Chapter 3

Fuzzy representations

3.1 Introduction

The theory of representations has been a powerful tool used in the study of groups. It is concerned with the classification of homomorphisms of abstract finite groups into groups of matrices or linear transformations. Frobenius developed the group representation theory at the end of the 19th century. The works of Burnside on representation theory mainly focus on the group theoretical calculations which are easier to carry out in group of matrices than in abstract groups. For deeper results in representation theory, module theoretic approach is more suitable and it gives more elegance to the theory. So, in the study of representations of finite groups, $G$-module structure has been widely used.
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The representation theory was developed using the notion of embedding a group \( G \) into a general linear group \( GL(V) \). The theory has important applications to physics, especially in quantum mechanics.

In this chapter, we study the fuzzy representations of fuzzy groups, \( M \)-fuzzy groups and fuzzy \( G \)-modules. A fundamental theorem of fuzzy representations on quotient groups is also introduced.

### 3.2 Fuzzy representations of fuzzy groups

#### 3.2.1 Definition [43]

Let \( G \) be a group, \( M \) be a vector space over \( K \) and \( T : G \to GL(M) \) be a representation of \( G \) in \( M \). Let \( \mu \) be a fuzzy group on \( G \) and \( \nu \) be a fuzzy group on \( T(G) \). Then the representation \( T \) is a fuzzy representation if \( T \) is a fuzzy homomorphism of \( \mu \) onto \( \nu \).

#### 3.2.2 Example

Let \( G = (\mathbb{Z}, +) \) and \( M \) be a vector space over \( R \).

Let \( T : G \to GL(M) \) be defined by \( T(x) = T_x \) where \( T_x : M \to M \), such that \( T_x(m) = xm \), for \( x \in G \) and \( m \in M \). Then \( T \) is a representation.

Now, define \( \mu \) on \( G \) by

\[
\mu(x) = \begin{cases} 
1, & \text{if } x \text{ is even} \\
1/2, & \text{if } x \text{ is odd}.
\end{cases}
\]
Then $\mu$ is a fuzzy group on $G$.

Let $\nu$ be the fuzzy group on the range of $T$ defined by
\[ \nu(T_{\text{even}}) = 1, \quad \nu(T_{\text{odd}}) = 1/2 \]
Then, we have,
\[ T(\mu)(T_{\text{even}}) = \vee \{ \mu(x) | x \in T^{-1}(T_{\text{even}}) \} = 1 \]
Similarly,
\[ T(\mu)(T_{\text{odd}}) = 1/2. \]
\[ \therefore T(\mu) = \nu. \text{ Hence } T \text{ is a fuzzy representation of } \mu \text{ onto } \nu. \]  

3.2.3 Theorem [27]

Let $\mu$ be a fuzzy group on $G$ and let $N$ be a normal subgroup of $G$. Define $\xi \in F(G/N)$ by $\xi([x]) = \vee \{ \mu(z) | z \in [x] \}, \forall x \in G$, where $[x]$ denotes the coset $xN$. Then $\xi$ is a fuzzy group on $G/N$.

3.2.4 Theorem (A fundamental theorem of fuzzy representations.)

Let $G$ be a group and $M$ be a vector space over a field $K$. If $T$ is a fuzzy representation of $G$, then $\psi : G/N \rightarrow GL(M)$ defined by $\psi([x]) = T(x), \quad x \in G$, is a fuzzy representation of $G/N$, where $N$ is a normal subgroup of $G$.

Proof. Let $\mu$ be a fuzzy group on $G$. Since $T$ is a fuzzy representation, $\exists$ a fuzzy group $\nu$ on $T(G)$ such that $T(\mu) = \nu$. We have to prove that $\psi$ is a fuzzy representation of $G$. 

Given that $\psi: G/N \to GL(M)$ defined by $\psi([x]) = T(x) = T_x, \forall x \in G$.

Then $\psi$ is a homomorphism of $G/N$ into $GL(M)$.

For $[x], [y] \in G/N$,

$$\psi([x][y]) = \psi([xy]) = T_{xy}, \quad x, y \in G$$

$$T_{xy}(m) = (xy)(m) = x(ym)$$

$$= T_x(ym)$$

$$= T_x(T_y(m))$$

$$= (T_xT_y)(m), \quad m \in M$$

$$\therefore T_{xy} = T_xT_y$$

$$\therefore \psi([xy]) = \psi([x])\psi([y])$$

$$\psi([\alpha x]) = T_{\alpha x}$$

$$T_{\alpha x}(m) = (\alpha x)(m)$$

$$= \alpha(xm)$$

$$= \alpha T_x(m)$$

$$= (\alpha T_x)(m), \quad m \in M, \alpha \in K$$
\[ T_{\alpha x} = \alpha T_x \]
\[ \therefore \psi([\alpha x]) = \alpha \psi([x]). \]

Hence \( \psi : G/N \to GL(M) \) is a representation.

For \( x \in G \), \( \exists \) an element \( xN = [x] \in G/N \).
\[
\psi(\xi)(y) = \vee \{\xi([x])| [x] \in \psi^{-1}(y), \ y \in \psi(G/N)\} \\
= \vee \{\vee \{\mu(z)| z \in [x], \ x \in G\}, \ y \in T(G)\} \\
= \vee \{\mu(z)| z \in [x], \ x \in G\}, \ y \in T(G)\} \\
= T(\mu)(y) \\
\therefore \psi(\xi) = T(\mu) = \nu.
\]

\( \therefore \psi \) is a fuzzy representation of \( \xi \) onto \( \nu. \)

3.2.5 Example

Let \( G = \{1, -1, i, -i\} \), a group under usual multiplication and \( M \) be a vector space over \( R \).

Let \( N = \{1, -1\} \). Then \( N \) is a normal subgroup of \( G \).

Let \( T : G \to GL(M) \) be defined by \( T(x) = T_x \), where \( T_x(M) = xM \), \( \forall x \in G \) and \( m \in M \). Define \( \mu \) on \( G \) by

\[
\mu(x) = \begin{cases} 1, & \text{when } x \text{ is } 1 \text{ or } -1 \\ 0.7, & \text{when } x \text{ is } i \text{ or } -i. \end{cases}
\]
Then \( \mu \) is a fuzzy group on \( G \).

Let \( \nu \) be a fuzzy group on \( T(G) \), defined by \( \nu(T_1) = 1, \nu(T_{-1}) = 1, \nu(T_1) = 0.7 \) and \( \nu(T_{-1}) = 0.7 \). Then

\[
T(\mu)(T_1) = \bigvee \{ \mu(x) / x \in T^{-1}(T_1) \} = 1
\]

Similarly we get,

\[
T(\mu)(T_{-1}) = 1, \quad T(\mu)(T(i)) = 1/2, \quad T(\mu)(T_{-i}) = 1/2
\]

\[\therefore T(\mu) = \nu\]

\[\therefore T \text{ is a fuzzy representation of } \mu \text{ onto } \nu.\]

3.2.6 Example

Let \( G = (R - \{0\}, \times) \) and \( N = \{1, -1\} \). Define \( T : G \to GL(M) \) by

\[
T(x) = T_x, \forall x \in G, m \in M, \text{ where } M \text{ is a vector space over } K.
\]
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Define $\mu$ on $G$ by

$$
\mu(x) = \begin{cases} 
  t, & \text{when } x = 1 \text{ or } -1 \\
  r, & \text{when } x \in Q - \{1, -1, 0\} \\
  s, & \text{when } x \in R - Q
\end{cases}
$$

where $t, r, s \in [0, 1]$ and $t > r > s$.

Then $\mu$ is a fuzzy group on $G$. Let $\nu$ be the fuzzy group on $T(G)$ defined by $\nu(T_1) = t, \nu(T_{-1}) = t, \nu(T_i) = r, i \in Q - \{1, -1, 0\}$ and $\nu(T_j) = s, j \in R - Q$. Then

$$
T(\mu)(T_1) = \bigvee \{\mu(x) | x \in T^{-1}(T_1)\} = t
$$

Similarly, $T(\mu)(T_{-1}) = t, T(\mu)(T_i) = r, T(\mu)(T_j) = s$.

