Chapter 2

Number of Cayley Graphs on Dihedral Group $D_{2n}$, Symmetric Group $S_n$ and Alternating Group $A_n$

2.1 Introduction

We begin with a short review of some of the basic concepts of graph theory and group theory. One of the most widely known and extensively studied family of vertex transitive graphs is the family of Cayley graphs. The construction of Cayley graphs is dependent upon groups. We restrict our attention only to finite groups.

Let $G$ denote an abstract group and let $S \subseteq G$ such that $1_G \notin S$. The Cayley graph of $G$ relative to $S$ is the graph $\Gamma = Cay(G, S)$ that
has as vertices the elements of $G$, and edge set the set $E(\Gamma(G, S)) = \{(g, h) \mid hg^{-1} \in S\}$. This construction will produce a directed graph, but if $S = S^{-1}$ we will obtain an undirected Cayley graph. The edges of $\Gamma$ represent the orbits of unordered pairs of elements of $G$ under the action of $S$. If $\langle S \rangle$ is a proper subgroup of $G$ and $g \in G$, $g \notin \langle S \rangle$, then the vertices $1_G, g \in V(\Gamma(G, S)) = G$ belong to different components of the graph and therefore the graph is not connected. We can observe that $S$ generates the whole group $G$ if and only if the corresponding Cayley graph is connected (or in other words, $S$ generates $G$ if and only if there is path which connects any two vertices of $V(\Gamma(G, S)) = G$). It easily follows that $Cay(G, S)$ has valency $|S|$. More details about Cayley graphs can be found in [19, 44, 45, 46].

**Example 2.1.1** The Cayley graph $\Gamma(S_3, \{(12), (123), (132)\})$. 

\[\begin{array}{c}
\begin{array}{c}
(12) \\
(132) \\
(13)
\end{array}
\end{array}\]
Example 2.1.2 Let $G$ be a finite group. Let $H$ be a subgroup of $G$, and define the set $S$ to be the subgroup $H$ with the identity removed. Let $gh$ be an edge of $\Gamma(G,S)$, then $gh^{-1}$ is an element of $S$, and hence is an element of $H$, which implies that the cosets $Hg$ and $Hh$ are equal. Thus $\Gamma(G,S)$ is a graph depicting the cosets of $H$ in $G$, where two vertices are adjacent if and only if they are in the same coset thus there are $|G:H|$ components.

Lemma 2.1.3 \cite{46} Let $G$ be a finite group. The Cayley graph $\Gamma(G,S)$ is a connected graph if and only if $S \subseteq G$ is a generating set for $G$.

Example 2.1.4 Three Cayley graphs $\Gamma(G,S)$ of the dihedral group $G = D_8$. The first two graphs are examples of a Cayley graph depicting the cosets of $D_8$ for the subgroups $\langle b \rangle$ and $\langle a \rangle$ respectively (where $S$ is the subgroup minus the identity in each case). In the final graph, $S = \{a, b, b^3\}$ is a generating set for $D_8$. Observe that this graph is connected.
A dihedral group $D_{2n}$ is a group with $2n$ elements such that it contains an element '$a'$ of order 2 and an element '$b'$ of order $n$ with $a^{-1}ba = b^{-1}$.

Thus $D_{2n} = \langle a, b | a^2 = ba^n = 1, a^{-1}ba = b^{-1} \rangle = \langle a, b | a^2 = ba^n = 1, a^{-1}ba = b^\alpha, \alpha \not\equiv 1 (mod \ n), \alpha^2 \equiv 1 (mod \ n) \rangle$. 

