Chapter 1

Introduction

The applications of graph theory are manifold; we can use graphs to model networks, for example, computer networks, road networks, electricity and water supplies. In a world where communication is of prime importance, the versatility of graphs makes them indispensable tools in the design and analysis of communication networks.

Algebraic graph theory is an important branch of mathematics that studies graphs by using algebraic properties of associated matrices and tools in algebra. More in particular, spectral graph theory studies the relation between graph properties and the spectrum of the adjacency matrix or Laplace matrix. And the theory of association
schemes and coherent configurations studies the algebra generated by associated matrices. The Cayley graph can be described as a graphical representation of a group. Each vertex of a Cayley graph corresponds to an element of the corresponding group, $G$ say. Also, there is a subset $S$ of $G$ such that if $s$ is an element of $S$, then its inverse $s^{-1}$ also belongs to $S$. Two vertices $g_1$ and $g_2$ are joined with an edge if and only if the element $g_2g_1^{-1}$ belongs to $S$. The result is a Cayley graph denoted by $Cay(G, S)$.

Graph theory has attracted mathematicians and scientists from diverse disciplines and, accordingly, is blessed (and cursed) with a proliferation of terminology and notations. Some words are used differently by different graph theory communities. For the history of early graph theory and terminologies of graph, see [2, 3, 4, 5, 6, 10, 12, 13, 15, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 30, 31, 32, 33, 34, 36, 37, 38, 39, 42, 43, 47, 48, 50, 54, 57, 60, 61, 63, 75, 76, 77, 79, 80, 81, 82, 83, 86].

Now we give some essential definitions which are useful in the succeeding chapters. Unless mentioned otherwise all graphs considered in the thesis are finite and simple.

**Definition 1.1.1** A **graph** $\Gamma$ **consists of a vertex set** $V(\Gamma)$ **and an edge**
set \( E(\Gamma) \), where an edge is an unordered pair of distinct vertices of \( \Gamma \). We will usually use \( xy \) rather than \( \{x, y\} \) to denote an edge. If \( xy \) is an edge, then we say that \( x \) and \( y \) are adjacent or that \( y \) is a neighbour of \( x \), and denote this by writing \( x \sim y \). A vertex is incident with an edge if it is one of the two vertices of the edge. Graphs are frequently used to model a binary relationship between the objects in some domain, for example, the vertex set may represent computers in a network, with adjacent vertices representing pairs of computers that are physically linked. A loop is an edge whose two end points are the same vertex. Edges that have same end points are called multiple edges. A graph with \( n \) vertices and \( m \) edges is called \((n, m)\) graph.

It is often convenient, interesting, or attractive to represent a graph by a picture, with points for the vertices and lines for the edges.

**Definition 1.1.2** Two graphs \( \Gamma \) and \( \Lambda \) are equal if and only if they have the same vertex set and the same edge set. Although this is a perfectly reasonable definition, for most purposes the model of a relationship is not essentially changed if \( \Lambda \) is obtained from \( \Gamma \) just by renaming the vertex set. This motivates the following definition: Two graphs \( \Gamma \) and \( \Lambda \) are isomorphic if there is a bijection, \( \varphi \) say, from \( V(\Gamma) \) to \( V(\Lambda) \) such that \( x \sim y \) in \( \Gamma \) if and only if \( \varphi(x) \sim \varphi(y) \) in \( \Lambda \). We say that \( \varphi \) is an
isomorphism from $\Gamma$ to $\Lambda$. Since $\varphi$ is a bijection, it has an inverse, which is an isomorphism from $\Lambda$ to $\Gamma$. If $\Gamma$ and $\Lambda$ are isomorphic, then we write $\Gamma \cong \Lambda$. It is normally appropriate to treat isomorphic graphs as if they were equal.

**Definition 1.1.3** The number of vertices in $\Gamma$ is called the **order of graph** and the number of edges in a graph is called the **size of $\Gamma$.**

**Definition 1.1.4** A **simple graph** $\Gamma$ is a graph having no loops or multiple edges. In this case, each edge $e$ in $E(\Gamma)$ can be specified by its endpoints $u, v$ in $V(\Gamma)$.

**Definition 1.1.5** A **trivial graph** (also called empty) is a graph consisting of one vertex and no edges. A graph with no vertices and no edges is called a **null graph**.

**Definition 1.1.6** A graph $\Gamma$ is **complete** if every pair of vertices are adjacent. The complete graph with $n$ vertices is denoted by $K_n$.

