CHAPTER 3

BIORCI'S CONJECTURE AND ITS PROOF

3.1 INTRODUCTION

It is a matter of considerable significance that the first of the panel discussions (1962) of the IRE Transactions on Circuit Theory, devoted to unsolved problems in circuit theory, should consider the realizability of the n-port networks without transformers and in particular, of n-port networks containing only real positive fixed resistors. The number of nodes of the n-port network cannot be lower than \((n+1)\) since the voltages across the ports must be independent. If the network turns out to have more than \(2n\) nodes, some nodes do not belong to any port, and they may be suppressed through star mesh transformations. Therefore the number of nodes may be any number between \((n+1)\) and \(2n\).

As in the case of realization of \((n+1)\) node n-port network from an \((nxn)\) short-circuit conductance matrix, Biorci (1962) conjectured in the panel discussion that a real symmetric \((nxn)\) short-circuit conductance matrix is realizable on \((n+p)\), \(p \geq 2\) nodes without ideal transformers, only a maximum of \(n(n+1)/2\) positive edge conductances will be required. This conjecture is neither proved nor disproved till now though some attempt in this direction was made by Biorci (1966) himself. In this chapter Biorci's conjecture is proved and this result is used in chapter 4.
3.2 GENERAL PROCEDURE FOR REALIZATION OF A SHORT-CIRCUIT CONDUCTANCE MATRIX

This section presents a general procedure for synthesis of $Y$-matrix as given by Subbarami Reddy and Thulasiraman (1972). It differs from that of Guillemin (1961) only in the method used to generate padding networks. The procedure is as follows:

**Step 1**

For the given $Y^*$ matrix obtain the unique network of departure $N_d$ having the prescribed port configuration. Let the port configuration $T^*$ corresponding to the given $Y^*$ be in $p$ connected parts. Obtain short circuit conductance matrix $Y$ corresponding to a linear tree port structure $T$ by a suitable transformation as suggested by Kim and Chien (1962) after properly introducing rows and columns of zeros in $Y^*$, corresponding to the non port tree branches. Then obtain the edge conductance matrix $G_d$ of the network of departure by using the procedure reported by Subbarami Reddy (1972).

**Step 2**

Assuming suitable values for non-zero non-unity potential factors $K_{(k)j}$’s and using Theorem 2 of Murti and Thulasiraman (1969) construct the modified cutset matrix $C$ appropriate to the given port configuration. Obtain a padding $n$-port network $N_p$ such that all $S_{(k)j}$’s are non-negative for all $i$ and $j$, $j \neq i$ ; $i, j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, n$. For this use the formulæ given in section 2.6. It is to be noted that in the $Y$ matrix synthesis problem with port configuration alone is specified the $n(p-1)+p(p-1)/2$ parameters namely $(S_0)_p$ and $K_{(k)j}$’s are to be assumed arbitrarily.
Step 3

If for the assumed values, no $N_p$ can be obtained such that the parallel combination $N$ of $N_p$ and $N_d$ contains no negative conductances, assume a different set of values for $K_{i0j}$'s and hence a different modified cut-set matrix.

Step 4

If an $N_p$ can be found which results in $N$ containing no negative conductances, then this network $N$ will be a proper realization of $Y$. If not, the matrix $Y$ is not realizable by resistive $n$-port networks having the prescribed port configuration. Thus step 4 results in an $n$-port network $N$ with all non-negative edge conductances. That is in the following equation,

$$i = 1, 2, \ldots, p$$
$$j = 1, 2, \ldots, p$$
$$k = 0, 1, \ldots, n_i$$
$$m = 0, 1, \ldots, n_j$$

(3.1)

$s_{k,m}^i$'s of $N$ are to be non-negative. Equation (3.1) can be rewritten as,

$$i = 1, 2, \ldots, p$$
$$j = 1, 2, \ldots, p$$
$$k = 0, 1, \ldots, n_i$$
$$m = 0, 1, \ldots, n_j$$

(3.2)
In the inequalities (3.2) formulae for \( g_{k,m}^{i} \)'s are given in Equations (2.26), (2.27) and (2.29). In these formulae potential factors K\(_{ikj} \)'s are involved. The values of set of K\(_{ikj} \)'s in terms of S\(_{ik} \)'s and S\(_{ij} \)'s can be obtained by solving Equation (2.22) There are \( n(p-1) \) Equations in (2.22) and the number of K\(_{ikj} \)'s are also \( n(p-1) \).

After obtaining K\(_{ikj} \)'s from Equation (2.22) in terms of S\(_{ik} \)'s and S\(_{ij} \) these are substituted for K\(_{ikj} \)'s in Equations (2.26), (2.27) and (2.29). The resulting formulae for \( g_{k,m}^{i} \)'s will be non linear functions of S\(_{ik} \)'s and S\(_{ij} \)'s.

Let \( g_{k,m}^{i} = f (S_{ik}, S_{ij}) \)

\[ i = 1, 2, \ldots, p \]
\[ j = 1, 2, \ldots, p; j \neq i \]
\[ k = 0, 1, \ldots, n_i \]
\[ m = 0, 1, \ldots, n_j \]  

(3.3)

Replace all S\(_{ik} \)'s; i, j = 1,2, ..., p; i ≠ j; in Equation (3.3) by S\(_{ij} \)'s and S\(_{ij} \)'s using Equation (2.23)

Inequalities (3.2) now become

\[ (g_{k,m}^{i}) + f (S_{ik}, S_{ij}) \geq 0 \]

\[ i = 1, 2, \ldots, p \]
\[ j = 1, 2, \ldots, p; i \neq j \]
\[ k = 1, 2, \ldots, n_i \]
\[ m = 1, 2, \ldots, n_j \]  

(3.4)
In Equation (3.4), $\left( g_{k,m} \right)_d$'s are known real numbers, being the edge conductances of unique network of departure $N_d$. The number of $S_{kj}$'s and $S_{ij}$'s in $f(S_{kj}, S_{ij})$ is $n(p-1)$ and $p(p-1)/2$ respectively. Any method of realizing the given $(nxn)$ matrix now boils down to selecting proper values for $S_{kj}$'s and $S_{ij}$'s such that inequalities (3.4) are satisfied. In the next section this realization problem is converted into one of optimization problem with $S_{kj}$'s and $S_{ij}$'s as decision variables. It has to be noted that in Equation (3.4) some selected sets of inequalities are added up the left hand side should yield a $S_{kj}$ since

$$\sum_{m=0}^{n_j} (g_{k,m})_d = 0$$

and the right hand side yields the sums of corresponding values of the non-negative edge conductances of $N$ as is apparent in Equation (3.2)

### 3.3 PROOF OF BIORCI'S CONJECTURE BY CONVERSION OF Y-MATRIX SYNTHESIS PROBLEM INTO AN OPTIMIZATION PROBLEM

In this section, the realization problem is converted into an optimization problem. Define a function

$$F = \sum_{i=1}^{P} \sum_{k=1}^{n_i} \sum_{j=1}^{p} \left( S_{kj} + S_{ij} \right)$$
as an objective function. Now minimize $F$ with inequalities in (3.2) as constraints. In the objective function $S_{k,j}$'s and $S_{ij}$'s are called decision variables. If many sets of $S_{k,j}$'s and $S_{ij}$'s are available, any set with $S_{k,j} \geq 0$ and $S_{ij} \geq 0$ can be selected and substituted in the formulae for the edge conductances to satisfy inequalities (3.2). So if the given matrix is realizable with the specified port configuration, this optimization technique with $S_{k,j}$'s and $S_{ij}$'s as decision variables will yield one such realization.

Let the decision variables be denoted as $x_i, i = 1, 2, \ldots, [n(p-1)+p(p-1)/2]$.