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If \( n = 2 \), then \( D_4 \) is abelian; for \( n \geq 3 \), \( D_{2n} \) is not abelian. The elements of dihedral group can be explicitly listed as

\[
D_{2n} = \{1, a, ab, ab^2, \ldots, ab^{n-1}, b, b^2, \ldots, b^{n-1}\}.
\]

In short, its elements can be listed as \( a^i b^k \) where \( i = 0, 1 \) and \( k = 0, 1, \ldots, (n - 1) \). It is easy to explicitly describe the product of any two elements \( a^i b^k a^j b^l = a^r b^s \) as follows:

1. If \( j = 0 \) then \( r = i \) and \( s \) equals the remainder of \( k + l \) modulo \( n \).
2. If \( j = 1 \), then \( r \) is the remainder of \( i + j \) modulo 2 and \( s \) is the remainder of \( k \alpha + l \) modulo \( n \).

The orders of the elements in the Dihedral group \( D_{2n} \) are: \( o(1) = 1 \), \( o(ab^i) = 2 \), where; \( 0 \leq i \leq n - 1 \), \( o(b^i) = n \), where; \( 0 < i \leq n - 1 \) and if \( n \) is even then \( o(b^{\frac{n}{2}}) = 2 \).

### 2.2 Number of Cayley Graphs on Dihedral Group \( D_{2n} \)

In this section we determine the number of Cayley graphs \( \Gamma \) on a dihedral group \( D_{2n} \) that are undirected.
The complement $\bar{S}$ of Cayley subset $S$ with respect to $G \setminus \{1_G\}$ is also a Cayley subset. Because if $x \in \bar{S}$ then $x \notin S$ and since $S$ is a Cayley subset $x^{-1} \notin S$. Hence $x^{-1} \in \bar{S}$ i.e. $\bar{S}$ is Cayley subset. It is clear The $\bar{\Gamma} = Cay(G, \bar{S})$ and $\Gamma = Cay(G, S)$ have the same vertex set as $G$, where vertices $g$ and $h$ are adjacent in $\bar{\Gamma} = Cay(G, \bar{S})$ if and only if they are not adjacent in $\Gamma = Cay(G, S)$. So the automorphism group of $\Gamma = Cay(G, S)$ is equal to the automorphism group of $\bar{\Gamma} = Cay(G, \bar{S})$.

Recently Gholamreza Aghababaei Beny and Zarullo Rakhmonov [14] have derived a formula for number of Cayley groups on $\mathbb{Z}_n$ that are undirected.

**Theorem 2.2.1** [14] Let $G$ be a finite group. Number of Cayley graphs undirected and contrasting with group of $\mathbb{Z}_n$ are:

i) If $n = 1$, then nonexist the undirected Cayley graph for group $\mathbb{Z}_n$.

ii) If $n = 2$ or $3$, then number of Cayley graph that undirected is one.

iii) If $n \geq 4$ and $n$ is even then number of Cayley graph that undirected is

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{\frac{n-2}{2}} + \binom{n}{\frac{n}{2}}.$$
iv) If \( n \geq 4 \) and \( n \) is odd then number of Cayley graphs that are undirected is

\[
\binom{n-1}{1} + \frac{n-1}{2} + \binom{n-1}{3} + \cdots + \frac{n-1}{n-3} + \frac{n-1}{n-2}.
\]

**Theorem 2.2.2**. Suppose \( |S_k| \) denotes the number of Cayley graphs \( \Gamma = \text{Cay}(D_{2n}, S) \) with \( |S| = k \) and \( n \) odd. Then

\[
|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k-2m} \binom{n}{m}.
\]

Here \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \).

**Proof.** Suppose \( n \) is odd. Then \( D_{2n} \setminus \{1\} = A \cup \bar{A} \), where \( A = \{a, ab, ab^2, \ldots, ab^{n-1}\} \) and \( \bar{A} = \{b, b^{n-1}, b^2, b^{n-2}, \ldots, b^{\frac{n-1}{2}}, b^{\frac{n+1}{2}}\} \). Note that \( |A| = n \) and \( |ar{A}| = n - 1 \). The orders of the elements in the set \( A \) is 2. Hence for every element of \( A \) its inverse is itself. If \( b^i, \; 0 < i \leq n - 1 \) is an arbitrary element of \( \bar{A} \), then \( o(b^i) = n \). So, its inverse is \( b^{n-i} \). If \( S \) is a Cayley subset of \( D_{2n} \) with \( k \) elements, then \( b^i \in S, \; 0 < i \leq n - 1 \) if and only if \( b^{n-i} \in S, \; 0 < i \leq n - 1 \). Since we have \( \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor \) such pairs in \( \bar{A} \), we can construct a Cayley subset with \( k \) elements by choosing \( m \) ( \( 0 \leq m \leq \lfloor \frac{k}{2} \rfloor \) ) pairs of \( \bar{A} \) and \( k - 2m \) elements of \( A \). This implies that the
number of Cayley graphs $\Gamma = Cay(D_{2n}, S)$ that undirected and having $k$ elements is