Therefore, $T(\mu) = \nu$. Hence $T$ is a fuzzy representation of $\mu$ onto $\nu$.

Given that $\psi : G/N \to GL(M)$ by $\psi([x]) = T_x = T(x), \forall x \in G$.

Then $\psi$ is a representation.

$$
G/N = \{N, -N, iN, jN\}, i \in Q - \{1, -1, 0\} \text{ and } j \in R - Q
$$

$$
= \{N, iN, jN\}
$$

Define $\xi$ on $G/N$ by $\xi(N) = t, \xi(iN) = r, \xi(jN) = s$.

Then $\xi$ is a fuzzy group on $G/N$.

$$
\psi(\xi)(T_1) = \bigvee \{\xi([1]) | [1] \in T^{-1}(T_1)\}, T_1 \in \psi(G/N)
$$

$$
= \bigvee \{\xi(N) | \psi(N) = T_1, T_1 \in T(G)\}
$$

$$
= t.
$$
\[
\psi(\xi)(T_{-1}) = \vee\{\xi[-1][-1] \in \psi^{-1}(T_{-1})\}, T_{-1} \in T(G)
\]
\[
= \vee\{\xi(N)|N \in \psi^{-1}(T_{-1}), T_{-1} \in T(G)\}
\]
\[
= t.
\]

For \(i \in Q - \{1, -1, 0\}\), \(\psi(\xi)(T_i) = r\).

For \(j \in Q'\), \(\psi(\xi)(T_j) = s\).

Hence \(\psi(\xi) = \nu\). Thus, \(\psi\) is a fuzzy representation of \(\xi\) onto \(\nu\).

3.2.7 Example

Let \(G = (Z, +)\) and \(N = \) set of all even integers. Then \(N\) is a normal subgroup of \(G\).

Define \(\mu\) on \(G\) by

\[
\mu(x) = \begin{cases} 
1, & \text{when } x \text{ is even} \\
1/2, & \text{when } x \text{ is odd}.
\end{cases}
\]

Define \(T : G \to GL(M)\), where \(M\) is a vector space over \(K\), by \(T(x) = T_x, \forall x \in G\).

Define a fuzzy group \(\nu\) on \(T(G)\) by \(\nu(T_{\text{even}}) = 1, \nu(T_{\text{odd}}) = 1/2\).

Then

\[
T(\mu)(T_{\text{even}}) = \vee\{\mu(x)|x \in T^{-1}(T_{\text{even}})\} = 1
\]

Similarly, \(T(\mu)(T_{\text{odd}}) = 1/2\).
\[ T(\mu) = \nu. \]
\[ \therefore T \text{ is a fuzzy representation of } \mu \text{ onto } \nu. \]

\[ G/N = \{x + N|x \in G\}. \]

Define \( \xi \) on \( G/N \) by

\[
\xi(x + N) = \begin{cases} 
1, & \text{when } x \text{ is even} \\
1/2, & \text{when } x \text{ is odd}
\end{cases}
\]

Then, we get

\[
\psi(\xi)(T_{\text{even}}) = \bigvee \{\xi[\text{even}][\text{even}] \in \psi^{-1}(T_{\text{even}})\}, T_{\text{even}} \in \psi(G/N) = 1
\]

A similar computation gives

\[
\psi(\xi)(T_{\text{odd}}) = 1/2.
\]

Hence, \( \psi(\xi) = \nu. \therefore \psi \) is a fuzzy representation of \( \xi \) onto \( \nu. \)

\[ \square \]

### 3.2.8 Remark

When the general linear space, in short, \( \text{gls} \), \( GL(M) \) is replaced by a group \( G' \), we get the following corollary, which may be called as a ‘fundamental theorem of fuzzy homomorphisms’.
3.2.9 Corollary 1

Let $T$ be a fuzzy homomorphism of $\mu$ onto $\nu$ where $\mu$ is a fuzzy group on $G$ and $\nu$ is a fuzzy group on $T(G)$. Then $\psi : G/N \rightarrow G'$, defined by $\psi([x]) = T(x)$, $x \in G$, is a fuzzy homomorphism of $\xi$ onto $\nu$ where $\xi$ is a fuzzy group on $G/N$, $N$ being a normal subgroup of $G$.

Proof. Straight forward. \qed

3.2.10 Corollary 2

Let $T$ be a fuzzy homomorphism of $G$ onto $G'$ and $K_T$ be the kernel of $T$. If $T$ is a fuzzy homomorphism of $\mu$ onto $\nu$ where $\mu$ is a fuzzy group on $G$ and $\nu$ is a fuzzy group on $G'$, then $\psi : G/K_T \rightarrow G'$ is a fuzzy isomorphism of $\xi$ onto $\nu$ where $\xi$ is a fuzzy group on $G/K_T$.

Proof. Straight forward. \qed

3.2.11 Example

Let $G = (Z, +)$, $N = \text{set of all even integers}$ and $G' = \{1, -1\}$, the group under multiplication. $N$ is a normal subgroup of $G$.

Define $T : G \rightarrow G'$ by

$$T(x) = \begin{cases} 1, \text{ when } x \text{ is even} \\ -1, \text{ when } x \text{ is odd.} \end{cases}$$
Then $T$ is a homomorphism of $G$ onto $G'$.

Let $\mu$ be the fuzzy group on $G$ defined by

$$\mu(x) = \begin{cases} t, & \text{when } x \text{ is even} \\ t_0, & \text{when } x \text{ is odd.} \end{cases}$$

$t > t_0$ and $t, t_0 \in [0, 1]$.

Let $\nu$ be the fuzzy group on $G'$ defined by $\nu(1) = t, \nu(-1) = t_0$.

Then, $T(\mu)(1) = t, T(\mu)(-1) = t_0$.

$\therefore T(\mu) = \nu$. $\therefore T$ is a fuzzy homomorphism of $\mu$ onto $\nu$.

$$G/N = \{x + N/x \in G\}.$$ 

Define $\xi$ on $G/N$ by

$$\xi(x + N) = \begin{cases} t, & \text{when } x \text{ is even} \\ t_0, & \text{when } x \text{ is odd.} \end{cases}$$

Then $\psi(\xi)(1) = \vee\{\xi(1)[1] \in \psi^{-1}(1); 1 \in G'\} = t, \psi(\xi)(-1) = t_0$.

Therefore $\psi(\xi) = \nu$. Hence $\psi$ is a fuzzy homomorphism of $\xi$ onto $\nu$. 

The fundamental theorem illustrates that every fuzzy representation or fuzzy homomorphism on $G$ gives rise to a fuzzy representation or fuzzy homomorphism on the factor group $G/N$ where $N$ is a normal subgroup of $G$. 

3.2.12 Proposition

Let $\mu$ and $\xi$ be fuzzy groups on $G$ and $G/N$ respectively where $N$ is a normal subgroup of $G$. If $\pi$ is the natural homomorphism from $G$ onto $G/N$, then $\pi$ is a fuzzy homomorphism of $\mu$ onto $\xi$.

Proof. Since $\pi$ is the canonical homomorphism of $G$ onto $G/N$, $\pi(g) = gN, g \in G$. We have to show that $\pi$ is a fuzzy homomorphism of $\mu$ onto $\xi$.

\begin{center}
\begin{tikzpicture}
    \node (G) at (0,0) {$G$};
    \node (G/N) at (2,0) {$G/N$};
    \node (N) at (1,-2) {$[0,1]$};
    \node (mu) at (1,2) {$\mu$};
    \node (xi) at (1,1) {$\xi$};
    \draw[->] (G) to node {$\pi$} (G/N);
    \draw[->] (G) to node[swap] {$\mu$} (N);
    \draw[->] (G/N) to node {$\xi$} (N);
\end{tikzpicture}
\end{center}

For every $[g] \in G/N$,

\[ \pi(\mu)([g]) = \bigvee \{ \mu(z) | z \in \pi^{-1}([g]) \} \]
\[ = \bigvee \{ \mu(z) | \pi(z) = [g] \} \]
\[ = \bigvee \{ \mu(z) | zN = [g] \} \]
\[ = \bigvee \{ \mu(z) | z \in [g] \} \]
\[ = \xi([g]). \]

\[ \therefore \pi(\mu) = \xi. \]

Hence $\pi$ is a fuzzy homomorphism of $\mu$ onto $\xi$. \qed
3.2.13 Example

Let \( G = \{1, -1, i, -i\} \); \( N = \{1, -1\} \); groups under usual multiplication. Then \( G/N = \{N, iN\} \). Consider the fuzzy group \( \mu \) on \( G \) defined by

\[
\mu(1) = 1, \quad \mu(-1) = 1, \quad \mu(i) = 1/2, \quad \mu(-i) = 1/2.
\]

Given that \( \pi : G \to G/N \) is the canonical homomorphism, so that \( \pi(g) = gN, g \in G \). Define \( \xi \) on \( G/N \) by

\[
\xi(N) = 1, \quad \xi(iN) = 1/2\]

Then,

\[
\pi(\mu)(N) = \vee\{\mu(z)|z \in \pi^{-1}(N)\}
\]

\[
= \vee\{\mu(z)|z \in \{1, -1\}\} = 1\]

And

\[
\pi(\mu)(iN) = \vee\{\mu(z)|z \in \{i, -i\}\}
\]

\[
= 1/2.
\]

\( \therefore \pi(\mu) = \xi \). Hence \( \pi \) is a fuzzy homomorphism of \( \mu \) onto \( \xi \). \( \square \)

Now we proceed to prove a proposition which gives us more insight into the relation between two fuzzy groups which are fuzzy homomorphic.

3.2.14 Proposition

Let \( T \) be a homomorphism from \( G \) onto \( \overline{G} \) and let \( \mu \) and \( \nu \) be fuzzy groups on \( G \) and \( \overline{G} \) respectively such that \( T(\mu) = \nu \). Let \( \pi \) be the natural homomorphism from \( G \) onto \( G/N \) where \( N \) is a normal subgroup of \( G \).
such that $N = \{ x \in G/T(x) \in \overline{N} \}$. Then there exists a homomorphism $\rho$ from $G/N$ to $\overline{G/N}$ and $\rho$ is a fuzzy homomorphism of $\xi$ onto $\overline{\xi}$ where $\xi$ and $\overline{\xi}$ are fuzzy groups on $G/N$ and $\overline{G/N}$ respectively.

**Proof.** Let $T : G \rightarrow \overline{G}$ be the homomorphism of $G$ onto $\overline{G}$, defined by $T(g) = \overline{g}$, $g \in G$, $\overline{g} \in \overline{G}$.

Let $\pi : \overline{G} \rightarrow \overline{G/N}$ be the natural homomorphism. Then $\pi \circ T = \psi$ is a homomorphism of $G$ onto $\overline{G/N}$ and

$$\psi(g) = (\pi \circ T)(g) = \pi(T(g)) = \pi(\overline{g}) = \overline{gN}.$$

The fuzzy group $\xi$ on $G/N$ is defined by

$$\xi([x]) = \vee\{\mu(z) | z \in [x]\}.$$

Now,

$$\psi(N) = (\pi \circ T)(N) = \pi(T(N))$$

$$= \pi(\overline{N})$$

$$= \overline{N}.$$

:. $N$ is the kernel of $\psi$.

:. $\psi$ is a homomorphism of $G$ onto $\overline{G/N}$ with kernel $N$. Hence there exists a homomorphism $\rho$ of $G/N$ onto $\overline{G/N}$, defined by $\rho(gN) = \overline{gN}$. i.e., $\rho([g]) = [\overline{g}]$.

We have to show that $\rho$ is a fuzzy homomorphism of $\xi$ onto $\overline{\xi}$. 
For $[g] \in \overline{G}/N$,

$$
\rho(\xi)(\overline{g}) = \vee \{ \xi[g] | [g] \in \rho^{-1}(\overline{g}) \} \\
= \vee \{ \mu(x) | x \in [g], g \in G, \rho(gN) = T(gN) \} \\
= \vee \{ \mu(x) | x \in [g], T(g) = \overline{g} \} \\
= T(\mu)(\overline{g}) \\
= \nu(\overline{g}), \quad \overline{g} \in \overline{G} \\
= \vee \{ \nu(\overline{g}) | \pi(\overline{g}) = \overline{gN} = [\overline{g}] \} \\
= \pi(\nu)[\overline{g}] \\
= \overline{\xi}[\overline{g}] \\
\text{By proposition 3.2.12, } \pi \text{ is a fuzzy homomorphism of } \nu \text{ onto } \overline{\xi}.
$$

\therefore \rho(\xi) = \overline{\xi}. \text{ Hence } \rho \text{ is a fuzzy homomorphism of } \xi \text{ onto } \overline{\xi}.
3.2.15 Example

Let $G = (\mathbb{Z}, +), \overline{G} = \{1, -1\}$, a group under multiplication.
Let $f : G \to \overline{G}$ be defined by

$$f(x) = \begin{cases} 
1, & \text{when } x \text{ is even} \\
-1, & \text{when } x \text{ is odd}. 
\end{cases}$$

Then $f$ is a homomorphism of $G$ onto $\overline{G}$. We have,

$$N = \{x \in G | f(x) \in \overline{N}\}.$$

Let $\overline{N} = \{1\}$. Then $\overline{N}$ is a normal subgroup of $G$.

$$\therefore \ N = \{x \in G | f(x) = 1\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}.$$

We know that $N$ is a normal subgroup of $G$. Define $\mu$ on $G$ by

$$\mu(x) = \begin{cases} 
1, & \text{when } x \text{ is even} \\
0.5, & \text{when } x \text{ is odd}. 
\end{cases}$$

Then $\mu$ is a fuzzy group on $G$. Define $\nu$ on $\overline{G}$ by

$$\nu(1) = 1, \nu(-1) = 1/2.$$
We have,

\[ G/N = \{ N, \pm 1 + N, \pm 2 + N, \ldots \} = \{ N, 1 + N \} \]

where \( N \) = set of even integers and \( 1 + N \) = set of odd integers. Define \( \xi \) on \( G/N \) by

\[ \xi([x]) = \begin{cases} 1, & \text{when } x \text{ is even} \\ 0.5, & \text{when } x \text{ is odd.} \end{cases} \]

\( \therefore \xi(N) = 1 \) and \( \xi(1 + N) = 0.5 \).

Then \( \xi \) is a fuzzy group on \( G/N \). Now,

\[ \overline{G/N} = \{ 1N, (−1)N \} = \{ \overline{N}, −\overline{N} \}. \]

Define \( \overline{\xi} \) on \( \overline{G/N} \) by

\[ \overline{\xi}(\overline{N}) = 1, \overline{\xi}(−\overline{N}) = 0.5. \]

Then \( \overline{\xi} \) is a fuzzy group on \( \overline{G/N} \).

Define \( \rho : G/N \to \overline{G/N} \) by

\[ \rho([x]) = [\overline{x}]. \]

\( \therefore \rho(N) = \overline{N}, \quad \rho(1 + N) = −\overline{N} \)
We have,
\[
\rho(\xi)(\overline{N}) = \vee \{\xi(N)/N \in \rho^{-1}(\overline{N})\} \\
= \vee \{\xi(N)/\rho(N) = \overline{N}\} \\
= 1
\]
Similarly, \(\rho(\xi)(-\overline{N}) = 0.5\).
\[
\therefore \rho(\xi) = \overline{\xi}. \text{ Hence } \rho \text{ is a fuzzy homomorphism of } \xi \text{ onto } \overline{\xi}. \quad \Box
\]

3.2.16 Proposition

Let \(\phi\) be a fuzzy homomorphism of \(\mu\) onto \(\nu\) where \(\mu\) and \(\nu\) are fuzzy groups on \(G\) and \(\overline{G}\) respectively. Let \(N\) be a normal subgroup of \(G\) and \(\overline{N} = \phi(N)\). Then the canonical homomorphism \(\pi : G \to \overline{G}/\overline{N}\) is a fuzzy homomorphism of \(\nu\) onto \(\overline{\xi}\) where \(\overline{\xi}\) is a fuzzy group on \(\overline{G}/\overline{N}\) and \(\psi = \theta \circ \phi\) is a fuzzy homomorphism of \(\mu\) onto \(\overline{\xi}\).

Proof.

\[
\begin{array}{c}
G \\
\downarrow \mu \\
[0,1] \\
\downarrow \psi \\
G/\overline{N} \\
\downarrow \pi \\
\overline{G}/\overline{N}
\end{array}
\]

\[
G \\
\downarrow \phi \\
\overline{G}
\]

\[
\nu \\
\overline{\xi}
\]
We know that $\psi = \pi \circ \phi$ is a homomorphism of $G$ onto $\overline{G}/\overline{N}$ where $\pi$ is the canonical homomorphism from $\overline{G}$ onto $\overline{G}/\overline{N}$. We have to show that $\pi \circ \phi$ is a fuzzy homomorphism of $\mu$ onto $\overline{\xi}$. i.e., to show that $(\pi \circ \phi)(\mu) = \overline{\xi}$.

For $[\overline{g}] \in \overline{G}/\overline{N}$,

$$(\pi \circ \phi)(\mu)([\overline{g}]) = \vee \{ \mu(z)/z \in (\pi \circ \phi)^{-1}([\overline{g}]) \}$$
$$= \vee \{ \mu(z)/z \in \phi^{-1}(\pi^{-1}([\overline{g}])) \}$$
$$= (\phi(\mu))(\pi^{-1}[\overline{g}])$$
$$= \nu(\pi^{-1}[\overline{g}]), \therefore \phi \text{ is a fuzzy homomorphism, } \phi(\mu) = \nu.$$
$$= (\pi^{-1})^{-1}(\nu)([\overline{g}])$$
$$= \pi(\nu)([\overline{g}])$$
$$= \overline{\xi}([\overline{g}]), \therefore \phi \text{ is a fuzzy homomorphism.}$$

$\therefore \pi \circ \phi$ is a fuzzy homomorphism of $\mu$ onto $\overline{\xi}$. $\square$

3.2.17 Example

Let $G = R - \{0\}$, $\overline{G} = \{1, -1\}$, groups under usual multiplication.

Define $f : G \rightarrow \overline{G}$ by

$$f(x) = \begin{cases} 
1, & \text{when } x \text{ is positive} \\
-1, & \text{when } x \text{ is negative.}
\end{cases}$$
Then $f$ is a homomorphism of $G$ onto $\overline{G}$. Define $\mu$ on $G$ by

$$
\mu(x) = \begin{cases} 
1, & \text{when } x \text{ is rational} \\
0.3, & \text{when } x \text{ is irrational.}
\end{cases}
$$

Define $\nu$ on $\overline{G}$ by $\nu(1) = 1$, $\nu(-1) = 0.3$. Then $\mu$ and $\nu$ are fuzzy groups on $G$ and $\overline{G}$ respectively. Let $\overline{N} = \{1\}$. Then $N = \{x \in G / f(x) = 1\} = Q - \{0\}$. $N$ and $\overline{N}$ are normal subgroups of $G$ and $\overline{G}$ respectively.

$$
G/N = \{\pm N, \pm 2N, \ldots, \pm xN, \pm yN | x \in Q - \{0\}, y \in R - Q \}
= \{xN, yN | x \in Q - \{0\}, y \in R - Q \}.
$$

Define $\xi$ on $G/N$ by $\xi(xN) = 1$ and $\xi(yN) = 0.3$. Then $\xi$ is a fuzzy group on $G/N$ and

$$
\overline{G}/\overline{N} = \{\overline{N}, -\overline{N}\}.
$$

Define $\overline{\xi}$ on $\overline{G}/\overline{N}$ by $\overline{\xi}(\overline{N}) = 1$, $\overline{\xi}(-\overline{N}) = 0.3$. Then $\overline{\xi}$ is a fuzzy group on $\overline{G}/\overline{N}$.

Define $\rho : G/N \to \overline{G}/\overline{N}$ by

$$
\rho(xN) = \overline{N}, \quad \rho(yN) = -\overline{N}, \quad x \in Q - \{0\}, y \in R - Q
$$
Then,
\[ \rho(\xi)(\overline{N}) = \bigvee \{ \xi([x])/[x] \in \rho^{-1}(\overline{N}) \} \]
\[ = 1 = \overline{\xi}(N). \]
\[ \rho(\xi)(-\overline{N}) = \bigvee \{ \xi([y])/[y] \in \rho^{-1}(-\overline{N}) \}, y \in R - Q \]
\[ = 0.3 = \xi(-\overline{N}). \]

\[ \therefore \rho(\xi) = \overline{\xi}. \text{ Hence } \xi \approx \overline{\xi}. \]

3.2.18 Cayley’s Theorem [26]

Every group is isomorphic to a subgroup of \( \mathcal{A}(S) \) for some appropriate set \( S \).

It may be recalled that \( \mathcal{A}(S) \) is the set of all one to one mapping of a set \( S \) onto itself.

3.2.19 Theorem

Any isomorphism \( \psi : G \approx \mathcal{A}(G) \) gives rise to a fuzzy isomorphism between a pair of fuzzy groups on \( G \) and \( \psi(G) \) respectively.

Proof.
Let $\mu$ be a fuzzy group on $G$. By Cayley’s theorem, there exists an isomorphism $\psi : G \rightarrow \mathcal{A}(G)$. Let $\psi(g) = tg$, $\forall g \in G$, where $tg \in \mathcal{A}(G)$. Define $\nu : \xi(G) \rightarrow [0, 1]$ by $\nu(tg) = \mu(g)$, $\forall tg \in \mathcal{A}(G)$. Then it is easy to verify that $\nu$ is a fuzzy group on $\psi(G)$.

Now, $\psi(\mu)(tg) = \vee \{\mu(g)/g \in \psi^{-1}(tg)\}, \ \forall tg \in \psi(G)$

$= \mu(g)$.

For $tg, th \in \psi(G)$,

$\psi(\mu)(tg \cdot th) = \psi(\mu)(tgh), \ \therefore t_g \cdot t_h = tgh$

$= \mu(gh)$

$\geq \mu(g) \land \mu(h)$

$\geq \psi(\mu)(tg) \land \psi(\mu)(th)$.

$\psi(\mu)(tg)^{-1} = \psi(\mu)(tg^{-1})$

$= \mu(g^{-1})$

$= \mu(g)$

$= \psi(\mu)(tg)$.

$\therefore \psi(\mu)$ is a fuzzy group on $\psi(G)$. Also, $\psi(\mu)(tg) = \mu(g) = \nu(tg), \ \forall tg \in \psi(G)$, and hence $\psi(\mu) = \nu$. $\therefore \psi$ is a fuzzy isomorphism of $\mu$ onto $\nu$. \qed
3.2.20 Example

Let $G = \{1, -1\}$. Define isomorphism $\psi : G \rightarrow \mathfrak{A}(G)$ by $\psi(g) = t_g$, \(\forall g \in G, t_g\) being an automorphism defined on $G$. Define $\mu$ on $G$ by $\mu(1) = 1$ and $\mu(-1) = 0.5$. Then $\mu$ is a fuzzy group on $G$.

