omega\}\). Thus we have,

\[ 1.1.9(b) \quad |x|_{S_P} \leq |x|_{K} = ||x|| \text{ for all } x \in B. \]

Since \(P \in \mathcal{S}_2(A)\), \(S_P \in \mathcal{B}(\tau)\); consequently, by Lemma 0.1.25,
\((B_P,|\cdot|_{S_P}) = (A(S_P),|\cdot|_{S_P})\) is a Banach \(^*\)-algebra. In particular, \(|\cdot|_{S_P}\) is \(^*\)-preserving on \(B_P\). Further, since \(B_P\) contains the \(C^*\)-algebra \(B\), by [65, Theorem 4.8.3], \(r_P(x) = r_{B_P}(x)\) for all
\( x \in B \), where \( r_B(\cdot) \) and \( r_{B_p}(\cdot) \) denote the spectral radius in \( B \) and \( B_p \) respectively. Thus by 1.1.9(b), for \( x \in B \),

\[
||x||^2 = ||x^*x|| = r_B(x^*x) = r_{B_p}(x^*x)
\]

\[
\leq ||x^*||_{B_p} \leq ||x^*||_{B_p} ||x||_{B_p} \leq ||x||_{B_p} ||x||.
\]

This gives,

1.1.9(c) \[ ||x|| \leq ||x||_{B_p} \text{ for all } x \in B. \]

Thus 1.1.9(b) and 1.1.9(c) give,

1.1.9(d) \[ ||x|| = ||x||_{B_p} \text{ for all } x \in B. \]

Now, let \( h \in A^h \) and \( (z_i)_{i \in I} \) be a net in \( B \) such that \( z_i \rightarrow h \). Taking \( h_i = \frac{z_i + z_i^*}{2} \), we get \( h_i \in B^h \), and \( h_i \rightarrow h \).

Since \( B \) is a \( C^\# \)-algebra, by [20, Proposition 12.20], \( B^h = H(B, ||\cdot||) \). By Theorem 1.1.6(1), \( V(A, P, h_i) \subseteq V(B_p, ||\cdot||_{B_p}, h_i) = V(B, ||\cdot||_{B_p}, h_i) = V(B, ||\cdot||, h_i) \subseteq \mathbb{R} \). Thus by Lemma 1.1.8 and by the fact that \( h_i \in H(A, P) \), we have \( h \in H(A, P) \), showing that \( A^h \subset H(A, P) \). Thus \( A = A^h + iA^h \subset H(A, P) + iH(A, P) \), a direct sum because \( H(A, P) \cap iH(A, P) = \{0\} \). Thus by Theorem 1.1.6(2), \( A \) is a pro-\( C^\# \)-algebra with the involution \( x = h + ik \mapsto h - ik = x^* \), \( (h, k \in H(A, P)) \) and \( P \subseteq S(A) \). Since \( ^* \) and \( \# \) agree on \( B \) and since \( B \) is dense in \( A \), \( ^* \) and \( \# \) agree on \( A \).
Finally, suppose that $K$ is closed. Then $B = A(K)$ is a $C^*$-algebra with $||\cdot|| = ||\cdot||_{s_p}$. Now, let $h \in B^h_p$. Then $h \in A^h$. Also, as noted earlier, $h_n = h(1 + \frac{1}{n}h^2)^{-1} \in B^h$ with $||h(1 + h^2)^{-1}|| \leq 1$ and $|h - h_n|_{s_p} \leq \frac{1}{\sqrt{n}} |h|^2 \to 0$ as $n \to \infty$, so that, $h \in B^h$. Hence $B^h = B^h_p$. Now, by Theorem 1.1.6(1) and Proposition 0.1.34, $B = B^h_p = \{ x \in A : v(A,P,x) < \infty \} = \{ x \in A : \sup_{p \in P} p(x) < \infty \} = b(A)$, which completes the proof.

This theorem is applied to obtain a number of other characterizations of pro-$C^*$-algebras. Next, we investigate a non-involutive case.