$$|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k-2m} \binom{\lfloor \frac{n}{2} \rfloor}{m}.$$ 

As the proof of the following theorem is similar to that of Theorem 2.2.2 we omit the proof.

**Theorem 2.2.3** Let $|S_k|$ be the number of Cayley graphs $\Gamma = Cay(D_{2n}, S)$ with $|S| = k$ and $n$ even. Then

$$|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n+1}{k-2m} \binom{\lfloor \frac{n-1}{2} \rfloor}{m}.$$ 

**Theorem 2.2.4** Let $N(D_{2n})$ denotes the number of Cayley graphs on a Dihedral group $D_{2n}$. Then

$$N(D_{2n}) = \begin{cases} 1 + 2 \sum_{k=1}^{n-1} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k-2m} \binom{\lfloor \frac{n}{2} \rfloor}{m} & \text{if } n \text{ is odd,} \\ 1 + 2 \sum_{k=1}^{n-1} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n+1}{k-2m} \binom{\lfloor \frac{n-1}{2} \rfloor}{m} & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** If $S$ is a Cayley subset of $D_{2n}$ with $k$ elements ($1 \leq k \leq n-1$),
then its complement \( \bar{S} \) with respect to \( D_{2n} \setminus \{1\} \) is also a Cayley subset of \( D_{2n} \) with \((2n - 1 - k)\) elements. Thus \(|S_k| = |S_{2n-1-k}|, 1 \leq k \leq n - 1.\)

Suppose \( n \) is odd. Then, we have

\[
N(D_{2n}) = \sum_{k=1}^{2n-1} |S_k|
\]

\[
= \sum_{k=1}^{n-1} |S_k| + \sum_{k=n}^{2n-2} |S_k| + |S_{2n-1}|
\]

\[
= 1 + 2 \sum_{k=1}^{n-1} |S_k|
\]

\[
= 1 + 2 \sum_{k=1}^{n-1} \left\lfloor \frac{k}{2} \right\rfloor \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{k-2m} \left( \frac{n}{m} \right), \text{ using Theorem 2.2.2}
\]

The proof of other case is similar.

**Theorem 2.2.5** Let \( N(D_{2n}) \) denotes the number of Cayley graphs on a Dihedral group \( D_{2n} \). Then

\[
N(D_{2n}) = \begin{cases} 
2^{\frac{(n-1)}{2}} - 1 & \text{if } n \text{ is odd,} \\
2^{\frac{n}{2}} - 1 & \text{if } n \text{ is even.}
\end{cases}
\]

**Proof.** Case (i) \( n \)-odd. If \( S \) is a Cayley subset of \( D_{2n} \) and \( b^i \in S, 0 < i \leq n - 1 \) if and only if \( b^{n-i} \in S. \) Also note that inverse of \( ab^i, 0 \leq i \leq n - 1 \) is itself. Hence \( N(D_{2n}) \) is the number of nonempty subsets
of a set with \(2n - 1 - \frac{n-1}{2}\) elements. i.e. \(N(D_2n) = 2^{\frac{(3n-1)}{2}} - 1\), if \(n\) is odd. The proof of the other case is similar.

