CHAPTER 0

INTRODUCTION AND PRELIMINARIES

0.1. Preliminaries

A topological algebra $A$ is a linear associative algebra, over the field $\mathbb{C}$ of all complex scalars, with a Hausdorff topology $\tau$ making $A$ a topological vector space such that the multiplication is jointly continuous. Unless otherwise reference is made to the contrary, we assume that $A$ is unital; the unit (also called the identity) of $A$ is denoted by $1$. A mapping $\ast : A \to A$ ($x \in A \mapsto x^* \in A$) is called an involution if for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^* x^*$,

(iii) $(x + y)^* = x^* + y^*$; and

(iv) $(\lambda x)^* = \overline{\lambda} x^*$.

A topological algebra with a continuous involution is called a topological $\ast$-algebra.

Definitions 0.1.1. Let $A$ be a topological algebra and $x \in A$.

(i) $x$ is called invertible if there exists $y \in A$, denoted by $x^{-1}$, such that $xy = yx = 1$. The set of all invertible
elements of \( A \) is denoted by \( A^{-1} \).

(ii) \( x \) is called quasiinvertible if there exists \( y \in A \), denoted by \( x_q \), satisfying \( x \circ y = y \circ x = 0 \), where the quasi-multiplication \( \circ \) is defined by \( x \circ y = x + y - xy \), \( (x, y \in A) \). \( A_q \) denotes the set of all quasiinvertible elements of \( A \).

Further, suppose that \( A \) is a topological \( ^* \) algebra.

(iii) \( x \) is called selfadjoint (respectively, normal) if \( x = x^* \) (respectively, \( x^* x = xx^* \)). The set of all selfadjoint elements of \( A \) is denoted by \( A^* \).

(iv) \( x \) is called positive if there exists \( y \in A \) satisfying \( x = y^* y \).

(v) \( x \) is called unitary if \( x^* x = xx^* = 1 \).

Definitions 0.1.2. A topological algebra \( A \) is called a \( Q \)-algebra if \( A^{-1} \) (equivalently, \( A_\omega \)) is open in \( A \). A topological \( ^* \) algebra \( A \) is called symmetric if \( 1 + x x \in A^{-1} \) (equivalently, \( -x^* x \in A_\omega \)) for all \( x \in A \). \( A \) is called hermitian if \( 1 + h^2 \in A^{-1} \) (equivalently, \( -h^2 \in A_\omega \)) for all selfadjoint elements \( h \in A \).

Definitions 0.1.3. Let \( A \) be a topological algebra. Then for \( x \in A \), we define the spectrum of \( x \) in \( A \) to be the set

\[
\text{sp}_A(x) = \{ \lambda \in \mathbb{C} : x - \lambda 1 \in A^{-1} \} \quad \text{if } A \text{ is unital, and}
\]

\[
\text{sp}_A(x) = \{ \lambda \in \mathbb{C} : \frac{x}{\lambda} \in A_\omega \} \cup \{0\} \quad \text{if } A \text{ is not unital.}
\]
Also, the spectral radius of $x$ in $A$ is

$$r_A(x) = \sup \{ |\lambda| : \lambda \in \text{sp}_A(x) \}.$$  

Whenever there is no room for confusion, we shall denote $\text{sp}_A(x)$ by $\text{sp}(x)$ and $r_A(x)$ by $r(x)$. $A$ is said to be spectrally bounded (sb) if $r(x) < \infty$ for each $x \in A$. A topological algebra $A$ is called spectrally bounded (sb) if $r(x^*x) < \infty$ for each $x \in A$.

Definitions 0.1.4. Let $A$ be a topological algebra and $B \subseteq A$.

(i) $B$ is said to be convex if $tx + (1 - t)y \in B$ whenever $x, y \in B$ and $t \in [0, 1]$.

(ii) $B$ is called balanced if $\lambda B = \{\lambda x : x \in B\} \subseteq B$ for each $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.

(iii) $B$ is called absolutely convex if $B$ is convex, as well as, balanced (equivalently, if for all $x, y \in B$ and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leq 1$, $\lambda x + \mu y \in B$).

(iv) Let $A(B) = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}$. Then the Hinkovski functional $|\cdot|_B$ of $B$ in $A(B)$ is defined by $|x|_B = \inf \{t > 0 : x \in tB\}, (x \in A(B))$.

(v) $B$ is called bounded if for every 0-neighbourhood $U$ in $A$, there exists $t > 0$ such that $B \subseteq tU$.

(vi) $B$ is called absorbing if $\cup \{tB : t > 0\} = A$.

(vii) $B$ is called idempotent if $B \cdot B = \{xy : x, y \in B\} \subseteq B$. 

3
(viii) we denote by $\mathcal{B}(\tau)$, the collection of all closed, absolutely convex, bounded, idempotent subsets $B$ of $A$ such that $1 \in B$.

(ix) If further, $A$ is a topological algebra, then $B$ is said to be preserving if $B^\ast = \{x^\ast : x \in B\} = B$.

(x) For a topological algebra $A$,

$$\mathcal{B}^*(\tau) = \{B \in \mathcal{B}(\tau) : B^\ast = B\}.$$ 

Definitions 0.1.5. Let $A$ be an algebra. A function $p : A \to \mathbb{R}$ is called a seminorm on $A$ if

(i) $p(x) \geq 0$ for all $x \in A$,

(ii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in A$; and

(iii) $p(\lambda x) = |\lambda|p(x)$ for all $x \in A, \lambda \in \mathbb{C}$.

Let $p$ be a seminorm on $A$. Then $p$ is called a norm on $A$ if

(iv) $p(x) = 0$ implies that $x = 0$.

$p$ is said to be an algebra seminorm (or a submultiplicative seminorm) if

(v) $p(xy) \leq p(x)p(y)$ for all $x, y \in A$.

$p$ is called a unital seminorm if

(vi) $p(1) = 1$.

$p$ is said to have the square property if

(vii) $p(x^2) = p(x)^2$ for all $x \in A$. 
p is called a uniform seminorm if p is an algebra seminorm having the square property. Further, suppose that A is a *algebra. Then p is called *preserving (or a *seminorm) if

(viii) $p(x^*) = p(x)$ for all $x \in A$.

p is called a $C^*$-seminorm if p is a *preserving algebra seminorm satisfying

(ix) $p(x^*) = p(x)^2$ for all $x \in A$.

Definitions 0.1.6. Let A be a topological algebra. A family P of seminorms on A is called a calibration on A if P determines the topology of A, i.e., for a net $(x_i)$ in A, $x_i \to 0$ in A if and only if $p(x_i) \to 0$ for each $p \in P$. A calibration P on A is said to be directed if P is directed under the order relation "≤" defined on P as follows:

For $p, q \in P$, $q \leq p$ if $q(x) \leq p(x)$ for all $x \in A$.

A calibration P on A is said to be unital if each $p \in P$ is a unital seminorm on A.

Definition 0.1.7. Let A be a topological algebra and p be a seminorm on A. Then for $n \in \mathbb{N}$, $U_p(n) = \{x \in A : p(x) \leq \frac{1}{n}\}$. We shall denote $U_p(1)$ by $U_p$.