1.2. A non-involutive case

Let us first note that if $A$ is a pro-$C^*$-algebra, then for any $P \in \mathcal{S}_2(A)$, $B^h_p = b(A)$ by Proposition 0.1.34 and Definition 0.1.35. It is proved in [45, Theorem 6], that an Lmc algebra $A$ with a directed $P \in \mathcal{S}_2(A)$ is a pro-$C^*$-algebra if and only if $B^h_p$ is a $C^*$-algebra. The following result improves this. In fact, [45, Theorem 6] follows as a corollary (Corollary 1.2.2).
Theorem 1.2.1 [15]. Let $A$ be a complete lcm algebra with a directed $P \in \mathcal{S}(A)$. Then $A$ is a pro-$C^*$-algebra with some involution and with $P \subseteq S(A)$ if and only if there exists a subalgebra $B$ of $B_p$ satisfying the following conditions.

(i) $B$ is a $C^*$-algebra with some involution and some norm $||\cdot||$.

(ii) $(B, ||\cdot||) \rightarrow A$ is the continuous inclusion with the dense range.

(iii) $|x|_p \leq ||x||$ for all $x \in B$.

Proof. The necessity is obvious by taking $B = B_p = \mathfrak{b}(A)$. Conversely, suppose that there exists a subalgebra $B$ of $B_p$ satisfying (i), (ii) and (iii) above. First, we show that $|x^*|_p = |x|_p$ for each $x \in B$. Since $B_p$ is a Banach algebra containing the $C^*$-algebra $B$, by [65, Theorem 4.8.3], $r_p(x) = r_p(x)$ for all $x \in B$. Hence, for all $x \in B^h$, $||x|| = r_p(x) = r_p(x) \leq |x|_p$, and consequently, for any $y \in B$,

$||y||^2 = ||y^*y|| \leq |y^*y|_p \leq |y^*|_p |y|_p \leq |y|_p ||y||$. Thus $|y|_p \leq ||y|| \leq |y|_p$ for all $y \in B$. Now, by the symmetry, $|y^*|_p \leq |y|_p$, showing that $|x^*|_p = |x|_p$ for all $x \in B$. This gives $||x||^2 = ||x^*x|| \leq |x^*x|_p \leq |x^*|_p |x|_p = |x|_p^2$, showing that $||x|| = |x|_p$ for all $x \in B$. Thus, as in the
proof of Theorem 1.1.9, it follows that \( A = \mathcal{H}(A, P) \ominus i\mathcal{H}(A, P) \).

So, by Theorem 1.1.6(2), \( A \) is a pro-\( C^\pi \)-algebra with \( P \subseteq S(A) \).

This completes the proof.

**Corollary 1.2.2** ([45, Theorem 6]). Let \( A \) be a complete lmc algebra. Let \( P \in \mathcal{E}(A) \). Then the following are equivalent.

1. There exists an involution \( * \) on \( A \), making \( A \) a pro-\( C^\pi \)-algebra with \( P \subseteq S(A) \).
2. \( (B_P, |^*|_{S_P}) \) is a \( C^\pi \)-algebra.

We obtain one more characterization of pro-\( C^\pi \)-algebras in the absence of the involution (Theorem 1.3.11), but the result also deals with the topological dual of \( A \) and hence fits more appropriately in the following section.

**1.3. Some dual characterizations**

It is known [46] that a unital Banach \( ^\pi \)-algebra \( A \) is a \( C^\pi \)-algebra if and only if every continuous hermitian linear functional on \( A \) is a difference of two continuous positive linear functionals on \( A \). We obtain an analogue of this result (Theorem 1.3.2).
Definition 1.3.1. Let $A$ be a complete lmc *algebra. Then $S = \{x \in A : f(x^*x) \leq f(1) \text{ for all } f \in P(A)\} = \{x \in A : ||\pi_f(x)|| \leq 1 \text{ for all } f \in P(A)\}$ is called the pro-$C^*$-ball of $A$.

Theorem 1.3.2 [15]. Let $A$ be a complete lmc *algebra. Then $A$ is a pro-$C^*$-algebra if and only if the following hold.

(i) $A$ is hermitian; and

(ii) every continuous hermitian linear functional on $A$ is a difference of two continuous positive linear functionals on $A$.

Proof. Suppose (i) and (ii) above hold. We first show that the pro-$C^*$-ball $S$ of $A$ is the greatest member of $B^*(\mathcal{H})$. Let $x \in S$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. Then for $f \in P(A)$, $||\pi_f(\lambda x)|| = |\lambda||\pi_f(x)|| \leq 1$. Hence $S$ is balanced. Also, for $x \in A$ and $f \in P(A)$, $||\pi_f(x^*)|| = ||\pi_f(x^*)|| = ||\pi_f(x)|| \leq 1$, showing that $S$ is a * preserving subset of $A$. Now, let $x, y \in S$ and let $t \in [0, 1]$, then for any $f \in P(A)$, by Cauchy-Schwarz inequality,

$$f((tx + (1 - t)y)^* (tx + (1 - t)y))$$

$$= t^2f(x^*x) + (1 - t)^2f(y^*y) + t(1 - t)f(x^*y) + f(y^*x))$$

$$\leq t^2f(x^*x) + (1 - t)^2f(y^*y) + t(1 - t)||x^*y|| + ||f(y^*x)||$$

$$\leq t^2f(x^*x) + (1 - t)^2f(y^*y) + 2t(1 - t)f(x^*x)^{1/2}f(y^*y)^{1/2}$$

$$= (tf(x^*x)^{1/2} + (1 - t)f(y^*y)^{1/2})^2$$

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Thus \( S \) is convex. Now, let \( x, y \in S \), \( u = xy \) and \( f \in P(A) \). Then \( f_y(z) = f(y^* z y) \), \( (z \in A) \), defines an element \( f_y \) of \( P(A) \). And \( f(u^* u) = f_y(x^* x) \leq f_y(1) = f(y^* y) \leq f(1) \), which shows that \( u \in S \). Thus \( S \) is an idempotent. Also, by the assumption (ii), \( A' \) is a linear span of \( P(A) \); hence \( S \) is \( \sigma(A, A') \)-bounded, and hence is bounded in the topology of \( A \). Now, let \( x \in \bar{S} \), the closure of \( S \) in \( A \). So, there exists a net \( (x_i) \) in \( S \) such that \( x_i \to x \) in \( A \). But then \( x_i^* x_i \to x^* x \), and consequently, for any \( f \in P(A) \), \( f(x_i^* x_i) = \lim f(x_i^* x_i) \leq f(1) \), showing that \( x \in S \).

Thus \( S \) is closed in \( A \). All this gives \( S \in \mathcal{B}^*(\tau) \). Now, let \( B \in \mathcal{B}^*(\tau) \), \( x \in B \) and \( f \in P(A) \). Then the automatic continuity of the positive linear functional \( f \) on the Banach \(^*\)-algebra \( (A(B), \| \cdot \|_B) \) implies that \( f(x^* x) \leq \| f \| \| x^* x \|_B \leq f(1) \| x \|_B^2 \leq f(1) \) since \( x \in B \). Thus \( x \in S \). Hence \( S \) is the greatest member of \( \mathcal{B}^*(\tau) \).

Next, we show that \((A(S), \| \cdot \|_S)\) is a \( C^* \)-algebra. Since \( S \in \mathcal{B}^*(\tau) \), by Lemma 0.1.25, \((A(S), \| \cdot \|_S)\) is a Banach \(^*\)-algebra. Let \( x \in A(S) \). Then \( \| x^* x \|_S \leq \| x \|_S \| x \|_S = \| x \|_S^2 \). On the other hand, let \( y = \frac{x^* x}{\| x^* x \|_S} \) (assuming that \( x \neq 0 \)). Then \( y \in S \).

So, \( f(y^* y) \leq f(1) \) and hence \( f \left( \frac{x^* x}{\| x^* x \|_S} \right) \leq f(1) \) for all \( f \in P(A) \).
This gives \( \frac{x}{|x^* x|^2} \in S \), and hence \( \frac{|x|}{|x^* x|^2} \leq 1 \), giving \( |x|^2 \leq |x^* x| \). Thus \( |\cdot| \) is a \( C^* \)-norm making \( (A(S), |\cdot|) \) a \( C^* \)-algebra. Also, by the assumption (i), for each \( h \in A^h \) and for each \( f \in P(A) \), \( \| (1 + \pi_i (h)^2)^{-1} \| \leq 1 \). Thus \( (1 + h^2)^{-4} \in S \) and \( h^2 (1 + h^2)^{-2} = (1 + h^2)^{-4} - (1 + h^2)^{-2} \in A(S) \) with \( |h^2 (1 + h^2)^{-2}| \leq 1 \). Hence \( |h (1 + h^2)^{-4}| \leq 1 \). As in the proof of Theorem 1.1.9, \( \lim_{n \to \infty} h (1 + \frac{1}{n} h^2)^{-4} = h \) in \( A \) with \( h (1 + \frac{1}{n} h^2)^{-4} \in A(S) \). Thus \( (A(S), |\cdot|) \) is a \( C^* \)-algebra which is dense in \( A \) and hence, by Theorem 1.1.9, \( A \) is a pro-\( C^* \)-algebra.