Let the constraints in (3.4) be written as $g_j \geq 0, j = 1, 2, \ldots, (n+p)\,(n+p-1)/2$ where $g_j$'s are the functions of the decision variables $x_i$'s.

In this section the Lagrange method of optimization is discussed. The constraints $g_j \geq 0$ can all be converted into equations by means of slack variables denoted by $y_j^2$ as $g_j - y_j^2 = 0, j = 1, 2, \ldots, (n+p)\,(n+p-1)/2$. Define the Lagrange function $L$ as

$$L = F(x_1, x_2, \ldots, x_D) + \sum_{j=1}^{C} \lambda_j (g_j - y_j^2)$$

Where $\lambda_j$'s are Lagrange multipliers; $C = (n+p)\,(n+p-1)/2$ and $D = n\,(p-1) + p\,(p-1)/2$. Formulate the following equations and solve for $x_i$'s and $\lambda_j$'s

$$\frac{\partial L}{\partial x_i} = 0; \ i = 1, 2, \ldots, D \quad (3.5a)$$

$$\frac{\partial L}{\partial \lambda_j} = 0; \ j = 1, 2, \ldots, C \quad (3.5b)$$
The equations in (3.5c) reduces into

$$\frac{\partial L}{\partial y_j} = 0; \ j = 1, 2, \ldots, C$$ \hspace{1cm} (3.5c)

The equations in (3.5c) reduces into

$$-2\lambda_j y_j = 0; \ j = 1, 2, \ldots, C$$ \hspace{1cm} (3.6)

In Equation (3.6) either $$\lambda_j = 0$$ or $$y_j = 0$$. Some of the $$\lambda_j$$'s are to be definitely non-zero. The question now is regarding the maximum number of $$\lambda_j$$'s that could be assumed to be non-zero. There is only one option as far as allowing certain number of $$\lambda_j$$'s to be non-zero is concerned. In fixing this number, Equations (3.5b) play a vital role. These equations are C in number and they are

$$g_j - y_j^2 = 0, \ j = 1, 2, \ldots, C$$ \hspace{1cm} (3.7)

It is known that $$g_j$$ contains the decision variables $$x_i$$'s which are D in number. To solve Equation (3.7) for $$x_i$$'s only a maximum of D number of these equations involving all $$x_i$$'s are needed. In Equation (3.6) if D number of $$y_j$$'s are made equal to zero the D number of $$\lambda_j$$'s will be non-zero. The D number of constraints of the type $$g_j = 0$$ can be solved for $$x_i$$. These values of $$x_i$$'s are then substituted in Equation (3.5a) and the resulting equations are solved for the corresponding non-zero $$\lambda_j$$'s. It may be mentioned here that the number of these linear equations are only D involving as many $$\lambda_j$$'s. The details of the proof of existence of real feasible solution for some D active constraints are presented in the Appendix. The discussion in the Appendix is based entirely on the fact that the Langrange method of optimization should yield a relative optimum for a function subject to a set of constraints, provided the constraints do have a feasible solution.
The crux of the problem in the discussions above is the presumption that a feasible solution for the inequalities (3.4) does exist. In fact the slack variables $y_j^2$'s used in Equation (3.7) are nothing but the edge conductances $g_y$'s of the resulting n-port network. When there is a solution for Equation (3.7) it implies that all $g_y$'s are non-negative, i.e., $g_y \geq 0$ for any $i$ and $j$. Since any $S_{k,j}$ or $S_j$ is equal to the sum of these $g_y$'s, the solution obtained is a feasible solution and it means that any $S_{k,j}$ or $S_j$ is non-negative.