The following proposition can be proved by using [61, Lemma 1.2] and [67, Theorem 1.34].
Proposition 0.1.8. Let $A$ be an algebra and $p$ be a seminorm on $A$. Then the following hold.

(i) $U_p(n)$ is a convex, balanced, absorbing subset of $A$ for each $n \in \mathbb{N}$.

(ii) $p$ is the Minkowski functional of $U_p$.

(iii) $p$ is unital if and only if $1 \in U_p$ and $1 \in tU_p$ whenever $|t| \in (0,1)$.

(iv) $p$ is submultiplicative if and only if $U_p$ (equivalently, $U_p(n)$ for each $n \in \mathbb{N}$) is an idempotent subset of $A$.

Further, suppose that $A$ is a topological algebra and $P$ is a calibration on $A$. Then the following hold.

(v) The collection $U_p = \{ \bigcap_{i=1}^{k} U_p(n_i) : p_i \in P, k, n_i \in \mathbb{N} \}$ is a local base at 0.

(vi) If $P$ is directed, then $\{ U_p(n) : n \in \mathbb{N}, p \in P \}$ is a local base at 0.

(vii) Suppose $P$ is directed. A seminorm $q$ on $A$ is continuous on $A$ if and only if there exist $p \in P$ and a constant $M > 0$ such that $q(x) \leq Mp(x)$ for all $x \in A$.

(viii) If $A$ is a *algebra, then the following statements are equivalent.

(a) $p$ is a *seminorm on $A$.

(b) $U_p(1)$ is a convex, balanced, absorbing, *preserving subset of $A$. 
(c) \( \bigcup_{n \in \mathbb{N}} U(n) \) is a convex, balanced, absorbing, preserving subset of \( A \) for each \( n \in \mathbb{N} \).

Unless otherwise mentioned, we shall assume that the calibrations under consideration are unital. However, it is easy to see that every non-zero \( C^* \)-seminorm is automatically unital.

**Definitions 0.1.9.** Let \( A \) be a topological algebra. A calibration \( \mathcal{P} \) on \( A \) is called an \( m \)-calibration if each \( p \in \mathcal{P} \) is an algebra seminorm. We denote the collection of all \( m \)-calibrations on \( A \) by \( \mathcal{E}_m(A) \). \( A \) is called a locally \( m \)-convex algebra (or an lmc algebra) if \( \mathcal{E}_m(A) \neq \emptyset \). A normed algebra \( (A, \| \cdot \|) \) is a topological algebra \( A \) with a norm \( \| \cdot \| \) on \( A \) such that \( \{ \| \cdot \| \} \in \mathcal{E}_m(A) \). If further, \( (A, \| \cdot \|) \) is complete, then \( A \) is called a Banach algebra.

**Definitions 0.1.10.** Suppose \( A \) is a topological algebra. An \( m \)-calibration \( \mathcal{P} \) on \( A \) is called an \( m^* \)-calibration if each \( p \in \mathcal{P} \) is also a \( * \) seminorm. We denote the collection of all \( m^* \)-calibrations on \( A \) by \( \mathcal{E}_m^*(A) \). \( A \) is called a locally \( m \)-convex * algebra (or an lmc * algebra) if \( \mathcal{E}_m^*(A) \neq \emptyset \). A normed * algebra is a normed algebra \( (A, \| \cdot \|) \) with \( \{ \| \cdot \| \} \in \mathcal{E}_m^*(A) \). A complete normed * algebra is called a Banach * algebra.
Definitions 0.1.11. Let $A$ be a topological algebra. A calibration $P$ on $A$ is called a $C^\infty$-calibration if each $p \in P$ is a $C^\infty$-seminorm. We denote the collection of all $C^\infty$-calibrations on $A$ by $\mathcal{C}_3(A)$. $A$ is called a pre-pro-$C^\infty$-algebra if $\mathcal{C}_3(A) \neq \emptyset$. A complete pre-pro-$C^\infty$-algebra is called a pro-$C^\infty$-algebra. A metrizable pre-pro-$C^\infty$-algebra (respectively, pro-$C^\infty$-algebra) is called a pre-$\sigma$-$C^\infty$-algebra (respectively, $\sigma$-$C^\infty$-algebra). A pre-$C^\infty$-algebra is a normed algebra $(A, ||\cdot||)$ such that $\{||\cdot||\} \in \mathcal{C}_3(A)$. A $C^\infty$-algebra is a complete pre-$C^\infty$-algebra.

Definition 0.1.12. Let $A$ be a topological algebra. Then $\mathcal{C}(A)$ (respectively, $\mathcal{S}(A)$) will denote the collection of all $C^\infty$-seminorms (respectively, continuous $C^\infty$-seminorms) on $A$.

Proposition 0.1.13. Let $A$ be an lmc algebra and $P \in \mathcal{C}_3(A)$. For $p \in P$, let $N_p = \{x \in A : p(x) = 0\}$.

1. $N_p$ is a closed two sided ideal in $A$. If further, $A$ is a topological algebra and if $p$ is also a seminorm, then $N_p$ is a ideal.

2. Let $A/N_p = \{x + N_p : x \in A\}$. Then $A/N_p$ is a normed algebra with the usual operations and with the norm $||x_p||_p = p(x)$, where $x_p = x + N_p$. Also, $\chi_p : A \to A_p$ defined by $\chi_p(x) = x_p$, $(x \in A)$, is a continuous homomorphism from $A$ to $A_p$, where $A_p$ is the completion of $(A/N_p, ||\cdot||_p)$. If
further, A is a topological algebra and if p is preserving, then \( \| \cdot \|_p \) is also a preserving norm making \((A/N_p, \| \cdot \|_p)\) a normed algebra. Also, if p is a C*-seminorm, then \((A/N_p, \| \cdot \|_p)\) is a pre-C*-algebra.

(3) Let \( p, q \in \mathcal{P} \) with \( q(x) \leq Mp(x) \) for all \( x \in A \) and for some constant \( M \) independent of \( x \). Then \( \chi_{p,q} : A/N_p \to A/N_q \) defined by \( \chi_{p,q}(x) = x_q \), \( x_q \in A/N_q \), is a continuous homomorphism in the respective norm topologies. Further, if A is an ImC algebra and if P is an \( m \)-calibration, then \( \chi_{p,q} \) is a \( m \)-homomorphism.

(4) Suppose A is a pro-C*-algebra and \( P \in \mathcal{Q}_2(A) \). If \( p \) and \( q \) are as in (3) above, then \( q \leq p \).

Proof. (1) For \( x, y \in N_p \) and \( \lambda \in \mathbb{C} \), we have,

\[
0 \leq p(x + y) \leq p(x) + p(y) = 0 = |\lambda| p(x) = p(\lambda x),
\]

showing that \( x + y, \lambda x \in N_p \) making \( N_p \) a subspace of \( A \). Also, for \( x \in A, y \in N_p \), \( 0 \leq p(xy) \leq p(x)p(y) = p(x)^*0 = 0 \), giving \( p(xy) = 0 \); and similarly \( p(yx) = 0 \). This shows that \( N_p \) is a two sided ideal. Since \( N_p = p^{-1}(\{0\}) \), and since \( p \) is continuous on \( A \), it follows that \( N_p \) is closed. Also, if \( p \) is preserving, then \( p(x) = 0 \) if and only if \( p(x^*) = 0 \). Thus, in this case, \( N_p \) is preserving and hence is a \( * \)ideal.