Conversely, a pro-\( C^* \)-algebra is always hermitian. Let \( f \in A^h \). Then for some \( q \in S(A) \) and some scalar \( M > 0 \), \( |f(x)| \leq Mq(x) \). Thus \( f \in A'^{(q)} \) and so, as in Theorem 0.1.31(3), \( f_q(x) = f(x), (x_q \in A_q) \), defines a hermitian linear functional on the \( C^* \)-algebra \( A_q \). Now, by the \( C^* \)-theory [46], there exist \( g_1, g_2 \in P(A_q) \) such that \( f_q = g_1 - g_2 \). But again, by Theorem 0.1.31(3), \( G_i(x) = g_i(x)_, (x \in A), (i = 1, 2), \) defines elements of \( P(A) \). It follows now, by the simple verification, that \( f = G_1 - G_2 \), a difference of two continuous positive linear functionals on \( A \). This completes the proof.
Definition 1.3.3. Let $A$ be a complete lmc \* algebra. An $f \in \mathcal{P}(A)$ is said to be representable if there exist a representation $\pi \in \mathcal{R}(A)$ and a cyclic vector $\xi \in H_\pi$ such that $f(x) = \langle \pi(x)\xi, \xi \rangle$ for all $x \in A$.

The following proposition is a particular case of [8, Theorem 2.2].

Proposition 1.3.4. Let $A$ be a complete lmc \* algebra and $f$ be a continuous positive linear functional on $A$. Then the following are equivalent.

$(1)$ $f$ is representable.

$(2)$ There exists a continuous positive linear functional $f_1$ on $A_1$, the unitization of $A$, such that $f_1|A = f$.

$(3)$ $|f(x)|^2 \leq Kf(x^*x)$ for all $x \in A$ and for some constant $K > 0$.

The following theorem is the non-unital version of Theorem 1.3.2.

Theorem 1.3.5. Let $A$ be a complete lmc \* algebra without unit. Then $A$ is a pro-C\* algebra if and only if the following hold.

$(i)$ $A$ is hermitian; and
(ii)' every continuous hermitian linear functional on $A$ is a difference of two continuous representable positive linear functionals on $A$.

Proof. Suppose (i)' and (ii)' hold. Let $A_1$ be the unitization of $A$ and $f_1$ be a continuous hermitian linear functional on $A_1$. Then $f_1 = f_1|A$ is a continuous hermitian linear functional on $A$ and hence $f = g - h$ on $A$ for some continuous representable positive linear functionals $g$ and $h$ on $A$. Let $K_g$ and $K_h$ be two constants such that for all $x \in A$, $|g(x)|^2 \leq K_g g(x^*x)$ and $|h(x)|^2 \leq K_h h(x^*x)$. Let $K = \max\{K_g, K_h\}$. Since $f_1$ is hermitian, $f_1(1) \in \mathbb{R}$. Suppose $f_1(1) \geq 0$. Then taking $K_1 = K + f(1) \geq K$, $|g(x^*x)| \leq K g(x^*x) \leq (K + f(1)) g(x^*x) = K_1 g(x^*x)$. For $x + \lambda 1 \in A_1$, we define $g_1(x + \lambda 1) = g(x) + K_1 \lambda$ and $h_1(x + \lambda 1) = h(x) + K \lambda$. Then,

$$g_1(x + \lambda 1) - h_1(x + \lambda 1) = g(x) - h(x) + f_1(\lambda 1)$$

$$= f(x) + f_1(\lambda 1)$$

$$= f_1(x) + f_1(\lambda 1)$$

$$= f_1(x + \lambda 1).$$

Similarly, if $f_1(1) < 0$, then replacing $K$ by $K_2 = K - f_1(1)$ in the definition of $h_1$ and replacing $K_1$ by $K$ in the definition of $g_1$, it follows that $f_1 = g_1 - h_1$, a difference of two continuous representable positive linear functionals on $A_1$. Thus by Theorem 1.3.2, $A_1$ is a pro-$C^\ast$-algebra and hence $A$ is a pro-$C^\ast$-algebra.
The converse is easy in the light of the fact that each continuous hermitian linear functional \( f \) on \( A \) can be extended as a continuous hermitian linear functional \( f_1 \) on \( A_1 \) defined by \( f_1(x + \lambda 1) = f(x) + \lambda, ((x + \lambda 1) \in A_1) \).

**Remark 1.3.6.** A decomposition stronger than that in Theorem 1.3.2(ii) is available in pro-\( C^\# \)-algebras, which alone is sufficient to characterize pro-\( C^\# \)-algebras. This is precisely what we show in the following proposition.

**Proposition 1.3.7 ([15],[49]).** Let \( A \) be a complete \( \text{Inc} \) algebra. Let \( P \in \mathcal{S}_2(A) \) be directed. Then \( A \) is a pro-\( C^\# \)-algebra with \( P \in \mathcal{S}_3(A) \) if and only if for each \( p \in P \) and each hermitian linear functional \( f \in A'(p) \), there exist \( g_1, g_2 \in P(A)_p \) such that \( f = g_1 - g_2 \).

**Proof.** Let \( p \in P \) and \( f_p \in A'^h_p \). Then by Theorem 0.1.31(3), \( f \in A'(p) \) and \( f \) is hermitian. So, there exist \( g_1, g_2 \in P(A)_p \) such that \( f = g_1 - g_2 \). But then again, by Theorem 0.1.31(3), \( g_{1p}, g_{2p} \in P(A_p) \) and \( f_p = g_{1p} - g_{2p} \), which makes \( A_p \) a \( C^\# \)-algebra by [46]. Thus \( A \) is a pro-\( C^\# \)-algebra with \( P \in \mathcal{S}_3(A) \).

Conversely, suppose that \( A \) is a pro-\( C^\# \)-algebra. Let \( p \in P \in \mathcal{S}_3(A) \) and \( f \in A'(p) \) be a hermitian linear functional on \( A \). So, by Theorem 0.1.31(3), \( f_p \in A'^h_p \). Now, by [46], there exist \( g_{1p}, g_{2p} \in P(A_p) \) such that \( f_p = g_{1p} - g_{2p} \).
Let us define \( g_i(x) = g_{ip}(x_p), \) \( (x \in A, \ i = 1, 2). \) Then one more application of Theorem 0.1.31(3) reveals that \( g_i \in A'(p), \) \( (i = 1, 2), \) are positive and \( f = g_1 - g_2, \) which completes the proof.

Before we proceed further, we note that the fact that the pro-\( C^* \)-ball of a pro-\( C^* \)-algebra \( A \) is the greatest member of \( B^*(r) \) has the following implication, which further justifies calling the elements of \( b(A) \) bounded.

**Proposition 1.3.8** ([15]). Let \( A \) be a pro-\( C^* \)-algebra. Suppose \( B \) is a *subalgebra of \( A \) which is also a Banach algebra under some norm \( ||·|| \). Then \( B \subset b(A) \) continuously.

**Proof.** Without loss of generality, we assume that \( 1 \in B. \) Let \( U = \{x \in B : ||x|| \leq 1\}. \) Then \( U \) is an absolutely convex, *preserving, idempotent subset of \( A \) containing \( 1. \) Now, let \( f \in P(A). \) Then by [20, §37], \( f|_B \) is a \( ||·|| \)-continuous positive linear functional on \( B \) satisfying \( |f|_B(x)| \leq f(1) \) for all \( x \in U. \) Hence as in the proof of Theorem 1.3.2, \( U \) is \( \sigma(A, A') \)-bounded, and so, \( U \) is bounded in the topology of \( A. \) Thus \( U \in \mathcal{B}(r), \) giving \( U \subset S. \) Now, it is obvious that on \( B, \)

\[
||·||_\infty \leq ||·||_E \leq ||·||_U = ||·||. \]

This completes the proof.

**Lemma 1.3.9.** Let \( A \) be a complete lmc algebra and \( P \in \mathcal{S}_1(A) \) be directed. Then \( A' \) is the linear span of \( D(A,P,1). \)
Proof. Let \( f \in A' \). So, there exists \( p \in P \) such that \( f \in A' (p) \).

So, by (19, p.100), there exist \( s \) in \([0,\infty)\) and \( g_{ip} \) in \( D(A_p,||\cdot||_p,1_p) \), \( (i = 1,2,3,4) \), such that \( f_p = s_i g_{ip} - s_i g_{ip} \) + \( i(s^*_ig_{ip} - s^*_ig_{ip}) \). Let \( g_i(x) = g_{ip}(x_p) \), \( (x \in A, i = 1,2,3,4) \).

Then by Remark 1.1.3(1), \( g_i \in D_p \), and hence \( g_i \in D(A,P,i) \) for each \( i = 1,2,3,4 \). Also, it is easily seen that \( f = s_1 g_1 - s_2 g_2 + i(s^*_1g_1 - s^*_2g_2) \), which completes the proof.

Now, we come to the final result of this section which, being a non-involutive version, also supplements the Vidav-Palmer Theorem. It generalizes the Moore dual characterization of \( C^* \)-algebras (19, §31), that is stated below.

Theorem 1.3.10 (19). Let \( (A,||\cdot||) \) be a Banach algebra. Then \( (A,||\cdot||) \) is a \( C^* \)-algebra with some involution defined on it if and only if \( A' = \text{H}(A') \oplus \text{iH}(A') \).

Theorem 1.3.11. Let \( A \) be a complete lmc algebra and \( P \subseteq \mathcal{B}(A) \) be directed. Then the following are equivalent.

1. There exists an involution on \( A \) making \( A \) a pro-\( C^* \)-algebra with \( P \subseteq \mathcal{S}(A) \).

2. \( A' = \text{H}(\mathcal{C}(A,P)' ) \oplus \text{iH}(\mathcal{C}(A,P)' ) \).

3. \( \mathcal{H}(\mathcal{C}(A,P)' ) \cap \text{iH}(\mathcal{C}(A,P)' ) = \{0\} \).
Proof. As can be seen from the proof of Lemma 1.3.9, $A' = H(A,Py) + iH(A,Py)$ is always true. Thus (2) $\iff$ (3) becomes obvious.

(1) $\Rightarrow$ (3). Suppose $f \in H(A,Py) \cap iH(A,Py)$. Since $A$ is a pro-$C^*$-algebra, $H(A,Py) = A^{h}$. Thus $f \in A^{h} \cap iA^{h}$. So, $f(x) = 0$ for all $x \in A$. This proves (3).

(3) $\Rightarrow$ (1). Let $h = sf - tg \in H(A,Py)$ with $f, g \in D(A,P,1)$ and $s, t > 0$. So, there exist $p_{1}, p_{2} \in P$ such that $f \in D_{p_{1}}, g \in D_{p_{2}}$. Since $P$ is directed, there exists $p \in P$ such that $p_{1} \leq p, p_{2} \leq p$. But then $f, g \in D_{p}$. Thus $h \in H^{p}$. This shows that $H(A,Py) = \bigcup_{p \in P} H^{p}$. Now, (3) implies that $H^{p} \cap iH^{p} = \{0\}$ for all $p \in P$ and hence by Remark 1.1.3(ii), $H(A') \cap iH(A') = \{0\}$. Applying [19, Theorem 31.10], it follows that for each $p \in P$, $(A_{p}, || \cdot ||_{p})$ is a $C^*$-algebra with the involution defined by the decomposition $A_{p} = H(A_{p}) \oplus iH(A_{p})$. The involution on the $C^*$-algebras $A_{p}$ are compatible with the inverse system $\{A_{p}\}$ and hence (1) follows. This completes the proof.

1.4. Local structure

A Banach algebra $A$ is called a $C^*$-equivalent algebra if $A$ is a $C^*$-algebra under some equivalent norm on $A$. It is shown in [26] that a Banach algebra $A$ is $C^*$-equivalent if and only if for each $h = h^{\infty}$ in $A$, the closed subalgebra...
generated by h is $C^\ast$-equivalent, which is true if and only if every maximal abelian $\ast$-subalgebra of A is $C^\ast$-equivalent. Such a phenomenon does not occur in pro-$C^\ast$-algebras (Example 1.5.1). However, we have the following theorem as an application of Proposition 1.3.8.

Theorem 1.4.1 [15]. Let A be a pro-$C^\ast$-algebra.

(1) If every maximal abelian $\ast$-subalgebra of A is a $C^\ast$-algebra, then A is a $C^\ast$-algebra.

(2) If every maximal abelian $\ast$-subalgebra of A is finite dimensional, then A is finite dimensional.

Proof. (1) Let $\mathfrak{M}$ be the collection of all maximal abelian $\ast$-subalgebras of A. Then by Proposition 1.3.8, each $M \in \mathfrak{M}$ is contained in $b(A)$, since each $M \in \mathfrak{M}$ is a $C^\ast$-algebra. Thus $A = b(A)$ algebraically. In view of [62, Proposition 1.14], it is enough to prove that A is a $Q$-algebra. For each $M \in \mathfrak{M}$, let $G(M) = \{x \in M : x$ is invertible in A and $x^{-1} \in M\} = \{x \in A : x$ is invertible in A and $x^{-1} \in M\}$ by the maximality of M. The set $Y(M) = \{x \in A : x$ has no inverse in M\} is closed and so $X(M) = A^h \cap Y(M)$ is closed in A. Thus $E = \{x \in A^h : x \in A^{-1}\} = \bigcap \{X(M) : M \in \mathfrak{M}\}$ is closed. Now, let $(x_\iota)_{\iota \in I}$ be a net of singular elements of A such that $x_\iota \to x$. Then $x_\iota^* x_\iota \to x^* x$ and $x_\iota x_\iota^* \to x x^*$. By the singularity of $x_\iota$, for each $i$, either
It is singular. Let $I_{x} = \{i \in I : x_{i}^{x} is not invertible\}$ and $I_{2} = \{i \in I : x_{i}x_{i}^{x} is not invertible\}$. Then $I = I_{1} \cup I_{2}$ and hence at least one of $I_{1}$ and $I_{2}$ is cofinal in $I$.

Without loss of generality, we assume that $I_{1}$ is cofinal in $I$. But then $(x_{i})_{i \in I_{1}}$ is a subnet of $(x_{i})_{i \in I}$. Thus $(x_{i}x_{i}^{x})_{i \in I}$ is a net in $E$, which converges to $x^{x}x$, showing that $x^{x}x \in E$. Thus $x^{x}x \in A^{-1}$; so $x \in A^{-1}$. Thus the set of all singular elements of $A$ is closed, i.e., $A^{-1}$ is open and hence $A$ is an $\mathbb{Q}$-algebra.

(2) As in the proof of (1), each $M \in \mathbb{R}$ is contained in $b(A)$ and so, each maximal commutative subalgebra of $b(A)$ is finite dimensional. Thus $b(A)$ is finite dimensional. Hence, by the denseness of $b(A)$ in $A$, $A = b(A)$ is finite dimensional. This completes the proof.

1.5. Example

The following example shows that there exists a complete $\mathcal{L}$-algebra $A$, in which, for each $h = h^{x} \in A$, the algebra generated by $h$ is $C^{*}$-equivalent but $A$ is not a pro-$C^{*}$-algebra.

Example 1.5.1. Let $A$ be the algebra of all continuous functions on $\mathbb{R}$ with compact support, with the finest locally convex topology on $A$, for which the inclusion mappings
\( \text{id: } A \rightarrow A \) become continuous, where \( A_n = \{ f \in A: \text{supp}(f) \subseteq [-n, n] \}, \ (n = 1, 2, \ldots) \). The topology of \( A_n \) is given by the norm \( \| f \|_n = \sup \{|f(t)| : t \in [-n, n]\} \). In fact, \( A \) is a complete local algebra with a local base at 0 consisting of idempotent sets of the form \( B(\{r_n\}) = \{ f \in A : \| f \|_n < r_n, \ n = 1, 2, \ldots \} \), where \( (r_n) \) is a non-decreasing sequence of positive real numbers [61]. \( A \) is not a pro-\( C^* \)-algebra as \( A \) is a \( \mathbb{Q} \)-algebra which is not normable [61] and a \( \mathbb{Q} \) pro-\( C^* \)-algebra must be a \( C^* \)-algebra [62, Proposition 1.14].