The objective function is defined as the sum of all the variables namely $S_y$'s and $S_{k,j}$'s which appear in the inequalities (3.4). For the sake of convenience in proving the conjecture all $S_{k,j}$'s and $S_y$'s are included in the objective function $F$. If the objective function contains only a few (not all) $S_y$'s and $S_{k,j}$'s then the feasible solution obtained is different from the previous one. But in this case the solution can be such that more $y_j^2$'s will be greater than zero. This means that more number of edge conductances $g_y$'s will be non zero. Therefore the number of non-negative edge conductances in the n-port network will exceed the Biorci's number namely $n(n+1)/2$. Hence the objective function should contain all $S_y$'s and $S_{k,j}$'s which are present in the inequalities (3.4). Thus the minimum value of the function $F$ corresponds to D number of constraints $g_j \geq 0$; $j = 1, 2, ..., (n+p)(n+p-1)/2$ being satisfied with strict equality. So if there is a minimum with $x_i \geq 0$ ($i = 1, 2, ..., D$) and all $g_j \geq 0$, $j = 1, 2, ..., (n+p)(n+p-1)/2$, a proper realization which contains utmost $[(n+p)(n+p-1)/2 - n(p-1) - p(p-1)/2]$ number of positive $g_y$'s is obtained. The number is $n(n+1)/2$. Hence the proof of Biorci's conjecture.
3.4 AN EXAMPLE

Example 3.1

This example is to illustrate the realization procedure. The matrix realized by Subbarami Reddy and Thulasiraman (1972) employing 13 edge conductances is considered for this purpose. Now this matrix can be realized using \( n(n+1)/2 = 4(4+1)/2 \) edge conductances. The matrix is

\[
Y = \begin{bmatrix}
67 & 45 & -6 & 2 \\
45 & 62 & 5 & 15 \\
-6 & 5 & 60 & 40 \\
2 & 15 & 40 & 55
\end{bmatrix}
\]

and the specified port configuration is shown in Fig. 3.1. The nodes are numbered 1, 2, 3, 4, 5 and 6. Any edge conductances connecting node \( i \) to node \( j \) is denoted as \( g_{ij} \). \( S_i \) represents the sum of all edge conductances connecting node \( i \) in \( T_1 \) (or \( T_2 \)) to all the nodes in \( T_2 \) (or \( T_1 \)). Let \( K_i \) denote the potential factor of ports in \( T_2 \) (or \( T_1 \)) with respect to 'i' in \( T_1 \) (or \( T_2 \)).

Let \( S_1 + S_2 + S_3 = S_4 + S_5 + S_6 \)

The conductances of the corresponding network of departure \( N_d \) are

\[
G_d = \text{Diag} \{ g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{23}, g_{24}, g_{25}, g_{26}, g_{34}, g_{35}, g_{36}, g_{45}, g_{46}, g_{56} \}_d
\]

\[
= \text{Diag} \{ 22, 45, 6, -8, 2, 17, -11, -2, 13, 5, 10, -15, 20, 40, 15 \}_d
\]

The formulae for \((g_{ij})_p\)'s are

\[
(g_{14})_p = S_4 K_1
\]
Fig. 3.1 Port configuration of Example 3.1
(g_{16})_p = S_6 K_1 \\
(g_{34})_p = S_4 K_2 \\
(g_{36})_p = S_6 K_2 \\
(g_{35})_p = S_3 - S_4 K_2 - S_6 K_2 \\
= [S_3 - K_2 (S_4 + S_6)] \\
= [S_3 + K_2 S_5 - K_2 S ] \\
(g_{24})_p = S_4 - S_4 K_1 - S_4 K_2 \\
(g_{26})_p = S_6 (1 - K_1 - K_2) \\
(g_{15})_p = S_1 - K_1 (S_4 + S_6) \\
(g_{25})_p = S_2 - S_4 (1 - K_1 - K_2) - S_6 (1 - K_1 - K_2) \\
= S_2 - (1 - K_1 - K_2) (S_4 + S_6) \\
(g_{13})_p = - S_3 K_1 \\
(g_{46})_p = - S_6 K_3 \\
(g_{12})_p = - S_2 K_1 \\
(g_{32})_p = - S_2 K_2 \\
(g_{45})_p = - S_5 K_3 \\
(g_{65})_p = - S_5 K_4

\text{(3.8a)}

\text{and}

S_1 = S K_1 \\
S_3 = S K_2 \\
S_4 = S K_3 \\
S_6 = S K_4 \\
S_2 = S - S_1 - S_3 \\
S_5 = S - S_4 - S_6

\text{(3.8b)}

Substituting Equation (3.8b) in Equation (3.8a) and replacing all potential factors

\[
(g_{ij})_p = \frac{S_i S_j}{S} \quad \text{for} \quad i = 1, 2, 3 \\
\text{or} \quad j = 3, 4, 5 \\
(g_{ij})_p = \frac{S_i S_j}{S} \quad \text{for} \quad i, j = 1, 2, 3 \quad \text{or} \quad i \neq j
\]

\text{(3.9)}
Thus the inequalities to be satisfied are as follows:

\[
\begin{align*}
S_1S_4 & \quad + \frac{6}{S} \geq 0 \\
S_1S_6 & \quad + \frac{2}{S} \geq 0 \\
S_3S_4 & \quad + \frac{5}{S} \geq 0 \\
S_5S_6 & \quad - \frac{15}{S} \geq 0 \\
S_3S_5 & \quad + \frac{10}{S} \geq 0 \\
S_2S_4 & \quad - \frac{11}{S} \geq 0 \\
S_2S_6 & \quad + \frac{13}{S} \geq 0 \\
S_1S_5 & \quad - \frac{8}{S} \geq 0 \\
S_2S_5 & \quad - \frac{2}{S} \geq 0 \\
S_1S_3 & \quad + \frac{45}{S} \geq 0 \\
S_4S_6 & \quad + \frac{40}{S} \geq 0 \\
S_1S_2 & \quad + \frac{22}{S} \geq 0
\end{align*}
\]
Five of the constraints in Equation (3.10) have been taken as active constraints and solved for $S_i$'s ($i = 1, 2, 3, 4, 5, 6$) and $S$

Let $\frac{S_i}{S} = s_i$, ($i = 1, 2, 3, 4, 5, 6$)

Substituting this in asterisked constraints of (3.10) with strict equality results in

$s_3s_6 = 15$
$s_2s_4 = 11$
$s_1s_5 = 8$
$s_3s_2 = 17$
$s_5s_6 = 15$

that is

\[
\begin{align*}
  s_6 &= \frac{15}{s_3}, & s_3 &= \frac{15}{s_6}, & s_1 &= \frac{8}{s_3} \\
  s_2 &= \frac{17}{s_3}, & s_4 &= \frac{11}{s_2}, & s_3 &= \frac{17}{s_3}
\end{align*}
\]

(3.11)
It is known that

\[ S_1 + S_2 + S_3 = S_4 + S_5 + S_6 \]  \hspace{1cm} (3.12)

Substituting Equations (3.11) in (3.12) and solving for \( S_3 \),

\[ \frac{8}{s_3} + \frac{17}{s_3} + s_3 = \frac{11}{s_3} + \frac{15}{s_3} \]

That is \( s_3^2 = \frac{170}{11} = 15.454 \); \( s_3 = 3.9312 \)

Substituting this value of \( s_3 \) in Equation (3.11) results,

\[ s_2 = 4.3243 \]
\[ s_5 = 3.9312 \]
\[ s_4 = 2.5437 \]
\[ s_6 = 3.8156 \]
\[ s_1 = 2.0350 \]

\[ S_1 + S_2 + S_3 = \sqrt{S} \]
\[ S_4 + S_5 + S_6 = \sqrt{S} = 10.2905 \]

Therefore \( s = 105.8943 \)

Besides

\[ S_1 = (\sqrt{S})s_1 = 20.941 \]
\[ S_2 = (\sqrt{S})s_2 = 44.4992 \]
\[ S_3 = (\sqrt{S})s_3 = 40.4543 \]
\[ S_4 = (\sqrt{S})s_4 = 26.173 \]
\[ S_5 = (\sqrt{S})s_5 = 40.4543 \]
\[ S_6 = (\sqrt{S})s_6 = 39.2644 \]  \hspace{1cm} (3.13)
It is found that these values of $S_*$'s and $S$ in Equation (3.13) when substituted in Equation (3.10), all of them are satisfied. Those of the constraints in Equation (3.10) which are strictly positive ($>$), give the values of the non-zero edge conductances. They are:

\begin{align*}
g_{14} &= 11.176 & g_{12} &= 13.2001 \\
g_{24} &= 0 & g_{46} &= 30.2954 \\
g_{34} &= 14.9997 & g_{16} &= 9.7647 \\
g_{13} &= 37 & g_{26} &= 29.4997 \\
g_{45} &= 10 & g_{36} &= 0 \\
g_{15} &= 0 & g_{23} &= 0 \\
g_{25} &= 14.9997 & g_{56} &= 0 \\
g_{35} &= 25.4546 \\
\end{align*}

All the conductances are in Siemens.

### 3.5 SIMULTANEOUS REALIZATION OF Y- AND K-MATRICES

This section addresses the realization problem of n-port resistive networks when both Y and potential factor matrix K are specified. Subbarami Reddy (1972) dealt with the case of (n+2)-node n-port realization when both Y- and K-matrices are specified. Now a method of realizing (n+p)-node n-port resistive network from Y- and K-matrices is given. Besides, the maximum number of positive edge conductances required is deduced.

When the complete K-matrix is specified, the corresponding port configuration can be obtained. Lempel and Cederbaum (1969) have dealt with this problem. Let the port configuration be in p connected parts. If each connected part is not in the form of a star tree as in Fig 2.3, it is always possible to make a transformation, as suggested by Subbarami Reddy (1972) and get the non unity potential factors corresponding to the (n+p)-node, p ≥ 2, n-port...
configuration where each connected part is in the form of a star tree port as in Fig 2.3.

Let these potential factors be denoted as $K_{(k),j}$'s. Let the given Y-matrix correspond to the port configuration determined by the K-matrix. In Equation (3.1) the values of $\left(\frac{g_{ki}}{m_d}\right)$'s are known since they are the conductances of the unique network of departure $N_d$. In Equation (3.1), let Equations (2.26), (2.27) and (2.29) be substituted for $\left(\frac{g_{ki}}{m_p}\right)$'s.

Let the resulting set of equations be numbered as Equation (3.14). These equations are $(n+p)(n+p-1)/2$ in number. Appending to these Equations (2.22) and (2.23) which are $n(p-1)$ and $p(p-1)$ in number respectively, a total of $(n+p)(n+p-1)/2 + n(p-1) + p(p-1) = E$ equations called Equation (3.15) involving $K_{(k),j}$'s, $\left(\frac{g_{ki}}{m_d}\right)$'s, $S_{ki}$'s and $S_{ij}$'s as unknowns, are obtained.

Since $K_{(k),j}$'s are specified they can be substituted in Equation (3.15) and the resulting equations are called as Equation (3.16). These equations are linear in $\left(\frac{g_{ki}}{m_d}\right)$'s and $S_{ij}$. The number of unknowns in these linear Equation (3.16) is $V = (n+p)(n+p-1)/2 + n(p-1) + p(p-1)/2$. By setting $V - E = p(p-1)/2$ variables zero and solving the constraint Equations (3.16) the objective function

$$F = \sum_{i=1}^{p} \sum_{k=1}^{n_i} \sum_{j=1}^{p} \left( S_{kij} + S_{ij} \right)$$

is minimized. If it is possible to get an optimum feasible solution, it leads to a basic feasible solution in which all $S_{ij}$'s and $S_{kij}$'s are positive. Out of the $\left(\frac{g_{ki}}{m_d}\right)$'s, $(V - E)$ many of them will be zero. So the number of positive conductances required is $(n+p)(n+p-1)/2 - p(p-1)/2 = n(n+1)/2 + n(p-1)$. 