(2) The proof is a simple verification.
(3) The well-definedness of $\chi_{p,q}$ follows from the fact that $N_p \subseteq N_q$. That $\chi_{p,q}$ is a homomorphism (respectively, a *homomorphism under the additional assumption), is again a straightforward verification. Continuity of $\chi_{p,q}$ is obvious at once from the inequality,

$$||\chi_{p,q}(x)||_q = ||x||_q = q(x) \leq M_p(x) = M||x||_p$$

for all $x \in A$. Next, let $A$ be an *algebra. Let us note that

$$\chi_{p,q}^{-1}(0) = \{x_p \in A/N_p : \chi_{p,q}(x_p) = 0\}$$

$$= \{\chi_p(x) : ||x||_q = 0\}$$

$$= \{\chi_p(x) : q(x) = 0\}$$

$$= \chi_p(N_q).$$

Since $N_q$ is a *ideal in $A$, and since $\chi_p$ is a *homomorphism, $\chi_{p,q}^{-1}(0)$ is a *ideal in $A/N_p$. Now, it follows that $\chi_{p,q}$ is a *homomorphism.

(4) First, we note that $1 = q(1) \leq M_p(1) = M$. Hence $M \geq 1$. Now, for $x \in A$, $q(x)^2 = q(x^2) \leq M_p(x^2) = M_p(x)^2$. This gives

$q(x) \leq M^{\frac{1}{2}} p(x)$. Repeating the same argument, we have,

$q(x) \leq M^{\frac{1}{2^n}} p(x)$ for each $n \in N$. Thus $q(x) \leq \inf \{M^{\frac{1}{2^n}} p(x) : n = 1, 2, \ldots\} = p(x)\inf \{M^{\frac{1}{2^n}} : n = 1, 2, \ldots\} = p(x)$. This proves (4).
Lemma 0.1.14. Suppose $A$ is a topological algebra and $p_1$ and $p_2$ are two continuous seminorms on $A$. Let us define $p(x) = \max \{p_1(x), p_2(x)\}$, ($x \in A$). Then the following hold.

1. $p$ is a continuous seminorm on $A$ satisfying $p_1 \leq p$ and $p_2 \leq p$.

2. If $p_1$ and $p_2$ are algebra seminorms (respectively, having the square property, uniform seminorms, unital seminorms), then $p$ is an algebra seminorm (respectively, having the square property, a uniform seminorm, a unital seminorm).

3. Further, if $A$ is a topological algebra and if $p_1$ and $p_2$ are seminorms (respectively, $C^*$-seminorms), then $p$ is a seminorm (respectively, $C^*$-seminorm).

In view of the above lemma, we have the following proposition.

Proposition 0.1.15. Suppose $A$ is a topological algebra. Suppose $P$ is a calibration on $A$. For a finite subset $J$ of $P$, let us define $p_J(x) = \max \{p(x) : p \in J\}$, ($x \in A$). Let $\hat{P} = \{p_J : J \subseteq P \text{ is finite}\}$. Then the following hold.

1. $\hat{P}$ is a directed calibration on $A$.

2. If $P \in \pi_1(A)$ (respectively, $\pi_2(A), \pi_3(A)$, in case if $A$ is a topological algebra), then $\hat{P} \in \pi_1(A)$ (respectively, $\pi_2(A), \pi_3(A)$).
(3) For \( x \in A \), \( \sup \{ p(x) : p \in P \} = \sup \{ p_j(x) : p_j \in \hat{P} \} \).

Proof. (1) Let \((x_\lambda)\) be a net in \( A \). Suppose \( p(x_\lambda) \to 0 \) for each \( p \in \hat{P} \). Since each \( p_j \in \hat{P} \) is continuous, \( p_j(x_\lambda) \to 0 \) for each \( p_j \). The converse is obvious because \( P \subseteq \hat{P} \). Now, let \( p_j, p_k \in \hat{P} \) with \( J, I \) finite subsets of \( P \). Let \( K = J \cup I \). Then \( p_k \in \hat{P} \) and \( p_k(x) = \max \{ p_j(x), p_k(x) \} \), \( x \in A \). So, \( \hat{P} \) is directed.

(2) The proof is a simple verification.

(3) The proof is a simple verification.

Remarks 0.1.16. (1) In view of Proposition 0.1.8(i), (iv), (v) and (viii), a topological algebra (respectively, a topological algebra) \( A \) is an \( \text{lmc algebra} \) (respectively, \( \text{lmc algebra} \)) if and only if there exists a local base at 0 consisting of closed, absolutely convex, idempotent (respectively, closed, absolutely convex, idempotent, preserving) subsets of \( A \). Thus our definitions of \( \text{lmc algebra} \) and \( \text{lmc algebra} \) are equivalent to those given in [61].

(2) We also note that \( A \) is a pro-\( C^\ast \)-algebra if and only if \( S(A) \in \mathcal{C}_3(A) \). Thus our definition of a pro-\( C^\ast \)-algebra is equivalent to that of [62].
(3) In view of the above proposition, a calibration \( P \) on a topological algebra can be replaced by a directed calibration of the same type.

Unless otherwise mentioned, we assume that the calibrations under consideration are directed. Pro-C\(^*\)-algebras have been studied under various names. They are called \( \mathcal{C}\)-algebras by C. Apostol [2], Locally C\(^*\)-algebras by A. Inoue [49], \( \Lambda \)-algebras by K. Schmüdgen [70]. Recently, N. C. Phillips (1621, [63]) called them pro-C\(^*\)-algebras. A pro-C\(^*\)-algebra is, in fact, a projective limit of C\(^*\)-algebras as we shall show in Theorem 0.1.24 and hence Phillips' nomenclature seems to be the most appropriate one. A projective limit is also called an inverse limit.

Definitions 0.1.17 [52, p.8]. By a category \( \mathcal{C} \) we mean a class \( \text{Ob}(\mathcal{C}) \) of objects and, for each pair of objects \( A \) and \( B \), a set \( \text{Mor}(A,B) \) of morphisms from \( A \) to \( B \) such that for all triplets \( (A,B,C) \) of objects, there exists a mapping

\[ o : \text{Mor}(A,B) \times \text{Mor}(B,C) \to \text{Mor}(A,C), \]

called the composition mapping, \( ((f,g) \mapsto g \circ f) \), satisfying:

(i) For the objects \( A, B, C \) and \( D \), and for \( f \in \text{Mor}(A,B), \)
\[ g \in \text{Mor}(B,C) \text{ and } h \in \text{Mor}(C,D), \]
\[ h \circ (g \circ f) = (h \circ g) \circ f. \]
(ii) For each object $A$, there exists $1_A \in \text{Mor}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \text{Mor}(B, A)$, $g \in \text{Mor}(A, B)$ and for all objects $B$.

Two objects $A$ and $B$ of the category $\mathcal{K}$ are called isomorphic if there exist $f \in \text{Mor}(A, B)$ and $g \in \text{Mor}(B, A)$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. An object $A$ of $\mathcal{K}$ is called universally attracting if $\text{Mor}(B, A)$ is precisely a singleton for each object $B$ of $\mathcal{K}$.