**Remark 2.2.6** Combining Theorem 2.2.4 and Theorem 2.2.5 we have the following beautiful identities:

1) \[1 + 2 \sum_{k=1}^{n-1} \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{n}{k-2m} \binom{\left\lfloor \frac{n}{2} \right\rfloor}{m} = 2^{(3n-1)/2} - 1, \text{ if } n \text{ is odd.}\]

2) \[1 + 2 \sum_{k=1}^{n-1} \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{n+1}{k-2m} \binom{\left\lfloor \frac{n-1}{2} \right\rfloor}{m} = 2^{3n/2} - 1, \text{ if } n \text{ is even.}\]

### 2.3 Number of Cayley Graphs on Symmetric Group \(S_n\)

In this section we investigate some properties of the Cayley graphs \(\Gamma\) on a symmetric group \(S_n\).

A permutation of a set \(\Omega\) is a bijection \(\alpha : \Omega \rightarrow \Omega\). If \(\Omega\) is a finite set and \(|\Omega| = n\), then an arrangement of \(\Omega\) is a list \(x_1, x_2, ..., x_n\) with no repetitions of all the elements of \(\Omega\).

The family of all the permutations of a set \(\Omega\), denoted by \(S_\Omega\), is called the symmetric group on \(\Omega\). When \(\Omega = \{1, 2, ..., n\}\), \(S_\Omega\) is usually denoted by \(S_n\), and it is called the symmetric group on \(n\) letters.
If $\alpha \in S_n$ and $i \in \{1, 2, ..., n\}$, then $\alpha$ fixes $i$ if $\alpha(i) = i$, and a moves $i$ if $\alpha(i) \neq i$.

Let $i_1, i_2, ..., i_r$ be distinct integers in $\{1, 2, ..., n\}$. If $\alpha \in S_n$ fixes the other integers (if any) and if

$$
\alpha(i_1) = i_2, \alpha(i_2) = i_3, ..., \alpha(i_{r-1}) = i_r, \alpha(i_r) = i_1,
$$

then $\alpha$ is called an $r$-cycle. One also says that $\alpha$ is a cycle of length $r$. A 2-cycle interchanges $i_1$ and $i_2$ and fixes everything else; 2-cycles are also called transpositions. A 1-cycle is the identity, for it fixes every $i$; thus, all 1-cycles are equal: $(i) = (1)$ for all $i$. We now introduce new notation: an $r$-cycle $\alpha$, as in the definition, shall be denoted by $\alpha = (i_1i_2...i_r)$.

Two permutations $\alpha, \beta \in S_n$ are disjoint if every $i$ moved by one is fixed by the other: if $\alpha(i) \neq i$, then, $\beta(i) = i$, and if, $\beta(j) \neq j$, then $\alpha(j) = j$. A family $\beta_1, ...\beta_t$ of permutations is disjoint if each pair of them is disjoint.

Disjoint permutations $\alpha, \beta \in S_n$ commute. Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles.

Every permutation is a product of 2-cycles. We shall refer to 2-cycles as transpositions. A permutation $\alpha \in S_n$ is said to be an even permutation if it can be represented as a product of an even number of transpositions.

We call a permutation odd if it is not an even permutation. $S_n$ has as a
normal subgroup of index 2 the Alternating group, \(A_n\), consisting of all even permutations. More details about Symmetric groups can be found in [78].

Now we proceed towards some theorems about the number of Cayley graphs.

**Theorem 2.3.1** Suppose \(|S_k|\) denotes the number of Cayley graphs \(\Gamma = Cay(S_n, S)\) with \(|S| = k\). Then

\[
|S_k| = \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{\gamma(S_n)}{k-2m} \left( \left\lfloor \frac{n!-1-\gamma(S_n)}{2} \right\rfloor m \right).
\]

Here

\[
\gamma(S_n) = \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{k!2^k(n-2k)!}.
\]

and \([x]\) denotes the greatest integer \(\leq x\).

**Proof.** Suppose \(S_n^* = S_n \setminus \{1_{S_n}\}\). Then \(S_n^* = A \cup \bar{A}\), where \(A = \{\rho_1\rho_2...\rho_t \mid \rho_i's\ are\ disjoint,\ \text{2-cycle},\ 1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor\} \) and \(\bar{A} = S_n^* \setminus A\). Since in the symmetric group \(S_n\), every \(k\)-cycle has order \(k\), and if \(\sigma\ is \ k\-cycle\ and\ \tau\ is\ an\ l\-cycle\ and\ these\ two\ cycles\ are\ disjoint\ then\ \sigma\tau = \tau\sigma\ and\ this\ element\ has\ order\ \text{lcm}(k,l)\). Therefore all of elements in \(A\ has\ order\ 2\). Hence for every element of \(A\ its\ inverse\ is\ itself.\ Also\ if\ 1 < r \leq n,\
there are \( \frac{1}{r} [n(n-1)...(n-r+1)] r \)-cycles in \( S_n \). And if \( kr \leq n \), where \( 1 < r \leq n \), then the number of \( \rho \in S_n \), where \( \rho \) is a product of \( k \) disjoint \( r \)-cycles, is \( \frac{1}{k!r} [n(n-1)...(n-kr+1)] \). So \( |A| = \sum_{k=1}^{\lfloor \frac{n}{r} \rfloor} \frac{n!}{k!2^k(n-2k)!} \), where we denotes \( |A| \) by \( \gamma(S_n) \). If \( S \) is a Cayley subset of \( S_n \) with \( k \) elements, \( \rho \in S \) if and only if \( \rho^{-1} \in S \). Since we have \( \lfloor \frac{n!-1-\gamma(S_n)}{2} \rfloor \) such pairs in \( \overline{A} \), we can construct a Cayley subset with \( k \) elements by choosing \( m \) ( \( 0 \leq m \leq \lfloor \frac{k}{2} \rfloor \) ) pairs of \( \overline{A} \) and \( k-2m \) elements of \( A \). This implies that the number of Cayley graphs \( \Gamma = Cay(S_n, S) \) that undirected and having \( k \) elements is

\[
|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\gamma(S_n)}{k-2m} \left( \frac{\lfloor \frac{n!-1-\gamma(S_n)}{2} \rfloor}{m} \right).
\]

**Theorem 2.3.2** Let \( N(S_n) \) denotes the number of Cayley graphs on a symmetric group \( S_n \). Then

\[
N(S_n) = 2 + 2 \sum_{k=1}^{\lfloor \frac{n!}{2} \rfloor} \sum_{m=0}^{\frac{n!-1-\gamma(S_n)}{2}} \binom{\gamma(S_n)}{k-2m} \left( \frac{\lfloor \frac{n!-1-\gamma(S_n)}{2} \rfloor}{m} \right).
\]

Here

\[
\gamma(S_n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!2^k(n-2k)!}.
\]

**Proof.** If \( S \) is a Cayley subset of \( S_n \) with \( k \) elements (\( 0 \leq k \leq n! - 1 \)), then its complement \( \overline{S} \) with respect to \( S_n^* = S_n \setminus \{1_{S_n}\} \) is also a Cayley subset of \( S_n \) with \( (n! - 1 - k) \) elements. Thus \( |S_k| = |S_{n!-1-k}| \),
$0 \leq k \leq n! - 1$. So

$$N(S_n) = \sum_{k=0}^{n!-1} |S_k|$$

$$= |S_0| + |S_{n!-1}| + \sum_{k=1}^{n!-2} |S_k|$$

$$= 2 + 2 \sum_{k=1}^{\frac{n!-2}{2}} |S_k|$$

$$= 2 + 2 \sum_{k=1}^{\frac{n!-2}{2}} \left\lfloor \frac{\gamma(A_n)}{k} \right\rfloor \left( \left\lfloor \frac{n! - \gamma(A_n)}{2} \right\rfloor \right),$$

using Theorem 2.3.1.