Let $\nu$ be the fuzzy group on $\psi(G)$ defined by $\nu(t_1) = 1$ and $\nu(t_{-1}) = 0.5$. Then

$$\psi(\mu)(t_1) = \vee\{\mu(x)/x \in \psi^{-1}(t_1)\} = 1$$

Similarly,

$$\psi(\mu)(t_{-1}) = 0.5.$$

\[ \therefore \psi(\mu) = \nu. \text{ Hence } \psi \text{ is a fuzzy isomorphism of } \mu \text{ onto } \nu. \]

If $T$ is a fuzzy representation of a group $G$ with representation space $M$ and $N$ is a subgroup of $G$, then $T$ on $N$ is (restriction of $T$ on $N$) denoted by $T_N$.

3.2.21 Proposition

If $T$ is a fuzzy representation of a group $G$ with representation space $M$ and $N$ is a subgroup of $G$, then $T_N$ is a fuzzy representation of $\mu_N$ onto $\nu_{T(N)}$. 
Proof.

Since $T$ is a fuzzy representation, $\exists$ fuzzy groups $\mu$ and $\nu$ on $G$ and $T(G)$ respectively such that $T(\mu) = \nu$.

We have to show that $T_N$ is a fuzzy representation.

For $y \in T(G)$,

\[
T_N(\mu)(y) = \forall \{\mu(x) | x \in T_N^{-1}(y)\}
\]
\[
= \forall \{\mu(x) | T(x) = y, x \in N, y \in T(N)\}
\]
\[
= \nu_{T(N)}(y).
\]

\[\therefore T_N(\mu) = \nu_{T(N)}.\]

\[\therefore T_N \text{ is a fuzzy representation of } \mu_N \text{ onto } \nu_{T(N)}.\]

\[\square\]

3.2.22 Definition [12]

Two representations $T$ and $T'$ with spaces $M$ and $M'$ are said to be equivalent if $\exists$ a $K$-isomorphism $S$ of $M$ onto $M'$ such that $T'(g)(S) = ST(g)$, $\forall g \in G$. i.e., $T'(g)S(m) = ST(g)(m)$, $\forall g \in G$ and $m \in M$. 

3.2.23 Definition

Let \( \mu \) be a fuzzy group on a group \( G \). Two fuzzy representations \( T \) and \( T' \) of \( G \) with spaces \( M \) and \( M' \) are said to be equivalent if \( T^{-1}(\nu)(x) = T'^{-1}(\eta)(x) \), \( \forall x \in G \). i.e., \( \nu[T(x)] = \eta[T'(x)] \), \( \forall x \in G \), where \( \nu \) and \( \eta \) are fuzzy groups defined on \( T(G) \) and \( T'(G) \) respectively.

\[
\begin{array}{ccc}
G & \xrightarrow{T} & GL(M) \\
\downarrow{T} & \mu & \downarrow{\nu} \\
GL(M') & \xrightarrow{\eta} & [0, 1]
\end{array}
\]

3.2.24 Remark

The ‘equivalence’ of fuzzy representations is an equivalence relation. \( \square \)

3.3 \( M \)-fuzzy representations

3.3.1 Definition

Let \( G \) be an \( M \)-group and let the vector space \( V \) over \( K \) be an \( M \)-group. An \( M \)-representation of \( G \) with representation space \( V \) is an
3.3.2 Definition

Let $\mu$ and $\nu$ be $M$-fuzzy groups on $G$ and $G'$ respectively. Let $f$ be an $M$-homomorphism from $G$ onto $G'$. Then $f$ is called an $M$-fuzzy homomorphism of $\mu$ onto $\nu$ if the $M$-homomorphism $f$ is a fuzzy homomorphism of $\mu$ onto $\nu$.

3.3.3 Example

Let $G = (\mathbb{C}, +)$, $G' = (\mathbb{R}, +)$ and $M =$ set of natural numbers. Define $f : G \to G'$ by $f(x + iy) = x + y$. Then $f$ is a homomorphism. For $m \in M$, $x + iy \in \mathbb{C}$,

$$f[(m(x + iy))] = mx + my$$

$$= m(x + y)$$

$$= mf(x + iy).$$

$\therefore f$ is an $M$-homomorphism. Let $\mu$ be the fuzzy group defined on $G$ by

$$\mu(x + iy) = \begin{cases} 1, & \text{if } x = y = 0 \\ 0.7, & \text{if } x \neq 0, y = 0 \\ 0.2, & \text{if } y \neq 0. \end{cases}$$
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Then

\[
\mu[m(x + iy)] = \mu(mx + imy) = \begin{cases} 
1, & \text{if } mx = my = 0 \\
0.7, & \text{if } mx \neq 0, my = 0 \\
0.2, & \text{if } my \neq 0.
\end{cases}
\]

\[
= \begin{cases} 
1, & \text{if } x = y = 0 \\
0.7, & \text{if } x \neq 0, y = 0 \\
0.2, & \text{if } y \neq 0.
\end{cases}
\]

\[
= \mu(x + iy).
\]

\[\therefore\] \mu is an M-fuzzy group on \(G\).

Let \(\nu\) be the fuzzy group on \(G'\), defined by \(\nu(0) = 1, \nu(x) = 1/2, x \neq 0\). Then

\[
\nu(mx) = \begin{cases} 
1, & \text{if } mx = 0 \\
0.7, & \text{if } mx \neq 0.
\end{cases}
\]

\[
= \begin{cases} 
1, & \text{if } x = 0 \\
0.7, & \text{if } x \neq 0.
\end{cases}
\]

\[
= \nu(x).
\]

\[\therefore\] \nu is an M-fuzzy group on \(G'\).
Now,

\[
f(\mu)(0) = \vee \{\mu(x + iy) | x + iy \in f^{-1}(0)\}
\]

\[
= 1.
\]

For \(0 \neq r \in R\), \(f(\mu)(r) = 0.7\).

\[\therefore f(\mu) = \nu. \text{ Hence } f \text{ is a fuzzy homomorphism of } \mu \text{ onto } \nu.\]

Also, \(f\) is an \(M\)-homomorphism.

Hence \(f\) is an \(M\)-fuzzy homomorphism of \(\mu\) onto \(\nu\).

\[\Box\]

### 3.3.4 Proposition

Let \(\mu\) be a \(M\)-fuzzy group on \(G\) and \(N\) be a normal \(M\)-subgroup of \(G\). Fuzzy subset \(\xi\) on \(G/N\) is defined as \(\xi([x]) = \vee \{\mu(z) | z \in [x]\}, \forall x \in G\), where \([x]\) denotes the coset \(xN\). Then \(\xi\) is a \(M\)-fuzzy group on \(G/N\).

**Proof.** For \(m \in M\) and \([x] \in G/N\),

\[
\xi(m[x]) = \xi([mx])
\]

\[
= \vee \{\mu(z) | z \in [mx], mx \in G\}
\]

\[
= \vee \{\mu(mg) | mg \in [mx], mx \in G\}
\]

\[
\geq \vee \{\mu(g) | g \in [x], x \in G\}
\]

\[
\geq \xi([x]).
\]

\[\therefore \xi \text{ is an } M\text{-fuzzy group on } G/N.\]

\[\Box\]
3.3.5 Theorem (A fundamental theorem of $M$-fuzzy homomorphisms.)

Let $\mu$ and $\nu$ be $M$-fuzzy groups on the $M$-groups $G$ and $G'$ respectively. Let $T$ be an $M$-fuzzy homomorphisms of $\mu$ onto $\nu$. Then $\psi : G/N \rightarrow G'$ defined by $\psi([x]) = T(x)$, $x \in G$, is an $M$-fuzzy homomorphism of $\xi$ onto $\nu$ where $\xi$ is an $M$-fuzzy group on $G/N$, $N$ being a normal $M$-subgroup of $G$.

Proof. Since $T$ is an $M$-fuzzy homomorphism from $\mu$ onto $\nu$, $T(\mu) = \nu$ and $T(mx) = mT(x)$, for $m \in M$ and $x \in G$.

We have to prove that $\psi : G/N \rightarrow G'$, defined by $\psi([x]) = T(x)$, is an $M$-fuzzy homomorphism.

Clearly $\psi$ is an $M$-homomorphism of $G/N$ into $G'$. Now it remains to prove that $\psi(\xi) = \nu$.

For $y \in G'$,

$$
\psi(\xi)(y) = \vee \{ \xi[x]| [x] \in \psi^{-1}(y) \}
= \vee \{ \xi[x]|\psi[x] = y \}
= \vee \{ \vee \{ \mu(z)|z \in [x] \in G/N \}, T(x) = y \}
= \vee \{ \mu(z)|z \in [x] \in G/N, x \in T^{-1}(y) \}
= T(\mu)(y).
$$

$\therefore \psi(\xi) = T(\mu) = \nu$.

$\therefore \psi$ is a fuzzy homomorphism of $\xi$ onto $\nu$ and hence $\psi$ is an $M$-fuzzy
homomorphism of $\xi$ onto $\nu$.

3.3.6 Example

Consider the example 3.3.3. Take $N = R$, set of real numbers. Note that $f : G \rightarrow G'$, defined by $f(x + iy) = x + y$ is an $M$-homomorphism. Define the $M$-fuzzy groups $\mu$ and $\nu$ on $G$ and $G'$ as in 3.3.3. Then

$$G/N = G/R = \{u + R|u \in G\}.$$ 

Define $\psi : G/R \rightarrow G'$ by $\psi(u + R) = f(u), u \in G$.

Then $\psi$ is an $M$-homomorphism. Further,

$$\psi(\xi)(u) = \vee\{\xi[u][u] \in \psi^{-1}(0)\}$$

$$= 1.$$ 

And for $0 \neq w \in R$, $\psi(\xi)(w) = 0.7$. 

$\therefore \psi(\xi) = \nu$. Hence $\psi$ is an $M$-fuzzy homomorphism of $\xi$ onto $\nu$.

3.3.7 Definition

Let $G$ be an $M$-group and $V$ be a vector space over $K$ which is also an $M$-group. Let $T : G \rightarrow GL(V)$ be an $M$-representation with representation space $\vee$. Let $\mu$ be an $M$-fuzzy group on $G$ and $\nu$ be an $M$-fuzzy group on the range of $T$. Then the $M$-representation $T$ is an $M$-fuzzy representation if $T$ is $M$-fuzzy homomorphism of $\mu$ onto $\nu$. 
3.3.8 Theorem

Let $G$ be an $M$-group and $V$ be a vector space over a field $K$ which is also an $M$-group. If $T$ is an $M$-fuzzy representation of $G$, then $\psi : G/N \to GL(V)$, defined by $\psi([x]) = T_x = T(x)$, $x \in G$, is a $M$-fuzzy representation of $G/N$ where $N$ is a normal $M$-subgroup of $G$.

Proof. By the fundamental theorem of fuzzy representations 3.2.4, we know that $\psi$ is a fuzzy representation. Also, $\psi : G/N \to GL(V)$, defined by $\psi([x]) = T(x)$, $x \in G$, is an $M$-homomorphism. Hence $\psi$ is an $M$-fuzzy representations of $\mu$ onto $\nu$. \qed

3.3.9 Example

Let $G = (R, +)$, $N = (Z, +)$, $M = Z^+$, the set of all (+)ve integers. Define $T : G \to GL(V)$ by $T(x) = T_x$, $x \in G$. Then $T$ is an $M$ representation. Define the $M$-fuzzy group $\mu$ on $G$ by

$$
\mu(x) = \begin{cases} 
1, & \text{when } x = 0 \\
0.5, & \text{when } x \in Q - \{0\} \\
0.2, & \text{when } x \in R - Q.
\end{cases}
$$

Define $\nu$ on $T(G)$ by
\[ \nu(T_x) = \begin{cases} 
1, & \text{when } x = 0 \\
0.5, & \text{when } x \in Q - \{0\} \\
0.2, & \text{when } x \in R - Q. 
\end{cases} \]

Then \( T(\mu) = \nu \). Hence \( T \) is a fuzzy representation of \( \mu \) onto \( \nu \).

Define \( \psi : G/N \rightarrow GL(V) \) by \( \psi(u + N) = T(u), \ u \in G \). Then \( \psi \) is an \( M \)-representation.

Define \( \xi \) on \( G/N \) by

\[ \xi([x]) = \begin{cases} 
1, & \text{when } x = 0 \\
0.5, & \text{when } x \in Q - \{0\} \\
0.2, & \text{when } x \in R - Q. 
\end{cases} \]

\[ \psi(\xi)(T_x) = \bigvee \{ \xi([x]) | [x] \in \psi^{-1}(T_x) \} = \begin{cases} 
1, & \text{when } x = 0 \\
0.5, & \text{when } x \in Q - \{0\} \\
0.2, & \text{when } x \in R - Q. 
\end{cases} = \nu(T_x). \]

\( \therefore \psi(\xi) = \nu \). Hence \( \psi \) is an \( M \)-fuzzy representation of \( \xi \) onto \( \nu \). \( \square \)
3.4 Fuzzy representations of fuzzy $G$-modules

3.4.1 Definition

Let $T$ be a $G$-module homomorphism of $M$ into $M^*$. Let $\mu$ and $\nu$ be fuzzy $G$-modules on the $G$-modules $M$ and $M^*$ respectively. Then $T$ is called a $G$-module fuzzy homomorphism if $T(\mu) = \nu$.

3.4.2 Example

The $G$-module homomorphism $f : M \to M^*$ defined in example 2.5.9 is a $G$-module fuzzy homomorphism of $\mu$ onto $\nu$ where $\mu$ is defined on $M$ by

$$\mu(x + iy) = \begin{cases} 
1, & \text{if } x = y = 0 \\
0.5, & \text{if } x \neq 0, y = 0 \\
0.2, & \text{if } y \neq 0
\end{cases}$$

and $\nu$ on $M^*$ by

$$\nu(z) = \begin{cases} 
1, & \text{if } z = 0 \\
0.5, & \text{if } z \neq 0.
\end{cases}$$
3.4.3 Example

Let $G = \{1, -1\}$, $M = R^4$ over $R$ and $M^* = R$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $M$. Define $\mu : M \rightarrow [0, 1]$ by

$$
\mu(k_1e_1 + k_2e_2 + k_3e_3 + k_4e_4) = \begin{cases} 
1, & \text{if } k_i = 0 \forall i \\
1/2, & \text{if } k_1 \neq 0, k_2 = k_3 = k_4 = 0 \\
1/3, & \text{if } k_2 \neq 0, k_3 = k_4 = 0 \\
1/4, & \text{if } k_3 \neq 0, k_4 = 0 \\
1/5, & \text{if } k_4 \neq 0.
\end{cases}
$$

Then $\mu$ is a fuzzy $G$-module on $M$. Define $\nu$ on $R$ by $\nu(0) = 1$, $\nu(x) = 1/2$, $x \neq 0$.

Define $f : R^4 \rightarrow R$ by $f(x) = \sum_{i=1}^{4} x_i$, $x = (x_1, x_2, x_3, x_4)$; $x_i \in R$.

For $a, b \in R$ and $x, y \in M$,

$$
f(ax + by) = \sum_{i=1}^{4} (ax_i + by_i); \ x = (x_1, x_2, x_3, x_4) \text{ and } y = (y_1, y_2, y_3, y_4)$$

$$= af(x) + bf(y).$$

For $g \in G$ and $x \in M$,

$$
f(gx) = \sum_{i=1}^{4} gx_i$$

$$= gf(x).$$
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\[ f(\mu)(0) = \vee \{ \mu(x) | x \in f^{-1}(0) \} = 1. \]

For \( 0 \neq w \in R, f(\mu)(w) = 1/2. \)
\[ \therefore f(\mu) = \nu. \] Hence \( f \) is a \( G \)-module fuzzy homomorphism of \( \mu \) onto \( \nu. \)

In group representation we are embedding a group into a general linear space \( GL(V) \). Here, we are extending this notion to the theory of \( G \)-modules.

3.4.4 Proposition

Let \( G \) be a group and \( V \) be a \( G \)-module over \( K \). Then \( GL(V) \) is a \( G \)-module.

Proof. For \( g \in G \) and \( f \in GL(V) \), define \( (gf)(v) = gf(v) \), \( v \in V \). Then it can be seen that \( GL(V) \) is a \( G \)-module.

3.4.5 Definition

Let \( G \) be a group and \( M \) and \( V \) be \( G \)-modules over \( K \). The representation \( T : M \to GL(V) \) is called a \( G \)-module representation if \( T \) is a \( G \)-module homomorphism of \( M \) into \( GL(V) \).
3.4.6 Definition

Let $G$ be a group and let $M$ and $V$ be $G$-modules over $K$. Let $T$ be a $G$-module homomorphism of $M$ into $GL(V)$. Let $\mu$ and $\nu$ be fuzzy $G$-modules on $M$ and $T(M)$ respectively. Then $T$ is called a $G$-module fuzzy representation if $T$ is a $G$-module fuzzy homomorphism of $\mu$ onto $\nu$.

3.4.7 Example

Let $G = \{1, -1\}$ and $M = C$. Define $T : M \to GL(V)$, where $V$ is a $G$-module over $K$, by $T(m) = T_m, m \in M$. Then $T$ is a $G$-module homomorphism. Define the fuzzy $G$-module $\mu$ on $M$ by

$$
\mu(x + iy) = \begin{cases} 
1, & \text{if } x = y = 0 \\
0.8, & \text{if } x \neq 0, y = 0 \\
0.4, & \text{if } y \neq 0.
\end{cases}
$$

Define $\nu$ on $T(M)$ by

$$
\nu(T_m) = \nu(T_{x+iy}) = \begin{cases} 
1, & \text{if } x = y = 0 \\
0.8, & \text{if } x \neq 0, y = 0 \\
0.4, & \text{if } y \neq 0.
\end{cases}
$$
Then $\nu$ is a fuzzy $G$-module on $T(M)$.

$$T(\mu)(T_{x+iy}) = \vee \{ \mu(z) | z \in T^{-1}(T_{x+iy}) \}$$

$$= \vee \{ \mu(x+iy) | T(x+iy) = T_{x+iy} \}$$

$$= \begin{cases} 
1, & \text{if } x = y = 0 \\
0.8, & \text{if } x \neq 0, y = 0 \\
0.4, & \text{if } y \neq 0.
\end{cases}$$

$$= \nu(T_{x+iy}).$$

$\therefore T(\mu) = \nu$. Hence $T$ is a $G$-module fuzzy representation of $\mu$ onto $\nu$. \hfill \Box$

### 3.4.8 Example

Let $M = C^n, V = C^n$ be the vector spaces over $C$. Let $G = \{1, -1, i, -i\}$.

Define $T : C^n \rightarrow GL(C^n)$ by $T(z) = T_z$ where $z = (z_1, z_2, \ldots, z_n) \in C^n$. Then $T$ is a $G$-module homomorphism.

Let $\mu : M \rightarrow [0, 1]$ be defined for every, $z = (z_1, z_2, \ldots, z_n) \in C^n,$
\[ \mu(z) = \begin{cases} 
1, & \text{if } z_i = 0 \ \forall i \\
1/2, & \text{if } z_1 \neq 0, z_2 = 0, \ldots, z_n = 0 \\
1/3, & \text{if } z_2 \neq 0, z_3 = 0, \ldots, z_n = 0 \\
\ldots & \text{...........} \\
1/n + 1, & \text{if } z_n \neq 0.
\end{cases} \]

Then \( \mu \) is a fuzzy \( G \)-module on \( M \).

Define \( \nu \) on \( T(M) \) by

\[ \nu(T_z) = \begin{cases} 
1, & \text{if } z_i = 0 \ \forall i \\
1/2, & \text{if } z_1 \neq 0, z_2 = 0, \ldots, z_n = 0 \\
1/3, & \text{if } z_2 \neq 0, z_3 = 0, \ldots, z_n = 0 \\
\ldots & \text{...........} \\
1/n + 1, & \text{if } z_n \neq 0.
\end{cases} \]

\[ T(\mu)(T_z) = \vee \{ \mu(z) | z \in T^{-1}(z) \} \]

\[ = \vee \{ \mu(z) | T(z) = T_z \} \]
\[ T \left( \frac{1}{i} \right) = \begin{cases} 
1, & \text{if } z_i = 0 \ \forall i \\
1/2, & \text{if } z_1 \neq 0, z_2 = 0, \ldots, z_n = 0 \\
1/3, & \text{if } z_2 \neq 0, z_3 = 0, \ldots, z_n = 0 \\
\vdots & \vdots \\
1/n + 1, & \text{if } z_n \neq 0. 
\end{cases} \]

\[ = \nu(T_z). \]

\( \therefore T(\mu) = \nu. \) Hence \( T \) is a \( G \)-module fuzzy representation of \( \mu \) onto \( \nu. \)

### 3.4.9 Proposition [43]

If \( M \) is a \( G \)-module and \( N \) is a \( G \)-submodule of \( M \), then \( M/N \) is a \( G \)-module.

### 3.4.10 Proposition

Let \( M \) be a \( G \)-module over \( K \) and \( N \) be a \( G \)-submodule of \( M \). Then the fuzzy subset \( \xi \) on \( M/N \), defined by \( \xi([x]) = \bigvee \{\mu(z)|z \in [x], x \in M\} \), where \( [x] = x + N \), is a fuzzy \( G \)-module on \( M/N \).

**Proof.** For \( a, b \in K \) and \( [x], [y] \in M/N \),

\[ \xi(a[x] + b[y]) = \xi([ax + by]) = \bigvee \{\mu(z)|z \in [ax + by]\} \]
\[ = \vee \{ \mu(au + bv) | au + bv \in [ax + by] \} \]
\[ = \vee \{ \mu(au + bv) | au \in [ax], bv \in [by] \} \]
\[ \geq \vee \{ \mu(u) \land \mu(v) | u \in [x], v \in [y] \} \]
\[ \geq [\vee \{ \mu(u) | u \in [x] \}] \land [\vee \{ \mu(v) | v \in [y] \}] \]
\[ \geq \xi([x]) \land \xi([y]) \].

For \( g \in G \) and \([x] \in M/N\),

\[ \xi(g[x]) = \xi([gx]) \]
\[ = \vee \{ \mu(z) | z \in [gx], gx \in M \} \]
\[ = \vee \{ \mu(gu) | gu \in [gx], gx \in M \} \]
\[ \geq \vee \{ \mu(u) | u \in [x], x \in M \} \]
\[ \geq \xi([x]). \]

\[ \because \xi \text{ is a fuzzy } G\text{-module on } M/N. \]  

### 3.4.11 Remark

The fuzzy \( G \)-module \( \xi \) defined on \( M/N \), as above, is called the quotient fuzzy \( G \)-module or factor fuzzy \( G \)-module of \( \mu \) of \( M \) relative to the \( G \) submodule \( N \) and is denoted by \( \mu/N \).
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3.4.12 Theorem

Let $G$ be a group and $M$ and $M^*$ be $G$-modules over $K$. Let $\mu$ and $\nu$ be fuzzy $G$-modules on $M$ and $T(M)$ respectively. Let $T : M \rightarrow M^*$ be a $G$-module fuzzy homomorphism of $\mu$ onto $\nu$. Then $\psi : M/N \rightarrow M^*$, defined by $\psi([x]) = T(x)$, $x \in M$, is a $G$-module fuzzy homomorphism of $\xi$ onto $\nu$ where $\xi$ is a fuzzy $G$-module on $M/N$ and $N$ is a $G$-submodule of $M$.

Proof.

\[
\begin{array}{c}
\text{Given that } T \text{ is a } G\text{-module fuzzy homomorphism of } \mu \text{ onto } \nu.
\end{array}
\]

\[
\therefore T(\mu) = \nu. \text{ We have to show that } \psi \text{ is a } G\text{-module fuzzy homomorphism of } \xi \text{ onto } \nu. \text{ Note that } \psi : M/N \rightarrow M^* \text{ is defined by } \psi([x]) = T(x), \forall x \in M \text{ and } [x] = x + N. \text{ Then } \psi \text{ is a } G\text{-module homomorphism. Now}
\]
to show that $\psi(\xi) = \nu$.

$$
\psi(\xi)(y) = \vee\{\xi[x][z] \in \psi^{-1}(y), \ y \in \psi(M/N)\}
= \vee\{\vee\{\mu(z)|z \in [x]\}, \psi([x]) = y, y \in T(M)\}
= \vee\{\mu(z)/z \in [x], T(x) = y \in T(M), x \in M\}
= T(\mu)(y)
$$

$\therefore \psi(\xi) = T(\mu) = \nu.$

$\therefore \psi$ is a $G$-module fuzzy homomorphism of $\xi$ onto $\nu$.

3.4.13 Theorem (A fundamental theorem of $G$-module fuzzy representations)

Let $G$ be a group and $M$ and $V$ be $G$-modules over $K$. Let $T : M \to GL(V)$ be a $G$-module fuzzy representation of $\mu$ onto $\nu$ where $\mu$ and $\nu$ are fuzzy $G$-modules on $M$ and $T(M)$ respectively. Then $\psi : M/N \to GL(V)$, defined by $\psi([x]) = T_x$, $x \in M$, is a $G$-module fuzzy representation of $\xi$ onto $\nu$ where $\xi$ is a fuzzy $G$-module on $M/N$, $N$ being a $G$-submodule of $M$. 

\[ \]
Proof.

\[
\begin{array}{ccc}
M & \xrightarrow{T} & GL(V) \\
\downarrow{\mu} & & \downarrow{\psi} \\
[0,1] & \xrightarrow{\xi} & M/N \\
\downarrow{\nu} & & \\
\end{array}
\]

Given that \( T \) is a \( G \)-module fuzzy representation of \( \mu \) onto \( \nu \) where \( \mu \) and \( \nu \) are fuzzy \( G \)-modules on \( M \) and \( T(M) \) respectively. We have to show that \( \psi : M \rightarrow GL(V) \) is a \( G \)-module fuzzy representation of \( \xi \) onto \( \nu \). By theorem 3.4.12, \( \psi \) is a \( G \)-module fuzzy homomorphism of \( \xi \) onto \( \nu \).

Now, for \( T_x \in T(M) \subset GL(V) \),

\[
\psi(\xi)(T_x) = \vee\{\xi([x])|[x] \in \psi^{-1}(T_x)\} \\
= \vee\{\vee\{\mu(z)|z \in [x]\}, \psi([x]) = T_x, x \in M\} \\
= \vee\{\mu(z)|z \in [x], \psi([x]) = T_x, x \in M\} \\
= T(\mu)(T_x) \\
\therefore \psi(\xi) = T(\mu) = \nu.
\]

Hence \( \psi \) is a \( G \)-module fuzzy representation of \( \xi \) onto \( \nu \).
3.4.14 Example

Let \( G = \{1, -1\} \), \( M = C \) and \( M^* = R \). Consider \( f, \mu \) and \( \nu \) defined as in example 3.4.2. Consider the \( G \)-submodule \( R \) of \( M \). Then \( M/R = \{u + R|u \in M\} \).

Define \( \psi : M/R \to R \) by \( \psi(u+R) = f(u), u \in M \). Then \( \psi \) is a \( G \)-module homomorphism. Define \( \xi \) on \( M/R \) by

\[
\xi([x]) = 0.5, \ x \neq 0, \text{ and } \xi([0]) = 1. 
\]

Then \( \psi(\xi)(0) = 1 \) and \( \psi(\xi)(x) = 0.5, x \neq 0. \)

\( \therefore \psi(\xi) = \nu. \therefore \psi \) is a \( G \)-module fuzzy homomorphism of \( \xi \) onto \( \nu. \)

3.4.15 Example

Let \( G = \{1, -1\} \), \( M = (R^4, +) \) and \( M^* = (R, +) \). Then \( M \) and \( M^* \) are \( G \)-modules over \( R \). Let \( N = \{z = (x_1, x_2, 0, 0)/x_1, x_2 \in R\} \).

\( M/N = \{x + N| x \in M\}. \) Take \( f, \mu \) and \( \nu \) as in example 3.4.3. \( \psi : M/N \to M^* \) defined by \( \psi([x]) = f(x), x \in M \), is a \( G \)-module homomorphism.

\[
\psi(\xi)(0) = \vee\{\xi([x])|[x] \in \psi^{-1}(0)\} \\
= 1 \\
\psi(\xi)(r) = 1/2, \ r \neq 0.
\]


\[ \psi(\xi) = \nu. \] Hence \( \psi \) is a \( G \)-module fuzzy homomorphism of \( \xi \) onto \( \nu. \)

\[ \square \]

3.4.16 Example

Let \( G = \{1, -1\}, M = (C, +) \). Let \( N = R \). Then \( N \) is a \( G \)-submodule of \( M \). Consider the \( G \)-module homomorphism. \( T : M \to GL(V) \), where \( V \) is a \( G \)-module over \( R \), defined by \( T(m) = T_m, \forall m \in M \). Define \( \mu \) and \( \nu \) as in example 3.4.7. Then \( T \) is a \( G \)-module fuzzy representation of \( \mu \) onto \( \nu \).

Define \( \psi : M/N \to GL(V) \) by \( \psi([u]) = T(u) = T_u, u \in M \).

Then \( \psi \) is a \( G \)-module homomorphism. For \( u = x + iy \), define \( \xi \) on \( M/N \) by

\[
\xi([u]) = \begin{cases} 
1, & \text{if } x = 0, y = 0 \\
0.8, & \text{if } x \neq 0, y = 0 \\
0.4, & \text{if } y \neq 0.
\end{cases}
\]

Then \( \psi(\xi)(T_m) = \nu(T_m) \).

\[ \therefore \psi(\xi) = \nu. \] Hence \( \psi \) is a \( G \)-module fuzzy representation of \( \xi \) onto \( \nu. \)

\[ \square \]

3.4.17 Example

Let \( M = C^n, V = C^n, G = \{1, -1, i, -i\} \).

Let \( N = \{ w = (z_1, z_2, \ldots, z_k, 0, 0, \ldots, 0) \} \). Then \( N \) is a \( G \)-submodule of \( M \). Take \( T, \mu \) and \( \nu \) as in example 3.4.8. Then \( T \) is a \( G \)-module fuzzy
representation of $\mu$ onto $\nu$. Now it remains to show that $\psi : M/N \rightarrow GL(V)$ is a $G$-module fuzzy representation of $\xi$ onto $\nu$ where $\xi$ is a fuzzy $G$-module on $M/N$.

For $T_z \in \psi(M/N)$,

$$\psi(\xi)(T_z) = \vee \{ \xi([u]) | [u] \in \psi^{-1}(T_z) \}$$

$$= \vee \{ \vee \{ \mu(v) | v \in [u], T[u] = T_z \} \}$$

$$= \vee \{ \mu(v) | v \in [u], u \in T^{-1}(T_z) \}$$

$$= \begin{cases} 
1, & \text{if } z_k = 0, \forall k \\
1/2, & \text{if } z_1 \neq 0, z_2 = \cdots, z_n = 0 \\
1/3, & \text{if } z_2 \neq 0, z_3 = \cdots, z_n = 0 \\
\cdots & \cdots \\
\cdots & \cdots \\
1/n + 1, & \text{if } z_n \neq 0.
\end{cases}$$

$$= \nu(T_z)$$

$\therefore \psi(\xi) = \nu$. Hence $\psi$ is a $G$-module fuzzy representation of $\xi$ onto $\nu$. $\blacksquare$