Let us note that two universally attracting objects of a category are always isomorphic [52].

Definitions 0.1.18. Let $\mathcal{K}$ be a category and $D$ be a directed set. Suppose $\{A_d : d \in D\}$ is a collection of the objects of $\mathcal{K}$. Suppose, for each pair $(d, e) \in D \times D$ with $d \geq e$, there is associated $f_{d, e} \in \text{Mor}(A_d, A_e)$ such that $f_{d, d} = 1_A$ for each $d \in D$, and such that the following diagram commutes whenever $c \geq d \geq e$,

![Figure 1](image)

i.e., $f_{d, e} \circ f_{c, d} = f_{c, e}$. Then $\{A_d : d \in D\}$ is called an inverse system (or a projective system) in $\mathcal{K}$, with the bonding maps $\{f_{d, e} : d \geq e\}$. We define the category $\mathcal{J}$, whose
objects are \((B, \{h_d : d \in D\})\), where \(B\) is an object of \(\mathcal{K}\) and, for each \(d \in D\), \(h_d \in \text{Mor}(B, A_d)\) such that the following diagram commutes whenever \(d \geq e\),

\[
\begin{array}{ccc}
B & \xrightarrow{h_d} & A_d \\
\downarrow & & \downarrow \phi_{d,e} \\
A_e & \xrightarrow{h_e} & A_e
\end{array}
\]

(Figure 2)

i.e., \(f_{d,e} \circ h_d = h_e\). For the objects \((B, \{h_d\})\) and \((C, \{g_d\})\) of \(\mathcal{J}\), let \(\text{Mor}((B, \{h_d\}), (C, \{g_d\}))\) consist of all those morphisms \(f \in \text{Mor}(B, C)\) for which the following diagram commutes whenever \((d, e) \in D \times D\) with \(d \geq e\),

\[
\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\downarrow h_d & & \downarrow g_e \\
A_d & \xrightarrow{f_{d,e}} & A_e
\end{array}
\]

(Figure 3)

The universally attracting object (up to isomorphism) of the category \(\mathcal{J}\) is called the inverse limit (or the projective limit) of \(\{A_d : d \in D\}\) in \(\mathcal{K}\). In this case, we write \(A = \lim_{\leftarrow d \in D} A_d\).

Remark 0.1.10. We shall occasionally consider, without stating explicitly, the following categories.

(i) The category \(\mathcal{A}\) of all algebras with the algebra homomorphisms as the morphisms.
(ii) The category $\mathcal{A}_T$ of all topological algebras with the continuous homomorphisms as the morphisms.

(iii) The category $\mathcal{A}_e$ of all $^\star$algebras with the $^\star$homomorphisms as the morphisms.

(iv) The category $\mathcal{A}_s$ of all topological $^\star$algebras with the continuous $^\star$homomorphisms as the morphisms.

(v) The category $\mathcal{A}_b$ of all Banach algebras with the continuous homomorphisms as the morphisms.

(vi) The category $\mathcal{A}_b^\star$ of all Banach $^\star$algebras with the continuous $^\star$homomorphisms as the morphisms.

(vii) The category $\mathcal{A}_c$ of all $C^*$-algebras with the continuous homomorphisms as the morphisms.

Proposition 0.1.20. Let $\mathcal{A} = \{A_d : d \in D\}$ be an inverse system in $\mathcal{A}$ with the bonding maps $\{\chi_{d,e} : d \geq e\}$. Then $\lim_{\mathcal{B}} A_d$ exists in $\mathcal{A}$.

Proof. Let $\prod_{d \in D} A_d$ be the cartesian product of $\{A_d : d \in D\}$ and $\prod_{d \in D} A_d \to A_e$ be the natural projection maps for all $e \in D$.

We define $A = \{x = (x_d) \in \prod_{d \in D} A_d : \chi_{d,e}(x) = \pi_e(x) \text{ whenever } d, e \in D \text{ with } d \geq e\}$. Then it is easy to see that $A$ is a subalgebra of $\prod_{d \in D} A_d$ with the pointwise operations. For $d \in D$, let $\chi_d = \pi_d|A$. We show that $(A, \{\chi_d\})$ is the universally attracting object of $\mathcal{A}$. 

16
Suppose \((B, \{f_d\})\) is an object of \(\mathcal{F}\). Then 
\[\chi_{d,e} \circ f_d = f_e \text{ for each } (d,e) \in D \times D \text{ with } d \geq e.\]
For \(b \in B\), let us define 
\[f : B \to \prod_{d \in B} A_d \text{ by } f(b) = (f_d(b)), \quad (b \in B).\]
Then for each \((d,e) \in D \times D \text{ with } d \geq e\), 
\[\chi_{d,e}(\pi_d(f(b))) = \chi_{d,e}(f_d(b)) = f_e(b) = \pi_e(f(b)) \text{ for each } b \in B.\]
Hence \(f(b) \in A\) for each \(b \in B\). Thus \(f(B) \subseteq A\). Now, let \(a, b \in B\) and \(d \in D\). Then
\[
\pi_d(f(a + b)) = f_d(a + b) \\
= f_d(a) + f_d(b) \\
= \pi_d(f(a)) + \pi_d(f(b)) \\
= \pi_d(f(a) + f(b)),
\]
giving \(f(a + b) = f(a) + f(b)\). Similarly, it can be verified that \(f(ab) = f(a)f(b)\) and \(f(\lambda a) = \lambda f(a)\) for \(\lambda \in \mathbb{C}\) and \(a \in B\). Thus \(f \in \text{Mor}(B, A)\). It is easily seen that the following diagram commutes.

(Figure 4)

Thus \(f \in \text{Mor}((B, \{f_d\}), (A, \{\chi_d\}))\). Now, suppose that \(g \in \text{Mor}((B, \{f_d\}), (A, \{\chi_d\}))\). Then the following diagram commutes.
Thus $g(b)_d = f_d(b)$ for all $d \in D$ and hence $f = g$, showing that $\text{Mor}(B, \{f_d\}, (A, \{x_d\}))$ is a singleton. Thus \( A = \lim_{d \in D} A_d \), which completes the proof.

We denote \( \{x = \{x_d\} \in \prod_{d \in D} A_d : x_d = \pi_d(x) \} \) whenever \( d \geq e \) by $\mathcal{P}(A)$.

Proposition 0.1.21. Let $\mathcal{T} = \{(A_d, \tau_d) : d \in D\}$ be an inverse system in $\mathcal{A}$ with the bonding maps $\{x_d : d \geq e\}$. Then $(\mathcal{P}(A), \tau) = \lim_{d \in D} (A_d, \tau_d)$, where $\mathcal{P}(A)$ carries the subspace topology $\tau$ inherited from $\prod_{d \in D} A_d$. Consequently, the topology of $\mathcal{P}(A)$ is the weakest topology determined by the family $\{x_d : d \in D\}$.