**Theorem 2.3.3** Suppose $|S_k|$ denotes the number of Cayley graphs $\Gamma = Cay(A_n, S)$ with $|S| = k$. Then

$$|S_k| = \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \left( \gamma(A_n) \right) \left( \left\lfloor \frac{n! - \gamma(A_n)}{2} \right\rfloor \right).$$

Here

$$\gamma(A_n) = \sum_{k \in U} \frac{n!}{k!2^k(n-2k)!}; U = \{2i \mid 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \}.$$ 

**Proof.** Suppose $A^*_n = A_n \setminus \{1_{A_n}\}$. Then $A^*_n = A \cup \tilde{A}$, where $A = \{\rho_1 \rho_2 \ldots \rho_t \mid \rho_i's \text{ are disjoint, 2-cycle and the number of those are even, } 1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor \}$ and $\tilde{A} = S^*_n \setminus A$. We note that $A_n \leq S_n$ and in the symmetric group $S_n$, every $k$-cycle has order $k$, and if $\sigma$ is $k$-cycle and $\tau$ is an is a $l$-cycle and these two cycles are disjoint then $\sigma \tau = \tau \sigma$ and this element has order $lcm(k, l)$. Therefore all of elements in $A$ has order
2. Hence for every element of $A$ its inverse is itself. If \( kr \leq n \), where \( 1 < r \leq n \), then the number of \( \rho \in S_n \), where \( \rho \) is a product of \( k \) disjoint \( r \)-cycles, is \( \frac{1}{k!r^n}[n(n - 1)...(n - kr + 1)] \). If \( r = 2 \) and \( k \) is even then \( \rho \in A \subseteq A_n \). So \( |A| = \sum_{k \in U} \frac{n!}{k!2^k(n - 2k)!} \); \( U = \{2i \mid 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \} \) where we denotes \( |A| \) by \( \gamma(A_n) \). If \( S \) is a Cayley subset of \( A_n \) with \( k \) elements, \( \rho \in S \) if and only if \( \rho^{-1} \in S \). Since we have \( \lfloor \frac{n!/2 - \gamma(A_n) - 1}{2} \rfloor \) such pairs in \( \bar{A} \), we can construct a Cayley subset with \( k \) elements by choosing \( m(o \leq m \leq \lfloor \frac{k}{2} \rfloor) \) pairs of \( \bar{A} \) and \( k - 2m \) elements of \( A \). This implies that the number of Cayley graphs \( \Gamma = Cay(A_n, S) \) that undirected and having \( k \) elements is

\[
|S_k| = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left( \frac{\gamma(A_n)}{k - 2m} \right) \left( \frac{n!/2 - \gamma(A_n) - 1}{2} \right) \left( \frac{n!}{m} \right).
\]

**Theorem 2.3.4** Let \( N(A_n) \) denotes the number of Cayley graphs on a Alternating group \( A_n \). Then

\[
N(A_n) = 2 \sum_{k=1}^{\lfloor \frac{n!}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left( \frac{\gamma(A_n)}{k - 2m} \right) \left( \frac{n!/2 - \gamma(A_n) - 1}{2} \right) \left( \frac{n!}{m} \right).
\]

Here

\[
\gamma(A_n) = \sum_{k \in U} \frac{n!}{k!2^k(n - 2k)!}; U = \{2i \mid 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \}.
\]
Proof. If $S$ is a Cayley subset of $A_n$ with $k$ elements ($0 \leq k \leq \frac{n!}{2} - 1$), then its complement $\overline{S}$ with respect to $A^*_n = A_n \setminus \{1, A_n\}$ is also a Cayley subset of $A_n$ with $(\frac{n!}{2} - 1 - k)$ elements. Thus $|S_k| = |S_{\frac{n!}{2} - 1 - k}|$, $0 \leq k \leq \frac{n!}{2} - 1$. So $N(A_n) = \sum_{k=0}^{\frac{n!}{2} - 1} |S_k| = |S_0| + \sum_{k=1}^{\frac{n!}{2} - 2} |S_k| = 2 + 2 \sum_{k=1}^{\frac{n!}{2} - 2} |S_k|$

$$= 2 + 2 \sum_{k=1}^{\frac{n!}{2} - 1} \left\lfloor \frac{k}{2} \right\rfloor \left( \frac{n! - \gamma(A_n) - 1}{2} \right) \sum_{m=0}^{\left\lfloor \frac{n! - \gamma(A_n) - 1}{2} \right\rfloor} \left( \frac{k}{2} - 2m \right), \text{ using Theorem 2.3.3}$$