1.6. The bounded part

As a byproduct of all that we have developed so far, we obtain certain description of the bounded part of a pro-\( C^* \)-algebra \( A \).

Proposition 1.6.1. Let \( A \) be a pro-\( C^* \)-algebra.

1. \( b(A) = \{ x \in A : V(A, S(A), x) \text{ is bounded} \} \).

2. \( b(A) = \{ \lambda x : \lambda \in \mathbb{C} \text{ and } f(x^*x) \leq f(1) \text{ for all } f \in P(A) \} \).

3. \( b(A) = \bigcup \{ A(B) : B \in \mathcal{B}(\tau) \} \).

Proof. \( 1 \) By Proposition 0.1.15(3), \( S(A) \) is directed. So, by Definition 1.1.2, \( b(A) = \{ x \in A : \sup p(x) < \infty \} = A(S(S(A))) \).

\( = B_{\mathcal{S}(A)} = \{ x \in A : V(A, S(A), x) \text{ is bounded} \} \).
(2) It can be seen from the proof of Theorem 1.3.2, that $S = \{ x \in A : f(x^*x) \leq f(1) \text{ for all } f \in P(A) \}$ is the greatest member of $\mathcal{B}(\tau)$ and that $(A(S), |\cdot|_S)$ is a $C^*$-algebra which is dense in $A$. Since $S = \{ x \in A(S) : |x|_S \leq 1 \}$ is closed in $A$, the result follows by taking $K = S$ in Theorem 1.1.9.

(3) For $B \in \mathcal{B}(\tau)$, $A(B) \subset A(S) = b(A)$ and hence $\bigcup \{ A(B) : B \in \mathcal{B}(\tau) \} \subset A(S) = b(A)$. Since $S \in \mathcal{B}(\tau)$, the reverse inclusion follows.

Proposition 1.6.2. Let $A$ be a pro-$C^*$-algebra and $P \in \mathcal{C}_1(A)$. Then $b(A)$ is the inverse limit (in the category $\mathcal{C}_1^*$) of the inverse system $\{ A_p : p \in P \}$ with the bonding maps $\{ x_{p,q} : p \geq q \}$.

Proof. Let $\chi_p : A \rightarrow A_p$ be the usual projection map and $\phi_p = \chi_p |_{b(A)}$ for each $p \in P$. Let $\mathfrak{S}_p$ be the category associated with the inverse system in $\mathcal{C}_1^*$. More explicitly, the objects of $\mathfrak{S}_p$ are of the form $(B, \{ h_p \})$, where $B$ is a $C^*$-algebra and $h_p : B \rightarrow A_p$ are continuous homomorphisms for all $p \in P$ such that the following diagram commutes for each pair $(p, q)$ of seminorms in $P$ with $p \geq q$.

![Diagram](image)

(Figure 6)
Thus \((b(A), \{\phi_p\})\) is an object of \(\mathcal{F}_p\). Now, let \((B, \{h_p\})\) be an object of \(\mathcal{F}_p\). So, \(B\) is a \(C^*\)-algebra and for each \(p \in P\), \(h_p : B \to A\) is a \(^*\)homomorphism such that the above diagram commutes. But every \(C^*\)-algebra is a pro-\(C^*\)-algebra and hence \(B\) is a pro-\(C^*\)-algebra with the family \(\{h_p\}_p\) of \(^*\)homomorphisms \(h_p : B \to A\) such that the above diagram commutes. Since \(A = \varprojlim A_p\), there exists a unique \(^*\)homomorphism \(\psi : B \to A\) such that the following diagram commutes.

![Diagram](image)

(Figure 7)

Thus we need only to show that \(\psi(B) \subseteq b(A)\). Let \(b \in B\). Then for \(p \in P\), \(p(\psi(b)) = ||\chi_p(\psi(b))||_p = ||h_p(b)||_p \leq ||b||\). The last inequality holds because \(h_p\), being a \(^*\)homomorphism between two \(C^*\)-algebras, is norm-decreasing. Thus taking supremum on the left hand side of the above over all \(p \in P\), we get \(||\psi(b)||_\infty \leq ||b||\), showing that \(\psi(B) \subseteq b(A)\). This completes the proof.