Proof. First, we note that $(\mathcal{P}(A), \tau, \{x_d\})$ is an object of $\mathcal{T}$. Let $(B, \tau', \{f_d\})$ be an object of $\mathcal{T}$. Then $B$ is an object of $\mathcal{A}$ and hence $(B, \{f_d\})$ is an object of $\mathcal{P}$ (where $\mathcal{P} = \{A_d : d \in D\}$ is the inverse system in $\mathcal{A}$ obtained by dropping the topolo-
gies on $A_d$ for all $d \in D$). Thus by Proposition 0.1.20, $\text{Mor}(B, \{f_d\}), (\mathcal{LP}(A), \{X_d\}) = \{f\}$, a singleton. We need only to show that $f$ is continuous. Let $b_\lambda \to 0$ in $(B, \tau')$. Then $X_d(f(b_\lambda)) = f_d(b_\lambda) \to f_d(0) = d = X_d(f(0))$ for each $d \in D$. Hence $f(b_\lambda) \to f(0)$, giving the continuity of $f$.

Corollary 0.1.22. Let $\sigma$ be as in Proposition 0.1.21. Then $\mathcal{B} = \{X_d^{-1}(U_d) = \pi_d^{-1}(U_d) \cap \mathcal{LP}(A) : d \in D, U_d$ is a $0$-neighbourhood in $A_d\}$ is a local base at $0$ in $\mathcal{LP}(A)$.

Proof. Since $\mathcal{LP}(A)$ carries the subspace topology from $\prod_{d \in D} A_d$, the family $\mathcal{B}' = \{\bigcap_{i=1}^n X_{d_i}^{-1}(U_{d_i}) = (\bigcap_{i=1}^n \pi_{d_i}^{-1}(U_{d_i})) \cap \mathcal{LP}(A) : d_i \in D$ and $U_d$ is a $0$-neighbourhood in $A_d\}$ is a local base at $0$ in $\mathcal{LP}(A)$. Let $U$ be a $0$-neighbourhood in $A$. Without loss of generality we assume that $U = \bigcap_{i=1}^n X_{d_i}^{-1}(U_{d_i}) \in \mathcal{B}'$. Since $D$ is directed, there exists $e \in D$ such that $d_i \leq e$ for each $i = 1, 2, \ldots, n$. Let $V_e = \bigcap_{i=1}^n X_{d_i}^{-1}(U_{d_i})$. Since $X_{e, d_i} : A_e \to A_{d_i}$ are continuous for all $i = 1, 2, \ldots, n$, $V_e$ is a $0$-neighbourhood in $A_e$. Let $z = (z_d) \in X_e^{-1}(V_e)$. Then $z_{d_i} = X_{e, d_i}(z) = X_{e, d_i}(z_d) \in X_{e, d_i}(V) \subset U_{d_i}$. So, $z_{d_i} \in U_{d_i}$ for all $i = 1, 2, \ldots, n$. Hence $z \in U$. Thus $X_e^{-1}(V_e) \subset U$, proving that $\mathcal{B}$ is a local base at $0$. 

164100
Corollary 0.1.23. In the notations of Proposition 0.1.21, if \((A_d, T_d)\) is complete for each \(d \in D\), then \(L^p(A)\) is also complete.

Proof. Let \((x_\lambda)\) be a Cauchy net in \(L^p(A)\). Then \((x_{\lambda_d})\) is a Cauchy net in \(A_d\) for each \(d \in D\). Let \(x_d = \lim_{\lambda} x_{\lambda_d}\), \((d \in D)\).

Then \(x_\lambda \to x = (x_d)\) in \(\prod_{d \in D} A_d\). Let \(d, e \in D\) with \(d \geq e\). Then,

\[
\chi_{d, e}(x_d) = \chi_{d, e}(\lim_{\lambda} x_{\lambda_d}) = \lim_{\lambda} \chi_{d, e}(x_{\lambda_d}) = \lim_{\lambda} x_{\lambda e} = x_e,
\]

showing that \(x \in L^p(A)\). Thus \(L^p(A)\) is complete and so is the proof.

Theorem 0.1.24. A topological algebra \(A\) is a pro-\(C^*\)-algebra if and only if \(A\) is an inverse limit of \(C^*\)-algebras in \(\mathcal{A}_t\).

Proof. Suppose \(A\) is a pro-\(C^*\)-algebra. Let \(P \in \mathcal{S}_3(A)\) be directed. For \(p, q \in P\) with \(p \geq q\), \(\chi_{p, q} : A_p \to A_q\) is norm decreasing and hence is continuous. Thus \(\{A_p : p \in P\}\) is an inverse system in \(\mathcal{A}_t\), and each \(A_p\) is a \(C^*\)-algebra by Proposition 0.1.13(2). Now, we show that \(A\) is topologically \(C^*\)-isomorphic to \(L^p(A) = \lim_{\mathcal{P}} A_p\). Let \(F : A \to L^p(A)\) be defined by

\[
F(x) = (x_p), \quad (x \in A).
\]

Then \(F\) is a \(C^*\)-isomorphism. Indeed, if \(F(x) = 0\), then for every \(p \in P\), \(p(x) = ||x_p||_p = 0\), giving \(x = 0\). Now, let \(x_\lambda \to 0\) in \(A\). Then \(||F(x_\lambda_p)||_p = ||x_{\lambda p}||_p = p(x_\lambda) \to 0\). Thus \(F(x_\lambda_p) \to 0\) in \(A_p\) for each \(p\), which gives \(F(x_\lambda) \to 0\) in \(L^p(A)\). Thus \(F\) is continuous. Now, suppose
F(x^) → 0 in F(A). Then \( p(x^) = \|x^p\|_p = \|F(x^)\|_p \to 0 \)
for each \( p \). This gives the continuity of \( F^{-1} : F(A) → A \).

Also, since \( A \) is complete, so is \( F(A) \). But \( (A_p, \|\cdot\|_p) \) is a
\( C^* \)-algebra for each \( p \in P \) and hence, by Corollary 0.1.23,
\( \mathcal{L}(A) \) is complete, giving the closedness of \( F(A) \) in \( \mathcal{L}(A) \).

Now, let \( y \in \mathcal{L}(A) \), \( p \in P \) and \( y \in \chi^{-1}_p(U_p) \) for some open set
\( U \) in \( A \). Since \( A_p \) is the completion of \( A/N \), there exists a
sequence \( (x_n) \) in \( A \) such that \( \|x_n - y\|_p \to 0 \). So, there
exists \( x \in A \) such that \( x \in U \). But then \( x_p = F(x)_p \in U_p \),
giving \( F(x) \in \chi^{-1}_p(U_p) \) and hence \( \chi^{-1}_p(U_p) \cap F(A) = \emptyset \). Since
\( \{\chi^{-1}_p(U_p) : p \in P \text{ and } U_p \text{ is an open set in } A_p \} \) is a base for
\( \mathcal{L}(A) \), it follows that every basic open set intersects \( F(A) \),
which, in turn, gives the denseness of \( F(A) \) in \( \mathcal{L}(A) \), which
is also closed in \( \mathcal{L}(A) \). Thus \( F \) is onto. This shows that \( A \)
is an inverse limit of \( C^* \)-algebras.