**Definition 1.1.7** The **complement** $\bar{\Gamma}$ of a graph $\Gamma$ has the same vertex set as $\Gamma$, where vertices $u$ and $v$ are adjacent in $\bar{\Gamma}$ if and only if they are not adjacent in $\Gamma$. 
Definition 1.1.8 A walk in a graph $\Gamma$ is an alternating sequence of vertices and edges, $W = v_1, e_1, v_2, e_2, \ldots, e_n, v_n$ such that for $j = 1, 2, \ldots, n$ the vertices $v_j$ and $v_{j+1}$ are adjacent. The vertex $v_1$ is called initial vertex and $v_n$ is called terminal vertex. The length of the walk is the number of edges in a walk. A walk is closed walk if the initial vertex is also the final vertex. A trail is a walk with no repeated edge.

A path is a walk with no repeated vertices. A cycle is a closed path of length at least one. The cycle with $n$ vertices is denoted by $C_n$ and the smallest cycle is complete graph $K_3$ (or $C_3$).

Definition 1.1.9 A graph $\Gamma$ is connected if for every $u, v \in V$, there exists a $u, v$ path in $\Gamma$. Otherwise $\Gamma$ is called disconnected. A connected graph with no cycle is called Tree.

Definition 1.1.10 The degree (or valency) of a vertex $v$ in a graph $\Gamma$, denoted by $\deg(v)$, is the number of neighbours of $v$. The degree sequence of a graph is the sequence formed by arranging the vertex degrees into non-decreasing order. Vertex with degree 0, is called isolated vertex.

Definition 1.1.11 A graph in which every vertex has equal degree $r$ is called regular graph of degree $r$.  

5
Definition 1.1.12 A bipartite graph $\Gamma$ is a graph whose vertex set $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every edge of $\Gamma$ joins $V_1$ and $V_2$. If every vertex of $V_1$ is joined to every vertex of $V_2$, then $\Gamma$ is called a complete bipartite graph. If $V_1$ and $V_2$ have $m$ and $n$ vertices, we write $\Gamma = K_{m,n}$. A graph is $k$-partite if its vertices can be partitioned into $k$ sets, in such a way that no edge joins two vertices in the same set.

Definition 1.1.13 An isomorphism from a graph $\Gamma$ to itself is called an automorphism of $\Gamma$. An automorphism is therefore a permutation of the vertices of $\Gamma$ that maps edges to edges and nonedges to nonedges. Consider the set of all automorphisms of a graph $\Gamma$. Clearly the identity permutation is an automorphism, which we denote by $e$. If $\alpha$ is an automorphism of $\Gamma$, then so is its inverse $\alpha^{-1}$ and if $\beta$ is a second automorphism of $\Gamma$, then the product $\alpha\beta$ is an automorphism. Hence the set of all automorphisms of $\Gamma$ forms a group, which is called the automorphism group of $\Gamma$ and denoted by $\text{Aut}(\Gamma)$. The symmetric group $\text{Sym}(V)$ is the group of all permutations of a set $V$, and so the automorphism group of $\Gamma$ is a subgroup of $\text{Sym}(V(\Gamma))$. If $\Gamma$ has $n$ vertices, then we will freely use $\text{Sym}(n)$ for $\text{Sym}(V(\Gamma))$.

In general, it is a nontrivial task to decide whether two graphs are
isomorphic, or whether a given graph has a nonidentity automorphism. Nonetheless there are some cases where everything is obvious. For example, every permutation of the vertices of the complete graph $K_n$ is an automorphism, and so $\text{Aut} K_n \cong \text{Sym}(n)$. And an automorphism on the star graph $K_{1,n}$ can permute any vertex of valency 1 with any other vertex of valency 1, but must keep the vertex of valency $n$ fixed. Therefore any permutation of the $n$ vertices of valency 1 is an automorphism of the graph, so its automorphism group is isomorphic to $S_n$.

**Example 1.1.14** The image of an element $v \in V(\Gamma)$ under a permutation $g \in \text{Sym}(V)$ will be denoted by $v^g$. If $g \in \text{Aut}(\Gamma)$ and $\Lambda$ is a subgraph of $\Gamma$, then we define $\Lambda^g$ to be the graph with $V(\Lambda^g) = \{x^g : x \in V(\Lambda)\}$ and $E(\Lambda^g) = \{\{x^g, y^g\} : \{x, y\} \in E(\Lambda)\}$. It is straightforward to see that $\Lambda^g$ is isomorphic to $\Lambda$ and is also a subgraph of $\Gamma$.

**Example 1.1.15** Let $V(K_4) = \{a,b,c,d\}$ and let $X = K_4 - \{a,c\}$, the result of removing edge $\{a,c\}$ from $K_4$. Then

$\text{Aut}(X) = \{i, \alpha, \beta, \alpha\beta\}$

7
where \( \alpha \) interchanges \( a \) and \( c \) but fixes both \( b \) and \( d \), while \( \beta \) fixes \( a \) and \( c \) but interchanges \( b \) and \( d \). Thus,

\[
\text{Aut}(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

**Definition 1.1.16** Let \( \mathbb{Z}_n \) denote the additive group of integers modulo \( n \). If \( C \) is a subset of \( \mathbb{Z}_n \setminus \{0\} \), then construct a directed graph \( \Gamma = \Gamma(\mathbb{Z}_n, C) \) as follows. The vertices of \( \Gamma \) are the elements of \( \mathbb{Z}_n \) and \( (i, j) \) is an arc of \( \Gamma \) if and only if \( j - i \in C \). The graph \( \Gamma(\mathbb{Z}_n, C) \) is called a **circulant** of order \( n \), and \( C \) is called its **connection set**. Suppose that \( C \) has the additional property that it is closed under additive inverses, that is, \( -c \in C \) if and only if \( c \in C \). Then \( (i, j) \) is an arc if and only if \( (j, i) \) is an arc, and so we can view \( \Gamma \) as an undirected graph.

Cayley graphs are examples where graph theory can be applied to groups. These graphs are useful for studying the structure of groups and the relationships between elements with respect to subsets of these groups (for example, generating sets). Cayley graphs are also useful in semigroup theory, but as the focus of this thesis is on group theory, we will go into details.

**Definition 1.1.17** Let \( G \) be a group and let \( S \) be a subset of \( G \) that is
closed under taking inverses and does not contain the identity. Then the
Cayley graph $\Gamma(G, S)$ is the graph with vertex set $G$ and edge set

$$E(\Gamma(G, S)) = \{gh : hg^{-1} \in S\}.$$ 

If $S$ is an arbitrary subset of $G$, then we can define a directed graph
$\Gamma(G, S)$ with vertex set $G$ and arc set $\{(g, h) : hg^{-1} \in S\}$. If $S$ is invers
closed and does not contain the identity, then this graph is undirected and
has no loops, and the definition reduces to that of a Cayley graph. Most
of the results for Cayley graphs apply in the more general directed case
without modification, but we explicitly use directed Cayley graphs only in
Chapter 4.

In [14], G. A. Beny, et al. have derived a formula for number of
Cayley graphs on $\mathbb{Z}_n$ that are undirected. Motivated by their findings ,
in the Chapter 2 of present thesis we determine the number of Cayley
graphs on dihedral group $D_{2n}$ and Symmetric Group $S_n$ and Alternating
Group $A_n$ that are undirected.

The automorphism group of a graph is very naturally viewed as a
group of permutations of its vertices, and so we now present some basic
information about permutation groups. This includes some simple but
very useful counting results. For a group theorist this result might be a
disappointment, but we take its lesson to be that interesting interactions between groups and graphs should be looked for where the automorphism groups are large. Consequently, we also take the time here to develop some of the basic properties of transitive groups.

**Definition 1.1.18** Let $\Omega$ be a set and $G$ be a finite group. Suppose that, for each $\alpha \in \Omega$ and $g \in G$, there corresponds a member of $\Omega$, denoted by $\alpha^g$. We say that this correspondence defines an **action** of $G$ on $\Omega$, or $G$ acts on $\Omega$, if for any $\alpha \in \Omega$ and $g, h \in G$ the following (i)-(ii) hold:

(i) $\alpha^1 = \alpha$, where $1$ is the identity of the group $G$;

(ii) $(\alpha^g)^h = \alpha^{gh}$.

In other words, an action of $G$ on $\Omega$ is a mapping $(\alpha, g) \rightarrow \alpha^g$ from $\Omega \times G$ to $\Omega$ which satisfies the conditions (i), (ii) above. In such a case, the degree of the action of $G$ on $\Omega$ is defined to be $|\Omega|$. We say that an element $g$ of $G$ fixes a point $\alpha$ of $\Omega$ if $\alpha^g = \alpha$. The kernel of the action of $G$ on $\Omega$ is defined to be the subgroup of all elements of $G$ which fix each point of $\Omega$. If this kernel is equal to the identity subgroup of $G$, then $G$ is said to act faithfully on $\Omega$. 
Example 1.1.19 Every permutation group $G$ on $\Omega$ acts naturally on $\Omega$, where $\alpha^g$ is the image of $\alpha$ under $g$, for $\alpha \in \Omega$ and $g \in G$. Clearly, such an action of $G$ on $\Omega$ is faithful. Except where stated otherwise, we will always assume that this is the action we are dealing with whenever we have a permutation group.

Naturally, an action of $G$ on $\Omega$ induces an equivalence relation $\sim_G$ on $\Omega$ defined by
\[
\alpha \sim_G \beta \text{ if and only if } \alpha^g = \beta \text{ for some } g \in G.
\]
The equivalence classes of $\sim_G$ are said to be $G$–orbits on $\Omega$. So any two $G$–orbits are either identical or disjoint, and the $G$–orbit containing a given point $\alpha$ of $\Omega$ is
\[
\alpha^G := \{\alpha^g : g \in G\}.
\]
We say that $G$ is transitive on $\Omega$ if there is only one $G$–orbit on $\Omega$, and otherwise $G$ is intransitive on $\Omega$.

Definition 1.1.20 For a positive integer $k$, we use $\Omega^{(k)}$ to denote the set of $k$–tuples of distinct members of $\Omega$. Let $G$ act on $\Omega$. Then $G$ induces a natural action on $\Omega^{(k)}$ defined by
\[(\alpha_1, \alpha_2, ..., \alpha_k)^g := (\alpha_1^g, \alpha_2^g, ..., \alpha_k^g)\]

for \((\alpha_1, \alpha_2, ..., \alpha_k) \in \Omega^{(k)}\) and \(g \in G\). If, under this action, \(G\) is transitive on \(\Omega^{(k)}\), then \(G\) is said to be \(k\)-transitive on \(\Omega\).

**Definition 1.1.21** Two elements \(a \text{ and } b\) of a group are **conjugate** to one another if there is an element \(g\) in the group such that

\[a = gbg^{-1}\]

(Since every element has an inverse, it also follows that \(b = g'ag'^{-1}\) where \(g' = g^{-1}\)).

**Definition 1.1.22** A **conjugate** of a subgroup \(H \leq G\) is a subgroup of \(G\) of the form

\[aHa^{-1} = \{aha^{-1} : h \in H\},\]

where \(a \in G\).

**Definition 1.1.23** The **normalizer** of \(H\) in \(G\) is the subgroup

\[N_G(H) = \{a \in G : aHa^{-1} = H\}.\]

It is obvious that \(H \triangleright N_G(H)\), and so the quotient group \(N_G(H)/H\) is defined.
1.2 Transitive Graphs

We are going to study the properties of graphs whose automorphism group acts vertex transitively. A vertex-transitive graph is necessarily regular. One challenge is to find properties of vertex-transitive graphs that are not shared by all regular graphs. We will see that transitive graphs are more strongly connected than regular graphs in general. Now we will define edge transitivity and vertex transitivity.

**Definition 1.2.1** A graph $\Gamma$ is vertex transitive (or just transitive) if its automorphism group acts transitively on $V(\Gamma)$. Thus for any two distinct vertices of $\Gamma$ there is an automorphism mapping one to the other.

**Definition 1.2.2** A graph $\Gamma$ is edge transitive if its automorphism group acts transitively on $E(\Gamma)$.

In [7], Alaeiyan, et al. gave all cubic, connected, and undirected edge-transitive Cayley graphs of dihedral groups, which are not normal edge-transitive. In [73], Praeger gave an approach to analyzing normal edge-transitive Cayley graphs. In [84], Talebi gave some normal edge-transitive Cayley graphs on dihedral groups. In **Chapter 3** we determine some normal edge-transitive Cayley graphs on Frobenius group $F_{p,3}$. 
1.3 Energy of a Graph

The study of spectral graph theory, in essence, is concerned with the relationships between the algebraic properties of the spectra of certain matrices associated with a graph and the topological properties of that graph. There are various matrices that are associated with a graph, such as the adjacency matrix, the incidence matrix, the Laplacian matrix and the distance matrix. The most common matrix investigated has been the $0-1$ adjacency matrix. The subject had its genesis with the early papers of L. M. Lihtenbaum (in, 1956) [65] and of L. Collatz and U. Sinogowitz in 1957 [36]. Since that time the subject has steadily grown and has shown surprising interrelationships with other mathematical areas. Let $\Gamma$ be a graph (assumed simple throughout) with $n$ vertices, $v_1, v_2, \ldots, v_n$ and $m$ edges. The adjacency matrix $A(\Gamma)$ of a graph $\Gamma$ is a square matrix $A(\Gamma) = [a_{ij}]$ of order $n$ in which

$$a_{ij} = \begin{cases} 
1 & \text{if } v_i \text{ and } v_j \text{ are adjacent}, \\
0 & \text{otherwise}.
\end{cases}$$

The eigenvalues of adjacency matrix $A(\Gamma)$ are the numbers $\lambda$ such that
\( Ax = \lambda x \) has a non zero solution vector: each solution is an eigenvector associated with \( \lambda \). The eigenvalues of a graph are the eigenvalues of its adjacency matrix \( A(\Gamma) \). These are the roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the characteristic polynomial

\[
\phi(\Gamma; \lambda) = det(\lambda I - A(\Gamma))
\]

\[
= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n, \quad (1.3.1)
\]

where \( I \) is the \( n \times n \) identity matrix. Since \( A(\Gamma) \) is a real symmetric matrix with zero trace, these eigenvalues are all real with sum equal to zero.

**Definition 1.3.2** The spectrum of a graph is the list of distinct eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_r \) of \( G \), with their multiplicities \( m_1, m_2, \ldots, m_r \): we write as

\[
Spec(G) = \begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_r \\
m_1 & m_2 & \cdots & m_r
\end{pmatrix}.
\]

The basic result concerning the characteristic polynomial is Sach’s formula which gives explicit expressions for the coefficient’s \( c_i \). An
immediate consequence of Sach’s formula is the following theorem.

**Theorem 1.3.3** Let \( \Gamma \) be a graph of order \( n \), with characteristic polynomial

\[
\phi(\Gamma; \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n.
\]

Then,

(i) \( c_1 = 0 \),

(ii) \( -c_2 = \) the number of edges in \( \Gamma \),

(iii) \( -c_3 = \) twice the number of triangles in \( \Gamma \).

The energy of a graph \( \Gamma \) was first defined by I. Gutman [52] in 1978 as the sum of the absolute values of its eigenvalues:

\[
E(\Gamma) = \sum_{i=1}^{n} |\lambda_i|.
\]

Ever since the graph energy \( E(\Gamma) \) of a simple graph \( \Gamma \) was introduced by I. Gutman [52], there is a constant stream of papers devoted to this topic. Survey of development before 2001 can be found in [53]. For recent developments one can consult [28].

If a graph is not connected, its energy is the sum of its connected components. Thus there is no loss in generality in assuming that graphs
are connected.

The Laplacian matrix is one of the important matrices that is naturally associated with a graph $\Gamma$. The Laplacian matrix of $\Gamma$ is $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma) = \text{diag}(d_1, d_2, \ldots, d_n)$ and $d_i$ is the degree of vertex $v_i$ for $i = 1, 2, \ldots, n$. $L(\Gamma)$ is a real and symmetric matrix. More details about the energy of a graph and the Laplacian matrix can be found in [64, 67, 68, 69, 71, 88].

I. Gutman and Bo. Zou [55] defined Laplacian energy $LE(\Gamma)$ of a graph and they found a great deal of analogy between the properties $E(\Gamma)$ and $LE(\Gamma)$. In [55] they have also obtained some new properties of $LE(\Gamma)$.

The eigenvalues of $L(\Gamma)$ are important in graph theory, because they have relations to numerous graph invariants including connectivity, expanding property, isoperimetric number, independence number, genus, mean distance and band width type parameters of a graph. See, for example [8, 9, 29, 35, 38, 39, 40, 49, 50, 56, 58, 74, 87].
In Chapter 4 of the present thesis motivated by the skew energy of a digraph as vitiated by C.Adiga, R.Balakrishnan and Wasin So[1], we introduce and investigate the skew energy of a Cayley digraphs of cyclic groups and dihedral groups and establish sharp upper bound for the same.