CHAPTER 4

NECESSARY AND SUFFICIENT CONDITIONS FOR THE REALIZATION OF Y-MATRICES OF (n+2)-NODE n-PORT NETWORK

4.1 INTRODUCTION

In this chapter necessary and sufficient conditions for the realization of an (nxn) real symmetric matrix as the short-circuit conductance matrix Y of a n-port, (n+2)-node resistive network containing no negative conductances are derived. Besides, Biorci's example is made realizable with number of edge conductances which are in accordance with his conjecture. Biorci's conjecture proved in Chapter 3 plays a vital role in arriving at the results obtained in the following sections.

4.2 A NEW ALGORITHM FOR (n+2)-NODE n-PORT REALIZATION

Let it be required to realize an (nxn) real symmetric matrix with a specified port configuration which is in two connected parts. From the given matrix, an (nxn) matrix Y corresponding to a port configuration whose two connected parts T₁ and T₂ are linear as in Fig. 4.1 can be got by using congruent transformation. The relevant procedure is given by Thulasiraman and Swamy (1981). Let the number of ports in T₁ be 'm' and the number of ports in T₂ be 'n-m'. Let the nodes be numbered as 1, 2, ..., (n+2). Let the set of vertices in T₁ be denoted by V₁ and those in T₂ be V₂. Let gᵢⱼ represent the conductance of the edge connecting vertices i and j in the network N realizing the Y-matrix. Following the realization procedure given in section (3.2)
Fig. 4.1 General $n$-port $(n+2)$ node linear 2-tree port configuration
the parallel combination of $N_p$ and $N_d$ will give the n-port network $N$ which contains no negative conductances. That is, if $(g_{ij})_d$'s are the edge conductances of the network of departure $N_d$ some of which are necessarily negative, the edge conductances of $(g_{ij})_p$'s of the padding network should be such that

$$
(g_{ij})_d + (g_{ij})_p = g_{ij} \geq 0 
$$

(4.1)

Subbarami Reddy and Thulasiraman (1972) derived the formulae for the edge conductances $(g_{ij})_p$'s and they are as follows:

$$(g_{ij})_p = \frac{-S_i S_j}{S} ; \quad i, j \in V_1 \cap V_2 (V_1 \cap V_2)$$

$$= \frac{S_i S_j}{S} ; \quad i \in V_1 \quad j \in V_2$$

Where

$$S_i = \sum_{j \in V_2(V_1)} (g_{ij})_p = \sum_{j \in V_2(V_1)} g_{ij} ; \quad i \in V_1 \cap V_2 (V_1 \cap V_2)$$

$$S = \sum_{i \in V_1(V_2)} S_i$$

(4.2)

In this section an alternative procedure for realizing an $(nxn)$ real symmetric matrix as the short-circuit conductance matrix $Y$ of a resistive $(n+2)$-node n-port network whose port configuration is as in Fig. 4.1. The procedure is as follows:
Step 1

Obtain the edge conductances \((g_{ij})_d\)'s of the unique network of departure \(N_d\) using the procedure given by Subbarami Reddy (1972). It may be noted that this \(N_d\) has \(Y\) as its short-circuit conductance matrix. Some of the \((g_{ij})_d\)'s are necessarily negative.

Step 2

Let \(s_i = \frac{s_i}{\sqrt{S}}\); \(i \in V_1(V_2)\) 

Substituting Equation (4.3) in Equation (4.2) obtain

\[(g_{ij})_p = -s_i s_j \quad ; \quad i, j \in V_1(V_2)\]

\[= s_i s_j \quad ; \quad i \in V_1\]

\[= s_j s_j \quad ; \quad j \in V_2\] (4.4)

Step 3

Let \((g_{ij})_p\)'s be a set of conductances in \(N_p\) whose corresponding conductances \((g_{ij})_d\)'s in \(N_d\) are of opposite sign. From such a set of conductances chose only \((n+1)\) of them such that these conductances form a tree in \(N\). Make these conductances in \(N\) zero. That is

\[(g_{ij})_d - s_i s_j = 0 \quad \text{for some} \quad 'i' \quad \text{and some} \quad 'j'\]

or

\[s_i s_j = (g_{ij})_d\] (4.5)
It is always possible to make this assumption since according to Biorci’s conjecture utmost \((n+2)(n+2-1)/2-(n+1) = n(n+1)/2\) conductances need be positive.

**Step 4**

Using Equation (4.5), obtain all other \(s_i\)'s in terms of one \(s_j\). This is always possible because in Step 3, it is to be noted that equations (4.5) covers all the nodes. It is known that

\[
\sum_{i=1}^{m+1} s_i = \sum_{j=m+2}^{n+2} s_j
\]

Using equation (4.6) replace all \(s_i\)'s in terms of a single \(s_j\). We will now have an equation involving \(s_j\) only and this can be easily solved for \(s_j\). After getting the values of \(s_j\) all other \(s_i\)'s can be obtained using Equations (4.5)

**Step 5**

Substitute these values of \(s_i\)'s \((i = 1, 2, \ldots, n+2)\) and \((g_{ij})_d\)'s in the equation

\[
| (g_{ij})_d | - s_i s_j = g_{ij}
\]

and get all the non-zero edge conductances \(g_{ij}\)'s of the final realization \(N\).
Step 6

In step 5, if some $g_{ij}$'s turn out to be negative, go back to step 3, and choose another set of $(n+1)$ edge conductances (forming a tree) and make them zero and go through step 4, 5 and 6. Repeat this process until all non-zero $g_{ij}$'s are non-negative. If this procedure fails, the given matrix $Y$ is not realizable as an $n$-port network with $(n+2)$-nodes. The necessary and sufficient conditions can be stated in the form of a theorem as follows:

Theorem 4.1

An $(nxn)$ real symmetric matrix $Y'$ is realizable as the short-circuit conductance matrix of an $(n+2)$-node $n$-port network with a specified port configuration and containing only positive conductances iff

1. the given $(nxn)$ matrix is paramount
2. after suitable congruent transformation of the type $A' Y' A$, the resulting matrix $Y$ which corresponds to a 2-tree linear tree port structure should be piece wise uniformly tapered.
3. steps 1 to 6 should yield a network $N$ containing no negative conductances. If not the matrix $Y'$ is not realizable.

4.3 EXISTING PROBLEM EXAMPLES AND THEIR SOLUTION USING NEW ALGORITHM

In this section two worked out examples were given. One of the matrices taken is that of Biorci's (1966)
Example 4.1

Considering first the same (4x4) Y-matrix realized in section (3.4), based on the new algorithm yet another (n+2)-node n-port realization containing only 10 positive conductances has to be deduced. Let it be required to realize with the port configuration in Fig. 4.2

\[
Y = \begin{bmatrix}
67 & 45 & -6 & 2 \\
45 & 62 & 5 & 15 \\
-6 & 5 & 60 & 40 \\
2 & 15 & 40 & 55
\end{bmatrix}
\]

The edge conductances of the unique network of departure \( N_d \) are

\[
G_d = \text{Dia} \{ g_{12}, g_{13}, g_{15}, g_{16}, g_{22}, g_{24}, g_{25}, g_{34}, g_{35}, g_{36}, g_{45}, g_{46}, g_{56} \} \quad d
\]

\[
\text{Dia} \{ 22, 45, 6, -8, 2, 17, -11, -2, 13, 5, 10, -15, 20, 40, 15 \} \quad d
\]

The conductances \( g_{12}, g_{23}, g_{24}, g_{15} \) and \( g_{56} \) are made zero in the final network \( N \). The Equations corresponding to Equation (4.5) are

\[
\begin{align*}
S_1 S_2 &= 22 \\
S_2 S_3 &= 17 \\
S_2 S_4 &= 11 \\
S_1 S_5 &= 8 \\
S_5 S_6 &= 15
\end{align*}
\]
Fig. 4.2 Linear 2-tree port configuration of Example 4.1
From the above equations

\[ s_2 = \frac{22}{s_1} \]
\[ s_3 = \frac{17}{22} s_1 \]
\[ s_4 = \frac{s_1}{2} \]
\[ s_5 = \frac{8}{s_1} \]
\[ s_6 = \frac{15}{8} s_1 \]

Substituting these values of \( s_i \)'s in the following equation

\[ s_1 + s_2 + s_3 = s_4 + s_5 + s_6 \]

That is

\[ \frac{s_1}{s_1} + \frac{22}{s_1} + \frac{17}{22} s_1 = \frac{s_1}{2} + \frac{8}{s_1} + \frac{15}{8} s_1 \]

\[ \frac{106}{176} s_1 = \frac{14}{s_1} \]

\[ s_1^2 = 23.24 \]

Hence

\[ s_1 = 4.82 \quad s_2 = 4.564 \quad s_3 = 3.724 \]
\[ s_4 = 2.41 \quad s_5 = 1.659 \quad s_6 = 9.0375 \]
Now the non-zero edge conductances may be calculated as follows:

\[ g_{ij} = (g_{ij})_p + (g_{ij})_d \]

\[ g_{14} = s_1s_2 + 6 = 4.82 \times 2.41 + 6 = 17.62 \]
\[ g_{16} = s_1s_6 + 2 = 4.82 \times 9.037 + 2 = 45.58 \]
\[ g_{26} = s_2s_6 + 13 = 4.564 \times 9.0375 + 13 = 54.25 \]
\[ g_{34} = s_3s_4 + 5 = 3.724 \times 2.41 + 5 = 13.98 \]
\[ g_{35} = s_3s_5 + 10 = 3.724 \times 1.659 + 10 = 16.178 \]
\[ g_{25} = s_2s_5 - 2 = 4.564 \times 1.659 - 2 = 5.571 \]
\[ g_{36} = s_3s_6 - 15 = 3.724 \times 9.0375 - 15 = 18.65 \]
\[ g_{46} = -s_4s_6 + 40 = -2.41 \times 9.0375 + 40 = 18.21 \]
\[ g_{13} = -s_1s_3 + 45 = -4.82 \times 3.724 + 45 = 27.04 \]
\[ g_{45} = -s_4s_5 + 20 = -2.41 \times 1.659 + 20 = 16 \]

All the conductances are in Siemens.

Thus the given Y- matrix has been realized by \( N \) containing just 10 positive edge conductances. It is to be noted that the corresponding \( S_i \)'s are:

\[ S_1 = 63.2 \quad S_2 = 59.82 \quad S_3 = 48.81 \]
\[ S_4 = 31.60 \quad S_5 = 21.751 \quad S_6 = 118.48 \]

**Example 4.2**

**Biorci's matrix**

Now the (6x6) matrix \( Y \) of Biorci (1966) is examined. He was not able to realize this with 21 edge conductances, in accordance with his conjecture, employing the method continuous reduction.
The specified port configuration is as shown in Fig. 4.3. The $Y'$-matrix corresponding to the linear 2-tree port structure as shown in Fig. 4.4 is to be obtained as follows. The procedure is based on the method explained by Thulasiraman and Swamy (1981) using congruent transformation.

The transformation matrix $A$ relating the vector of port voltages in Fig. 4.4 to the port voltages in Fig. 4.3 is given by

$$A = \begin{bmatrix}
-1 & -1 & -1 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}$$

The congruent transformation $Y = A^t Y'A$ yield $Y'$-matrix with 2-tree linear port configuration as in Fig. 4.4.
Fig. 4.3 The 2-tree port configuration of Example 4.2

Fig. 4.4 The linear 2-tree port configuration of Example 4.2
The conductances of unique network of departure $N_d$ are given in the compact form as given below.

$$
\begin{bmatrix}
3002.300 & 2001.620 & 1000.920 & 0000.690 & -0000.065 & -0000.150 \\
1000.920 & 3001.800 & 4002.790 & 2002.018 & -0000.190 & -0000.375 \\
0000.690 & 1001.340 & 2002.018 & 3002.246 & 0000.122 & 0000.242 \\
-0000.065 & -0000.160 & -0000.190 & 0000.122 & 1002.730 & 1001.510 \\
-0000.150 & -0000.260 & -0000.375 & 0000.242 & 1001.510 & 1003.000
\end{bmatrix}
$$

In the final network $N$ seven conductances namely, $g_{10}$, $g_{23}$, $g_{36}$, $g_{40'}$, $g_{16}$, $g_{50}$ and $g_{56}$ are made equal to zero. This yields the following equations corresponding to Equation (4.5).

$$
\begin{align*}
s_1s_6 &= 0.69 \\
s_3s_4 &= 0.312 \\
s_2s_6 &= 0.11 \\
s_4s_{6'} &= 0.312 \\
s_1s_6 &= 0.15
\end{align*}
$$
Using the above equations all \( s_j \)'s can be expressed in terms of say \( s_3 \) as in Example 1. It is to be noted that

\[ s_1 + s_2 + s_3 + s_4 + s_6 = s_0' + s_5 + s_6 \]

Substituting in the above all \( s_j \)'s in terms of \( s_3 \), and solving for \( s_3^2 \) yields \( s_3^2 = 0.158 \). Therefore \( s_3 = 0.3975 \). Using this value of \( s_3 \) the numerical values for all other \( s_j \)'s can be calculated. Following are the values of the non-negative conductances in Siemens.

\[
\begin{align*}
g_{45} &= s_4 s_5 - 0.000305 = 0.7849 \times 3.0690 - 0.000305 = 0.002103 \\
g_{06} &= s_1 s_6 - 0.000242 = 2.2331 \times 0.4853 - 0.000242 = 0.000841 \\
g_{12} &= -s_1 s_2 + 1000.680 = -0.3089 \times 0.2264 + 1000.680 = 1000.610 \\
g_{13} &= -s_1 s_3 + 1000.700 = -0.3089 \times 0.3975 + 1000.700 = 1000.577 \\
g_{14} &= -s_1 s_4 + 1000.230 = -0.3089 \times 0.7849 + 1000.230 = 0.999987 \\
g_{24} &= -s_2 s_4 + 1000.230 = -0.2264 \times 0.7849 + 1000.230 = 1000.050 \\
g_{20} &= -s_2 s_0 + 1000.650 = -0.2264 \times 2.2331 + 1000.650 = 1000.144 \\
g_{30} &= -s_3 s_0 + 1001.678 = -0.3975 \times 2.2331 + 1000.678 = 0.999790 \\
g_{40} &= -s_4 s_0 + 1000.228 = -0.7849 \times 2.2331 + 1000.228 = 0.998475 \\
g_{06'} &= -s_0 s_6 + 1001.510 = -0.3975 \times 0.4853 + 1001.510 = 1000.320 \\
g_{23'} &= -s_2 s_3 + 0.000680 = -0.2264 \times 0.3975 + 0.000680 = 0.000590 \\
g_{36'} &= s_3 s_6 - 0.000115 = 0.3975 \times 0.4853 - 0.000115 = 0.000080 \\
g_{10'} &= s_1 s_0' + 0.000065 = 0.3089 \times 0.3975 + 0.000065 = 0.000188 \\
g_{15'} &= s_1 s_5 + 0.000085 = 0.3089 \times 3.0690 + 0.000085 = 0.001033 \\
g_{20'} &= s_2 s_0' + 0.000095 = 0.2264 \times 0.3975 + 0.000095 = 0.000184
\]

\[ s_5 s_6 = 1.49 \]

\[ s_0' s_5 = 1.22 \]
\[ g_{25} = s_2s_5 + 0000.015 = 0.2264 \times 3.0690 + 0000.015 = 0000.709 \]
\[ g_{30'} = s_3s_{0'} + 0000.030 = 0.3975 \times 0.3975 + 0000.030 = 0001.305 \]
\[ g_{35} = s_3s_5 + 0000.085 = 0.3975 \times 3.0690 + 0000.085 = 0001.305 \]
\[ g_{46} = s_4s_6 + 0000.617 = 0.7849 \times 0.4853 + 0000.617 = 0000.998 \]
\[ g_{00'} = s_0s_{0'} + 0000.122 = 2.2331 \times 0.3975 + 0000.122 = 0001.010 \]
\[ g_{05} = s_0s_5 + 0000.120 = 2.2331 \times 3.0690 + 0000.120 = 0006.970 \]

Thus the \( Y' \)-matrix has been realized by means of a network \( N \) containing only 21 positive edge conductances in accordance with Biorci's conjecture.

### 4.4 COMMENTS

In this chapter, using the results of Chapter 3 deduced a procedure for realizing a given \((n \times n)\) real symmetric matrix as short-circuit conductance matrix of an \((n+2)\)-node \( n \)-port resistive network with a specified port configuration. If this procedure fails then the given matrix is not realizable by an \((n+2)\)-node \( n \)-port network. Thus theorem 1 gives the necessary and sufficient conditions for such a realization. It is interesting to note that if an \((n \times n)\) real symmetric matrix is realizable by an \( n \)-port resistive network on \((n+2)\)-nodes and containing \( n(n+1)/2 \) conductances, it is quite possible that to have more than one such realization unlike in \((n+1)\)-node realization which is unique.