Conversely, suppose that \( \mathcal{L}(A) \) is an inverse limit
of \( C^* \)-algebras, i.e., suppose that \( \{(A_p, \|\cdot\|_p) : p \in P\} \) is an
inverse system in \( \mathcal{T} \) with the bonding maps \( \{\chi_{p,q} : p \geq q\} \) and
that each \( (A_p, \|\cdot\|_p) \) is a \( C^* \)-algebra. For \( x = (x_p) \in \mathcal{L}(A) \),
let \( p(x) = \|x_p\|_p \). Then \( p \) is a \( C^* \)-seminorm on \( \mathcal{L}(A) \). Also,
a net \( (x_\lambda, x) \) in \( \mathcal{L}(A) \) converges to \( x \) in \( \mathcal{L}(A) \) if and only if
\( x_\lambda \to x_p \) in \( A_p \) for all \( p \in P \) which holds if and only if
\( \|x_\lambda - x_p\|_p \to 0 \) for all \( p \in P \). Thus \( x_\lambda \to x \) in \( \mathcal{L}(A) \) if
and only if \( p(x_\lambda - x) \to 0 \) for all \( p \in P \). This shows that \( P \)
determines the topology of $A$. Thus $LP(A)$ is a pre-pro-$C^*$-algebra. Since each $A$ is a $C^*$-algebra, Corollary 0.1.23 guarantees the completeness of $LP(A)$ which makes $LP(A)$ a pro-$C^*$-algebra. This completes the proof.

We note that the similar result is also true for complete lmc algebras [61, Theorem 5.1]. Now onwards, for a complete lmc algebra or a pro-$C^*$-algebra $A$, we shall identify topologically, $A$ with $LP(A)$.

In passing, we note the following consequence of [69, Theorem 1.6].

**Lemma 0.1.25.** Let $A$ be a topological algebra. Let $B \in B(\tau)$. Then $(A(B),|\cdot|_B)$ is a normed algebra. If further, $A$ is complete, then $(A(B),|\cdot|_B)$ is a Banach algebra. If $A$ is a complete topological $^*$-algebra, then $(A(B),|\cdot|_B)$ is a Banach algebra for each $B \in B^*(\tau)$.

**Proof.** That $(A(B),|\cdot|_B)$ is a normed algebra, is a straightforward verification. We suppose that $A$ is complete and prove that $(A(B),|\cdot|_B)$ is also complete. Let $(x_n)$ be a Cauchy sequence in $(A(B),|\cdot|_B)$. Let $U$ be a $0$-neighbourhood in $A$. By the boundedness of $B$ in $A$, there exists $t > 0$ such that $tB \subset U$. Let $n_1 \in \mathbb{N}$ be such that $|x_n - x_m|_B < t$ whenever $n, m \geq n_1$. Thus $x_n - x_m \in tB \subset U$ for all $n, m \geq n_1$, showing
that \( (x_n) \) is Cauchy in \((A,T)\). By the completeness of \( A \), \( x_n \to x \) in \( A \) for some \( x \in A \). Now, since each Cauchy sequence is bounded, \( \{x_n\} \subset sB \) for some \( s > 0 \). Since \( sB \) is closed and \( x_n \to x, x \in sB \). This gives \( x \in A(B) \). Now, let \( \varepsilon > 0 \) be given. Let us choose \( n_0 \in \mathbb{N} \) such that \( |x_n - x_m|_B \leq \varepsilon \) for all \( n, m \geq n_0 \), and fix \( k \geq n_0 \). Defining \( y_n = x_k - x_n, (n \in \mathbb{N}) \), \( y_n \to x_k - x \) in \( A \). But for \( n \geq n_0 \), \( \frac{y_n}{\varepsilon} = \frac{x_k - x_n}{\varepsilon} \in B \). Since \( B \) is closed in \( A \), \( \frac{x_k - x}{\varepsilon} \in B \), which gives \( |x_k - x|_B \leq \varepsilon \) for all \( k \geq n_0 \), showing that \( x_n \to x \) in \((A(B), \| \cdot \|_B)\). This completes the proof.

Definition 0.1.26. (1), (47, Definition 2.1). A topological algebra \( A \) is said to be pseudocomplete if for every \( B \in \mathcal{B}(r) \), \((A(B), \| \cdot \|_B)\) is a Banach algebra.

Definitions 0.1.27. Let \( A \) be a topological algebra. Then \( A^0 = \{f : A \to \mathbb{C} \text{; } f \text{ is a linear map} \} \) is called an algebraic dual of \( A \) and \( A' = \{f \in A^0 \text{; } f \text{ is continuous} \} \) is called the topological dual of \( A \). For a seminorm \( p \) on \( A \), we define \( A'(p) = \{f \in A' \text{; } |f(x)| \leq kp(x) \text{ for some constant } k > 0 \text{ and for all } x \in A \} \). Also, \( A'_p \) will denote the topological dual of the Banach space \( A_p \). The set \( U'_p = \{f \in A'(p) \text{; } |f(x)| \leq p(x) \text{ for all } x \in A \} \) is called the polar of \( U_p \). If further, \( A \) is commutative, then we define \( A^0(A) = \{f \in A^0 \text{; } f(xy) = f(x)f(y) \text{ for all } x, y \in A \} \) and the Gelfand space \( \mathcal{M}(A) = A^0(A) \cap A' \).
Definitions 0.1.28. Let $A$ be a topological algebra. A linear functional $f \in A^0$ is said to be hermitian if $f^* = f$, where $f^* \in A^0$ is defined by $f^*(x) = f(x^*)$, $(x \in A)$. The collection of all hermitian linear (respectively, continuous hermitian linear) functionals on $A$ is denoted by $A^{oh}$ (respectively, by $A^{h}$). A linear functional $f \in A^0$ is said to be positive if $f(x^*x) \geq 0$ for all $x \in A$, in this case, we write $f \geq 0$. Also, the collection of all such $f$ will be denoted by $\mathcal{P}(A)$. An $f \in \mathcal{P}(A)$ is called admissible if $\sup \{f(y^*x^*y) : y \in N_f\} < \infty$ for all $x \in A$, where $N_f = \{x \in A : f(x^*x) = 0\}$. An $f \in \mathcal{P}(A)$ is called extendable if there exists $f_1 \in \mathcal{P}(A_1)$ such that $f_1|_A = f$, where $A_1$ is the unitization of $A$ if $A$ is not unital, and $A_1 = A$ if $A$ is unital.

Throughout the thesis, we fix up the following notations.

$\mathcal{P}(A) = A' \cap \mathcal{P}(A)$.

$\text{AP}(A) = \{f \in \mathcal{P}(A) : f \text{ is admissible}\}$.

$\text{AP}(A) = A' \cap \text{AP}(A)$.

$\text{EP}(A) = \{f \in \mathcal{P}(A) : f \text{ is extendable}\}$.

$\mathcal{E}(A) = \mathcal{P}(A) \cap \mathcal{P}(A)$.

$\mathcal{E}(A) = \{f \in \mathcal{P}(A) : |f(x)|^2 \leq f(x^*x) \text{ for all } x \in A\}$. 

24
\[ p^0(A) = P(A) \cap p^0(A). \]

\[ M^0(A) = M^0(A) \cap p^0(A). \]

\[ P^0_p(A) = P^0_p(A) \cap p^0(A). \]

\[ p^0(A) = P^0_p(A) \cap p^0(A). \]

where \( p \) is some continuous seminorm on \( A \). Further, if \( A \) is commutative, then \( \mathcal{H}(A) = \mathcal{M}(A) \cap A^h \) is called the hermitian Gelfand space of \( A \).

Definitions 0.1.29. Let \( A \) be a complete lcsc algebra. A continuous \(^*\)homomorphism \( \pi : A \to \mathcal{B}(H_\pi) \) is called a bounded representation of \( A \) (when there is no confusion, we omit the word "bounded"), where \( \mathcal{B}(H_\pi) \) denotes the \( C^* \)-algebra of all bounded linear operators on the Hilbert space \( H_\pi \), with the operator norm. For a subset \( M \) of a Hilbert space \( H \), let \([M]\) denote the closed linear span of \( M \) in \( H \). For a subalgebra \( \mathcal{Y} \) of \( \mathcal{B}(H) \), a subspace \( M \) of \( H \) is said to be invariant under \( \mathcal{Y} \) if \( \mathcal{Y}(M) = \bigcup \{ T(M) : T \in \mathcal{Y} \} \subseteq M \). A bounded representation \( \pi : A \to \mathcal{B}(H_\pi) \) is called topologically irreducible if there exists no closed subspace \( M \) of \( H_\pi \) other than \( \{0\} \) and \( H_\pi \) which is invariant under \( \pi(A) \). Further, \( \pi \) is called algebraically irreducible if the only invariant subspaces for \( \pi(A) \) are \( \{0\} \).
and \( H_n \). We denote by \( \mathcal{R}(A) \), the collection of all bounded representations of \( A \) and by \( \mathcal{R}'(A) \), the collection of all topologically irreducible \( \pi \in \mathcal{R}(A) \). Let \( P \in \mathcal{S}(A) \) and \( p \in P \). Then we define,

\[
\mathcal{R}_p(A) = \{ \pi \in \mathcal{R}(A) : ||\pi(x)|| \leq kp(x) \text{ for all } x \in A \text{ and for some constant } k > 0 \},
\]

and \( \mathcal{R}'_p(A) = \mathcal{R}'(A) \cap \mathcal{R}_p(A) \). Also, \( \mathcal{R}(A_p) \) and \( \mathcal{R}'(A_p) \) will denote respectively, the collections of all bounded representations and all topologically irreducible bounded representations of the Banach algebra \( A_p \).

Lemma 0.1.30. Let \( A \) be a complete lmc algebra and \( f \in \mathcal{A}'(p) \). If \( f_p : A/N_p \to \mathbb{C} \) is defined by \( f_p(x_p) = f(x), (x_p \in A/N_p) \), then \( f_p \in (A/N_p)' \). Moreover, \( f \in \mathcal{A}'(p) \leftrightarrow f_p \in (A/N_p)' \) is a one to one and onto correspondence.

We note that each \( f_p \) defined on \( A/N_p \) can be uniquely extended continuously to \( \mathcal{A}'_p \). Thus, in fact, the above correspondence identifies \( \mathcal{A}'(p) \) with \( \mathcal{A}'_p \). We record the following theorem, which we shall use frequently.

Theorem 0.1.31. Let \( A \) be a complete lmc algebra. Let \( P \in \mathcal{S}(A) \) and \( p, q \in P \).

1. The map \( f \mapsto f_p \) is an isomorphism of \( \mathcal{A}'(p) \) onto \( \mathcal{A}'_p \).
(2) \[ \| f_p \| = \sup_{x \in \mathcal{U}_p} |f(x)| \] for each \( f \in A'(p) \).

Suppose further, that \( A \) is a complete \( \text{lmc}^\ast \) algebra and \( p \in \mathcal{P} \in \mathcal{C}_2(A) \).

(3) \( f \in A'(p) \) is positive (respectively, hermitian) if and only if \( f \) is positive (respectively, hermitian).

**Definitions 0.1.32.** Let \( A \) be a complete \( \text{lmc}^\ast \) algebra. Let \( f \in \mathcal{S}(A) \). Then \( A/N_f \) is an inner product space with the inner product defined as

\[
\langle x + N_f, y + N_f \rangle = f(y^*x), \ (x + N_f, y + N_f \in A/N_f).
\]

We denote by \( D_f \) the inner product space \( A/N_f \) and by \( H_f \) the Hilbert space completion of \( D_f \). For \( x \in A \), we define a linear operator \( \pi_f(x) : D_f \rightarrow D_f \) as \( \pi_f(x)(y + N_f) = xy + N_f \), \((y + N_f \in D_f)\). Then \( \pi_f : A \rightarrow \mathcal{L}(D_f), \ (x \rightarrow \pi_f(x)) \), is a \( \ast \) representation of \( A \) into the algebra of all linear operators on \( D_f \). The representation \( \pi_f \) of \( A \) is called the GNS-representation of \( A \) and is denoted by \( (\pi_f, D_f, H_f) \).

**Proposition 0.1.33.** Let \( A \) be a complete \( \text{lmc}^\ast \) algebra and \( f \in \mathcal{S}(A) \). Then \( f \) is admissible if and only if \( \pi_f(x) \) is continuous for each \( x \in A \). In this case, each \( \pi_f(x) \) can be extended to an element \( \tilde{\pi}_f(x) \) of \( B(H_f) \) and \( \tilde{\pi}_f \) becomes a bounded representation of \( A \).
Proof. Let $x \in A$. Then

$$\sup \left\{ \left\| \pi_f(x)(y + N_f) \right\| : y \in N_f \right\}$$

$$= \sup \left\{ \frac{\langle xy + N_f, xy + N_f \rangle}{\langle y + N_f, y + N_f \rangle} : y \neq N_f \right\}^{1/2}$$

$$= \sup \left\{ \frac{f(y^* x y)}{f(y^* y)} : y \neq N_f \right\}^{1/2}.$$ 

Thus $\pi_f(x)$ is continuous if and only if the last quantity is finite. Hence, if and only if $f$ is admissible. It follows now, that

$$\left\| \pi_f(x) \right\| = \sup \left\{ \frac{f(y^* x y)}{f(y^* y)} : y \neq N_f \right\}^{1/2}.$$ 

The rest of the proof is straightforward.

We note that, by [21, Theorem 6.1], each $f \in P(A)$ is admissible.

Now, finally, we recall some elementary facts about pro-$C^*$-algebras.

Proposition 0.1.34. Let $A$ be a pro-$C^*$-algebra and $P \in S_g(A)$. Then for any $x \in A$,

$$\sup \{ p(x) : p \in P \} = \sup \{ p(x) : p \in S(A) \}.$$
Proof. In view of Proposition 0.1.15(3), we assume that $P$ is directed. Also, since $P \subseteq S(A)$, we have,

$$\sup \{p(x) : p \in P\} \leq \sup \{p(x) : p \in S(A)\} \text{ for all } x \in A.$$  

Let $q \in S(A)$. Since $q$ is continuous, by Proposition 0.1.8 (viii), there exist $p_x \in P$ and a constant $M > 0$ such that $q(x) \leq Mp_x(x)$ for all $x \in A$. So, by Proposition 0.1.13(4), $q(x) \leq p_x(x)$ for all $x \in A$. This gives $q(x) \leq \sup \{p(x) : p \in P\}$ for all $x \in A$. The result follows now, by taking supremum on the left of this inequality over all $q \in S(A)$.

