COMMON CAUSE FAILURES WITH REPAIR PROVISION-
OPTIMUM REPLACEMENT POLICY
CHAPTER 5

COMMON CAUSE FAILURES WITH REPAIR PROVISION - OPTIMUM REPLACEMENT POLICY

5.1 INTRODUCTION

So far the traditional system adopted by majority is a single unit system on the replacement policy. For obtaining optimal replacement time $T^*$ and optimal number of units $N^*$ for a parallel redundant system that consists of $N$-identical units Nakagawa (1978, 1984) suggested an optimal procedure.

The models reported considers $N$-identical units under assumption that the components fail independently. Subsequent to this work fewer authors also reported similar models incorporating per unit time repair cost, maintenance cost into Nakagawa's model. Treating the assumption that components fail independently. But in practice the independent assumption may not hold good, generally for most of the systems for the simple fact that the system failures also takes place as a result due to maintenances errors, abnormal environmental conditions equipment design deficiency, functional deficiency, operator error etc., which are known as Common Cause Failures (CCFs). The Common Cause Failures has to be taken care of in addition to the random failures to develop an optimal replacement policy for complex systems.


Considering technological development E. J.Elton and N.J.Gruber (1976) developed the optimality of an equal life policy for equipment. Developing optimal

Keeping in view of the reported literature the author in this chapter proposes a novel concept of consideration of a N - unit parallel system subject to Common Cause Failures (CCF’s) and in addition to random failures could derive an optimal replacement policy using the measure of optimality.

The proposed model is demonstrated through the case problem and the utility value of the established model is presented in the section 5.5 (discussion).

A N-Unit parallel system within the limits or common cause failures and random failures is discussed here under by presenting some numerical results to illustrate the procedure.

5.2 THE MODEL

Consider a N-unit parallel system subject to two modes of failures, Common Cause Failures (CCFs) and random failures. That is, the system can fail in either of two mutually exclusive ways. Whenever the system fails, we can repair it.

ASSUMPTIONS:

i) The time span is infinite.

ii) At the beginning, a new system is used.

iii) The system becomes new after each repair in the sense that the mean time to failure (MTTF) of the system remain same.
(iv) The successive repair times \( (Z_i, i = 1, 2, ... ) \) constitute a non-decreasing Geometric Process with parameter \( 0 < b \leq 1 \) and \( E(Z_i) = \tau \geq 0, \tau = 0 \) means that repair time is negligible. For a sequence of random variables \( \{Z_i, Z_j, \ldots\} \) and for some \( b > 0 \), if \( \{b^{-1}Z_i, i = 1, 2, \ldots\} \) forms a renewal process, then \( \{Z_i, i = 1, 2, 3, \ldots\} \) is called a Geometric process (G.P) and \( b \) is the parameter of the Geometric Process. The G.P is said to be non-increasing if \( b \geq 1 \) and non-decreasing if \( b \leq 1 \).

(v) Common Cause failures (CCFs) and random failures are statistically and mutually independent.

(vi) Each component has the same failure time distribution.

**NOTATION:**

- \( C_1 \) : Acquisition cost of each unit
- \( C_2 \) : System repair cost per unit time
- \( \lambda_r \) : Hazard rate of random failure
- \( \lambda_c \) : Common cause failure rate
- \( \beta \) : \( \lambda_c / \lambda_r \)
- \( R_N(t) \) : Reliability of the N-Unit parallel system subject to CCF's and random failures.
- \( \mu_N \) : \( (= \int_0^\infty R_N(t) \, dt) \) MTTF of the N-unit parallel system subject to CCFs and random failures.
- \( Z_i \) : Random repair time after \( i \) th system failure \( i = 1, 2, 3, \ldots, n \) and \( E(Z_i) = \tau \)
- \( n \) : Number of failures
For the above reliability system, we determine an optimal replacement policy

\( n^* \) (A replacement policy \( n \) is a policy of which we replace the system at the time of \( n \) th failure) such that the long run average cost per unit time is minimised.

### 5.3 THE PROCEDURE

The assumption (iv) on the model implies

\[
E(Z_i) = E(bZ_i) = E(b^2Z_i) = \ldots = \tau \quad \ldots 5.1
\]

Since \( \{b^{-1}Z_i, i = 1, 2, \ldots \} \) forms renewal process, therefore

\[
E(Z_1) = \tau, \quad E(Z_2) = \tau / b, \quad E(Z_3) = \tau / b^2
\]

and \( E(Z_i) \leq E(Z_{i+1}) \leq E(Z_{i+j}) \ldots \) for \( b \leq 1 \quad \ldots 5.2
\]

The expected cost incurred in a cycle (upto \( (n-1) \) repairs) for the system is given by

\[
C_2 = \tau \sum_{i=1}^{n-1} \frac{1}{b^i} \quad \ldots 5.3
\]