2.4 Non-Isomorphic Cayley graphs of Symmetric groups

A fundamental problem in graph theory is the so-called isomorphism problem, that is, to decide whether two given graphs are isomorphic or not. In this section we investigate the isomorphism problem for finite Cayley graphs on symmetric group $S_n$.

The following Theorems are basic for Cayley graphs and Cayley subsets.

Theorem 2.4.1 [41] Let $S_n$ be Symmetric group and $n > 2$ then $\text{Aut}(S_n) = S_n$ except when $n = 6$. In the exceptional case $\text{Aut}(S_6) = S_6 \rtimes C_2$.

If $\delta = (i_1i_2...i_r) \in S_n$, $\alpha(j_1, j_2, ..., j_k) \in \text{Aut}(S_n)$ is an arbitrary automorphism of Symmetric group $S_n$, then it can be expressed by
$(i_1i_2...i_r) \xrightarrow{\alpha(j_1,j_2,...,j_k)} (t_1t_2...t_r)$ where for $1 \leq m \leq r$,

$$t_m = \begin{cases} 
i_m & \text{if } i_m \notin \{j_1, j_2, ..., j_k\}, \\
 j_{l+1} & \text{if } i_m = j_1, \text{ for some } 1 \leq l < k, \\
 j_1 & \text{if } i_m = j_k.
\end{cases}$$

**Theorem 2.4.2** [46] If $\alpha$ is an automorphism of group $G$, then $\text{Cay}(G, S)$ and $\text{Cay}(G, \alpha(S))$ are isomorphic.

The converse of the Theorem 2.4.2 is not true. Two Cayley graphs for a group $G$ can be isomorphic even if there is no automorphism of $G$ relating their connection sets.

The converse of Theorem 2.4.2 about Cayley Graphs of Alternating group $A_4$ and all disconnected Cayley graphs of $A_5$ [51] and all Cayley graphs of $A_5$ of valency 4 is true [70] which was a conjecture posed by Li and Praegar [62].

In general it is difficult to compute the number of Cayley graphs on a group $G$ that are undirected up to isomorphism. In this section we determine the number of non-isomorphic Cayley graphs of symmetric group $S_n$ and Alternating group $A_n$ for $n = 3, 4$.

First we prove two lemmas which are necessary to prove our results.

**Lemma 2.4.3** Let $S_n$ be Symmetric group. Then $\text{Cay}(S_n, \{(ij)\})$
i, j \in \{1, 2, ..., n\} are all isomorphic to each other. i.e up to isomorphism, there is exactly one Cayley graph on Symmetric group $S_n$ of valency 1.

**Proof.** Since $\{ij\}^{\alpha(i,k)(j,l)} \{kl\}$ that $i, j, k, l \in \{1, 2, ..., n\}$ it follows from Theorem 2.4.2 that $Cay(S_n, \{(ij)\})$ and $Cay(S_n, \{(kl)\})$ are isomorphic. Thus all Cayley graphs of $S_n$ of valency 1 are isomorphic.

**Lemma 2.4.4** Let $S_n(n \geq 4)$ be Symmetric group. Then

$Cay(S_n, \{(12), (13)\}), Cay(S_n, \{(12), (34)\}), Cay(S_n, \{(12), (13)(24)\})$ $Cay(S_n, \{(123), (132)\})$ are all Non-Isomorphic to each other. i.e up to isomorphism, there are at least 4 Cayley graph on Symmetric group $S_n$ of valency 2.