**Definition 0.1.35.** Suppose $A$ is a pro-$C^\infty$-algebra. Then the set, $b(A) = \{x \in A : \|x\|_\infty = \sup \{p(x) : p \in S(A)\} < \infty\}$ is called the bounded part of $A$. In this case, $(b(A), \|\cdot\|_\infty)$ is a $C^\infty$-algebra.

**Theorem 0.1.36** (82). Let $A$ be a pro-$C^\infty$-algebra.

1. If $x \in A^h$, then $\text{sp}(x) \subseteq \mathbb{R}$.
2. If $x \geq 0$, then $\text{sp}(x) \subseteq [0, \infty)$.
3. If $x$ is unitary, then $\text{sp}(x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

**Theorem 0.1.37** (82). Let $A$ be a pro-$C^\infty$-algebra and let $x \in A$ be normal. Then there exists a unique $^*$homomorphism $\Psi$ from the pro-$C^\infty$-algebra $C_0(\text{sp}(x)) = \{f \in C(\text{sp}(x)) : f(0) = 0\}$ to $A$,
Proposition 0.1.38. Let $A$ be a pro-$C^*$-algebra.

(1) Let $x \in A$ be normal and $f \in C_0(sp(x))$ be bounded. Then $f(x) \in b(A)$. In particular, $x \in b(A)$ if and only if $sp(x)$ is bounded.

(2) $(b(A), ||\cdot||_\infty)$ is a $C^*$-algebra and is dense in $A$.

(3) For $x \in b(A)$, $sp_{b(A)}(x) = sp_A(x)$.

(4) If $p \in S(A)$, then $\chi_p(b(A)) = A/N_p$; consequently, $A/N_p$ is automatically complete (and hence, is a $C^*$-algebra).

(5) Let $B$ be a pro-$C^*$-algebra. Let $\phi : A \rightarrow B$ be a $^*$-homomorphism (not necessarily continuous). Then $\phi(b(A)) \subset b(B)$ and consequently, $\phi|_{b(A)}$ is a $^*$-homomorphism from $b(A)$ to $b(B)$.

Proposition 0.1.39 [62, Proposition 1.14]. Let $A$ be a pro-$C^*$-algebra. Then $A$ is a $C^*$-algebra if and only if $A$ is a $Q$-algebra.

We now end the present chapter with a brief chapter-wise summary of the thesis.
0.2. Summary

The study of pro-$C^\#$-algebras dates back to the middle of this century. They had been considered by Michael [61]. The theory has been found to be important due to the occurrence of the pro-$C^\#$-algebras in various fields; for details we refer to [2], [22], [25], [30], [31], [34], [36], [37], [40], [44], [49], [58], [59], [60], [62], [63], [64], and references therein. However, very little attention seems to have been paid to developing the theory for its own sake. There are not many results in the literature characterizing the pro-$C^\#$-algebras. So, it is of interest to obtain various characterizations of these algebras, and this is what we do in the first chapter. The central result of the first chapter is Theorem 1.1.9 which characterizes these algebras intrinsically. In passing, we also investigate the local structure of pro-$C^\#$-algebras in the last but one section and the structure of the bounded part of a pro-$C^\#$-algebra in the final section. The results of the first chapter have appeared in [15].

The chapter 2 deals with the tensor products and the completely positive maps between two pro-$C^\#$-algebras. Even though, proofs of some of the results, included in the second chapter may seem to be simple, they are of importance in view of their applications in the investigation of the nuclear pro-$C^\#$-algebras in chapter 3. In second chapter, we
also introduce four major tensor product topologies on the tensor products and obtain their inverse limit versions in the form of tensor products of $C^*$-algebras. There are number of conditions on various tensor products associated with the pro-$C^*$-algebras $A$ and $B$ (e.g., $A \otimes B$, $A \otimes b(B)$, $b(A) \otimes B$, $b(A) \otimes b(B)$ and so on) characterizing the commutativity of $A$ or $B$. Further, we also investigate the admissible topologies (Definitions 2.2.1) on $A \otimes B$. In the second half of chapter 2, we concentrate on the completely positive maps between two pro-$C^*$-algebras. It is proved (Corollary 2.4.14) that two pro-$C^*$-algebras are homeomorphically isomorphic if and only if there exists a linear homeomorphism between them, which is completely positive with completely positive inverse.

The third chapter is devoted to the nuclear pro-$C^*$-algebras. Our definition of nuclear pro-$C^*$-algebras is along the line of $C^*$-algebra (55), which, in general, is weaker than the linear topological nuclearity introduced by A. Grothendieck (169). The nuclear pro-$C^*$-algebras are precisely the inverse limits of the nuclear $C^*$-algebras with the surjective bonding maps. A commutative pro-$C^*$-algebra turns out to be a nuclear pro-$C^*$-algebra (Corollary 3.1.3(2)). Later in this chapter, we also investigate certain permanence properties of nuclear pro-$C^*$-algebras. As an application of what is developed in the earlier sections we
investigate the multipliers on a pro-$C^*$-algebra. The results of this chapter together with those in the second chapter have appeared in [12].

Associated to each Banach $^*$ algebra $(A, ||*||)$, there is a $C^*$-seminorm $m(*)$ on $A$ called the Gelfand Naimark pseudo-norm [20]. The Hausdorff completion of $A$ with this seminorm is a $C^*$-algebra, called the enveloping algebra of $A$. The point is that each (bounded) representation of $A$ factors through this enveloping algebra. The enveloping pro-$C^*$-algebras have emerged while looking for the objects through which all the continuous representations of a complete Imc algebra $A$ factor [22], [34], [49]. The question has been asked in [36], for a complete characterization of those (complete) Imc $^*$-algebras, whose enveloping algebras are barrelled $Q$-algebras. However, the pro-$C^*$-algebras that are $Q$-algebras are just the $C^*$-algebras [62, Proposition 1.14] (Proposition 0.1.39) and barrelledness becomes redundant. The main theme of chapter 4 is to characterize all those Imc $^*$-algebras $A$ whose enveloping algebras become $C^*$-algebras. This answers the question posed in [36]. We also present several examples of such algebras from various branches of Functional Analysis. The results of this chapter have appeared in [14].
In passing, we study the automatic continuity of homomorphisms between two non-normed topological algebras in Appendix A, thereby, improving a recent result of M. Fragoulopoulou [43]. The results of this appendix will appear in [18].

We investigate the problem of the uniqueness of the uniform norm on a commutative unital Banach algebra in Appendix B. The results of appendix B have appeared in [13].

The main results of the Thesis believed to be new are contained in Theorem 1.1.9, Theorem 1.2.1, Theorem 1.3.2, Theorem 1.3.5, Theorem 1.3.11, Proposition 1.6.1, Theorem 2.2.2, Theorem 2.2.3, Proposition 2.3.2, Corollary 2.4.11, Corollary 2.4.14, Theorem 3.1.4, Theorem 3.2.2, Theorem 3.4.3, Theorem 3.5.5, Theorem 3.6.3, Theorem 4.2.1, Corollary 4.2.6, Theorem 4.2.8, Corollary 4.2.13, Theorem 4.4.6, Theorem 4.4.8, Theorem 4.6.3, Lemma A.3, Theorem A.4, Theorem A.5, Theorem B.4, Corollary B.5 and Theorem B.8.