The expected length of a cycle is

\[
n\mu_n + \tau \sum_{i=1}^{n-1} \frac{1}{b^i} \quad \ldots 5.4
\]

From the renewal argument, the long run expected cost per unit time is given by:
\[ C_2 \tau \sum_{i=1}^{n-1} (1/b)^{i-1}, \]

\[ n \mu_N + \tau \sum_{i=1}^{n-1} (1/b)^{i-1}, \]

if repair is carried out \[ \ldots 5.5 \]

On the other hand if replacement of the system is restored to at the time of \( n \)th failure, the expected cost per unit time is \( NC_1 / \mu_N \) \[ \ldots 5.6 \]

We now propose the measure \( C(n) \) defined as

\[
C(n) = \left| \frac{NC_1}{\mu_N} - \frac{C_1 \tau \sum_{i=1}^{n-1} (1/b)^{i-1}}{n \mu_N + \tau \sum_{i=1}^{n-1} (1/b)^{i-1}} \right| \quad \ldots 5.7
\]

as the measurement of optimality.

**THEOREM 1.** The \( n^* \) which minimises \( C(n) \) given in (5.7) satisfies the pair of inequalities.

\[
D(n) > C_1 / C_2 \quad \ldots 5.8
\]

and \[
D(n-1) < C_1 / C_2 \quad \ldots 5.9
\]

where

\[
D(n) = \frac{\mu_N}{2N} \left[ \frac{\tau \sum_{i=1}^{n} (1/b)^{i-1}}{(n+1) \mu_N + \tau \sum_{i=1}^{n} (1/b)^{i-1}} + \frac{\tau \sum_{i=1}^{n-1} (1/b)^{i-1}}{n \mu_N + \tau \sum_{i=1}^{n-1} (1/b)^{i-1}} \right] \]

Further \( n^* \) is finite and unique \[ \ldots 5.10 \]
**Proof:** First we observe that $C(n)$ is discrete in $n$. Hence to obtain the $n^*$ that minimises $C(n)$, we form the pair of inequalities:

\[ C(n+1) > C(n) \quad \ldots (5.11) \]

and

\[ C(n) < C(n-1) \quad \ldots (5.12) \]

Using (5.7) in (5.11), we get:

\[
\left[ \frac{N C_1}{\mu_N} - \frac{C_2 \tau \sum_{i=1}^{n} (1/b)^{-i}}{(n+1) \mu_N + \tau \sum_{i=1}^{n} (1/b)^{-i}} \right] > \]

\[
\left[ \frac{N C_1}{\mu_N} - \frac{C_2 \tau \sum_{i=1}^{n-1} (1/b)^{-i}}{n \mu_N + \tau \sum_{i=1}^{n-1} (1/b)^{-i}} \right] \quad \ldots (5.13) \]

After some simplifications, (5.13) gives

\[
\frac{\mu_N}{2N} \left[ \frac{\tau \sum_{i=1}^{n} (1/b)^{-i}}{(n+1) \mu_N + \tau \sum_{i=1}^{n} (1/b)^{-i}} + \frac{\tau \sum_{i=1}^{n-1} (1/b)^{-i}}{n \mu_N + \tau \sum_{i=1}^{n-1} (1/b)^{-i}} \right] > \frac{C_1}{C_2} \quad \ldots (5.14) \]

i.e., $D(n) > C_1 / C_2 \quad \ldots (5.15)$

Similar procedure suing (5.7) in (5.12), we get

\[ D(n-1) < \frac{C_1}{C_2} \quad \ldots (5.16) \]

The results in (5.15) and (5.16) together establish the first part of the theorem.
The second part of the theorem is proved, once it is established that \( D(n) \) strictly increases with \( n \). This is done in the following:

\[
D(n+1) - D(n) = \frac{\mu_n}{2N} \left[ \frac{\tau \sum_{i=1}^{n+1} (1/b)^{i-1}}{(n+2) \mu_n + \tau \sum_{i=1}^{n+1} (1/b)^{i-1}} + \frac{\tau \sum_{i=1}^{n} (1/b)^{i-1}}{(n+1) \mu_n + \tau \sum_{i=1}^{n} (1/b)^{i-1}} \right] - \theta n
\]

After some simplifications, we get:

\[
\frac{\mu_n}{2N} \left[ \frac{\tau \sum_{i=1}^{n} (1/b)^{i-1}}{(n+1) \mu_n + \tau \sum_{i=1}^{n} (1/b)^{i-1}} + \frac{\tau \sum_{i=1}^{n-1} (1/b)^{i-1}}{n \mu_n + \tau \sum_{i=1}^{n-1} (1/b)^{i-1}} \right] > 0
\]

for every \( n \)

Hence \( D(n) \) is strictly increasing in \( n \) and thus crosses the value \( C_1/C_2 \), being finite, just once.