**Proof.** Let $H = \langle (12), (13) \rangle$ be a subgroup of $S_n$. Let $gh$ be an edge of $Cay(S_n, \{(12), (13)\})$, then $gh^{-1}$ is an element of $H \setminus \{1_S_n\}$, which implies that cosets $Hg$ and $Hh$ are equal. So two vertices are adjacent if and only if they are in the same coset. Thus there are $|S_n : H|$ components; each of which is a clique of size $|H|$. But $H = \langle (12), (13) \rangle = \{(), (12), (13), (23), (123), (132)\}$
\[ K = \langle (12), (34) \rangle = \{(), (12), (34), (12)(34)\} \]
\[ M = \langle (12), (13)(24) \rangle = \{(), (12), (34), (12)(34), (13)(24), (1423), (1324), (14)(23)\} \]
\[ N = \langle (123), (132) \rangle = \{(), (123), (132)\}. \]

and \(|S_n : H| = n!/6\), \(|S_n : K| = n!/4\), \(|S_n : M| = n!/8\), \(|S_n : N| = n!/3\).

This implies that Cayley graphs of \(S_n\) of valency 2 as above are all non-isomorphic to each other.

**Theorem 2.4.5** Up to isomorphism, there are exactly 8 Cayley graphs of \(S_3\).

**Proof.** From the definition of Cayley subsets it easily follows that a subset \(S\) of \(S_3^* = S_n \setminus \{1_{S_n}\}\) is a Cayley subset of \(S_n\) if and only if \(S_3^* \setminus S\) is a Cayley subset of \(S_n\). So we shall always assume that \(|S| \leq 2\). Since \(\{(12), (13)\} \xrightarrow{\alpha(1,2,3)} \{(12), (23)\} \xrightarrow{\alpha(1,2,3)} \{(13), (23)\} \) and \(\{(12), (13)\} = S_3\), and \(Cay(S_3, \{12\}, (13))\) is connected also \(\{(123), (132)\}\) = \(A_3\) so \(Cay(S_3, \{12\}, (13))\) is disconnected. Now by Lemma 2.4.3 there are exactly 1,1,2 non-isomorphic Cayley graphs of \(S_3\) of valency 0,1,2 respectively. Thus there are exactly 1,1,2 non-isomorphic Cayley graphs of \(S_3\) of valency 5,4,3 respectively. Therefore up to isomorphism, there are exactly 8 Cayley graphs of \(S_3\).
**Theorem 2.4.6** [51] Up to isomorphism, there are exactly 22 Cayley graphs of \( A_4 \).

**Theorem 2.4.7** Up to isomorphism, there are exactly 4 Cayley graphs of \( S_4 \) of valency 2.

**Proof.** By Lemma 2.4.4 and Sylow Theorem every Cayley subsets of Symmetric group \( S_4 \) can be expressed \( T_1 = \{(12), (13)\} \), \( T_2 = \{(12), (34)\} \), \( T_3 = \{(12), (13)(24)\} \), \( T_4 = \{(123), (132)\} \). And \(|\langle T_1 \rangle| = 6\), \(|\langle T_2 \rangle| = 4\), \(|\langle T_3 \rangle| = 8\), \(|\langle T_4 \rangle| = 3\). Since \( Cay(S_4, T_1) \cong 4C_6 \), \( Cay(S_4, T_2) \cong 6C_4 \), \( Cay(S_4, T_3) \cong 3C_8 \), \( Cay(S_4, T_4) \cong 8C_3 \). So there are exactly 4 Cayley graphs of \( S_4 \) of valency 2.

### 2.5 Open problems

We conclude this chapter with four open problems.

(1) Determine the number of Cayley graphs of Dihedral Groups up to isomorphism?

(2) Determine the number of Cayley graphs of \( \mathbb{Z}_n \) and \( \mathbb{Z}_{2n} \) up to isomorphism, where \( n \) is odd square-free integer?

(3) Determine the number of Cayley graphs of Symmetric groups \( S_n \) up to isomorphism?
(4) Determine the number of Cayley graphs of Alternating groups $A_n$ up to isomorphism?