Therefore, \( n^* \) is finite and unique.

The proof is complete.
In the following section, we present numerical work to illustrate the application value of the theorem.

5.4 NUMERICAL RESULTS:

We now consider, probability distribution function of CCF's and random failure times are exponential with parameters \( \lambda_c \) and \( \lambda_r \), respectively. For illustration then.

\[
R_n(t) = \sum_{j=1}^{N} \binom{N}{j} (e^{\lambda r})^j (1 - e^{\lambda r})^{N-j} e^{\lambda t}
\]

and \( \mu_c(N, \beta) = \int_0^\infty R_n(t) \, dt \)

\[
= \sum_{i=1}^{N} \int_0^\infty \binom{N}{i} (e^{\lambda r})^i (1 - e^{\lambda r})^{N-i} e^{\lambda t} \, dt
\]

\[
= \frac{N!}{\lambda_c \Gamma(N + \beta + 1)} \sum_{i=1}^{N} \frac{\Gamma(i + \beta)}{i!}
\]

Let us choose \( b = 0.6 \) and \( \tau = 0.2 \).

Using theorem 1, we compute the optimal \( n^* \) corresponding to different values of \( C_1 / C_2 \), \( \beta \) and different \( N \)'s are present in these tables 1, 2, 3, 4 and 5.
Table 1: \( C_1 / C_2 \), \( \beta \) and \( n^* \)

\[ \begin{array}{cccc}
\beta = 0.2 & \beta = 0.4 & \beta = 0.6 & \beta = 0.8 \\
0.0677 & 0.0674 & 0.0671 & 0.0668 \\
0.1083 & 0.1075 & 0.1067 & 0.1059 \\
0.1595 & 0.1578 & 0.1562 & 0.1545 \\
0.2294 & 0.2260 & 0.2225 & 0.2191 \\
0.3263 & 0.3193 & 0.3124 & 0.3057 \\
0.4592 & 0.4455 & 0.4323 & 0.4196 \\
\end{array} \]

\[ N = 2 \]

Table 2: \( C_1 / C_2 \), \( \beta \) and \( n^* \)

\[ \begin{array}{cccc}
\beta = 0.2 & \beta = 0.4 & \beta = 0.6 & \beta = 0.8 \\
0.0341 & 0.0339 & 0.0338 & 0.0336 \\
0.0546 & 0.0543 & 0.0539 & 0.0535 \\
0.0839 & 0.0861 & 0.0793 & 0.0784 \\
0.1170 & 0.1154 & 0.1137 & 0.1120 \\
0.1678 & 0.1645 & 0.1611 & 0.1577 \\
0.2390 & 0.2322 & 0.2255 & 0.2190 \\
\end{array} \]

\[ N = 4 \]

Table 3: \( C_1 / C_2 \), \( \beta \) and \( n^* \)

\[ \begin{array}{cccc}
\beta = 0.2 & \beta = 0.4 & \beta = 0.6 & \beta = 0.8 \\
0.0228 & 0.0227 & 0.0226 & 0.0224 \\
0.0366 & 0.0363 & 0.0361 & 0.0358 \\
0.0542 & 0.0537 & 0.0532 & 0.0525 \\
0.0786 & 0.0775 & 0.0764 & 0.0752 \\
0.1131 & 0.1109 & 0.1086 & 0.1063 \\
0.1618 & 0.1573 & 0.1527 & 0.1482 \\
\end{array} \]

\[ N = 6 \]
Table 4: $C_1 / C_2$, $\beta$ and $n*$

<table>
<thead>
<tr>
<th>$\beta$ = 0.2</th>
<th>$\beta$ = 0.4</th>
<th>$\beta$ = 0.6</th>
<th>$\beta$ = 0.8</th>
<th>$n^*$</th>
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</tr>
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</table>

Table 5: $C_1 / C_2$, $\beta$ and $n*$

<table>
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<th>$\beta$ = 0.2</th>
<th>$\beta$ = 0.4</th>
<th>$\beta$ = 0.6</th>
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<th>$n^*$</th>
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<td>0.0927</td>
<td>0.0901</td>
<td>7</td>
</tr>
</tbody>
</table>

5.5 DISCUSSION

It is observed from the tables 1,2,3,4 and 5 that for same $n^*$ value $C_1 / C_2$ decreases with increasing common cause failure rate $\lambda_c$. Also as number of units $N$ increases $C_1 / C_2$ decreases for the same $n^*$ value. For example, there is a decrease of about 50 per cent in the $C_1 / C_2$ values when $N = 4$ compared to when $N = 2$. The numerical work reported above is aimed at to provide useful idea about the pattern of increases in the various parameters.

Final inference is that the optimal replacement policy based on repairs ($n$) is operationally more convenient rather than the approach through optimal replacement policies in terms of time ($T$).
REFERENCES:


