Chapter 1

Preliminaries

1.1 Introduction

Zadeh [56] introduced the concept of fuzzy sets as a generalization of the concept of characteristic functions representing subsets in the classical set theory. Rosenfeld [41] defined the notion of fuzzy group and obtained some of its properties. Das [13] described the interrelationship between fuzzy groups and its level subgroups.

A classical set \( A \) is defined as a collection of objects \( x \) which belongs to a universal set \( X \). Each member of \( X \) can either belong to or not belong to \( A \). In the first case, the statement “\( x \) belongs to \( A \)” is true, and in the later case this statement is false.

Such a classical set can be defined in different ways. Either we
can list the objects that belong to the set or describe the set analytically or define the member elements by using the characteristic function $\chi_A$ defined from $X$ to \{0, 1\} in which 1 indicates the membership and 0 non-membership. In the general context, a membership function allows various degrees of membership for the elements of a given set. The range of the membership function is usually taken as \([0, 1]\). It has been noticed that the membership function is not limited to the values between 0 and 1. The set defined by a membership function is called a fuzzy set. In short, a fuzzy set is a generalization of a classical set and the membership function that of the characteristic function. In fact, we do not distinguish between these two notions.

This chapter contains some basic definitions and results which are required in the sequel.

### 1.2 Fuzzy groups and operations

A fuzzy set on $X$ is a function $\mu : X \to [0, 1]$. The set of all fuzzy sets on $X$ is called the fuzzy power set of $X$ and is denoted by $F(X)$.

#### 1.2.1 Definition [15]

Let $\mu, \nu \in F(X)$. Then

(i) $\mu \subseteq \nu \iff \mu(x) \leq \nu(x), \ \forall \ x \in X$

(ii) $\mu = \nu \iff \mu(x) = \nu(x), \ \forall \ x \in X$. 
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1.2.2 Definition [15]

Let $\mu, \nu \in F(X)$. The union $\mu \cup \nu$ and intersection $\mu \cap \nu$ in $F(X)$ are defined as follows.

\[
(\mu \cup \nu)(x) = \mu(x) \lor \nu(x), \\
(\mu \cap \nu)(x) = \mu(x) \land \nu(x), \ \forall x \in X
\]

where $\lor$ and $\land$ denote maximum (supremum) and minimum (infimum) respectively.

In the context of fuzzy sets, we can define $\cup$ and $\cap$ in several ways. Throughout this work, we consider only the above definitions of $\cup$ and $\cap$, which are called the standard fuzzy union and standard fuzzy intersection respectively. We can extend the above definitions to the arbitrary case as follows.

For any family $\{\mu_i | i \in I\}$, of fuzzy sets on $X$, where $I$ is an arbitrary index set, $\bigcup_{i \in I} \mu_i$ and $\bigcap_{i \in I} \mu_i$ are defined by

\[
(\bigcup_{i \in I} \mu_i)(x) = \lor_{i \in I} \mu_i(x) \text{ and } \\
(\bigcap_{i \in I} \mu_i)(x) = \land_{i \in I} \mu_i(x), \ \forall x \in X.
\]

1.2.3 Definition [15]

The fuzzy complement $\overline{\mu}$ of $\mu \in F(X)$ is defined by $\overline{\mu}(x) = 1 - \mu(x)$, $\forall x \in X$. 
1.2.4 Definition [3]

Let $f : X \rightarrow Y$ be a function and $\mu \in F(X)$. The image of $\mu$ under $f$ is the fuzzy set $f(\mu)$ on $Y$ defined by

$$f(\mu)(y) = \vee \{ \mu(x) | x \in f^{-1}(y) \}, \forall y \in R(f)$$

$$= 0, \quad \text{otherwise.}$$

The pre-image of $\nu \in F(Y)$ under $f$ is the fuzzy set $f^{-1}(\nu)$ on $X$ defined by $f^{-1}(\nu)(x) = \nu\{f(x)\}, \forall x \in X$.

1.2.5 Definition [41]

A fuzzy set $\mu$ on a group $G$ is called a fuzzy group on $G$ if $\forall x, y \in G$,

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(x^{-1}) = \mu(x)$

1.2.6 Result [41]

If $\mu$ is fuzzy group on $G$, then $\mu(e) \geq \mu(x)$, for all $x \in G$.

1.2.7 Definition [27]

A fuzzy group $\mu$ on $G$ is said to be fuzzy normal if $\mu(xy) = \mu(yx)$, $\forall x, y \in G$. 
1.2.8 Definition [27]

Let \( f : G \to G' \) be a group homomorphism. Then for every fuzzy group \( \mu \) on \( G \), \( f(\mu) \) is a fuzzy group on \( G' \).

1.2.9 Definition [27]

Let \( f : G \to G' \) be a group homomorphism. Then for every fuzzy group \( \nu \) on \( G' \), \( f^{-1}(\nu) \) is a fuzzy group on \( G \).

1.2.10 Example

Consider the group \( G = \{1, -1, i, -i\} \) under usual multiplication.
Define \( \mu \) on \( G \) by

\[
\mu(x) = \begin{cases} 
1, & \text{when } x = 1 \\
1/2, & \text{when } x = -1 \\
1/3, & \text{when } x = i \text{ or } -i.
\end{cases}
\]

It can be easily verified that \( \mu \) is a fuzzy group on \( G \).

1.2.11 Example

Let \( G = (\mathbb{R}, +) \). Define \( \mu \) on \( G \) by
\[ \mu(x) = \begin{cases} 
1, & \text{when } x = 0 \\
0.7, & \text{when } x \in Q - \{0\} \\
0.3, & \text{when } x \in R - Q 
\end{cases} \]

Then \( \mu \) is a fuzzy group on \( G \).

1.2.12 Example

Let \( G = (Z, +) \) and \( H = \{1, -1\} \), a group under multiplication. Define \( f : G \to H \) by

\[ f(x) = \begin{cases} 
1, & \text{when } x \text{ is even} \\
-1, & \text{when } x \text{ is odd.} 
\end{cases} \]

Then \( f \) is a group homomorphism. Define \( \mu \) on \( G \) by

\[ \mu(x) = \begin{cases} 
1, & \text{when } x \text{ is even} \\
1/2, & \text{when } x \text{ is odd.} 
\end{cases} \]

Then \( \mu \) is a fuzzy group on \( G \). We have

\[ f(\mu)(1) = \vee \{\mu(x) \mid x \in f^{-1}(1)\} \]
\[ = \vee \{\mu(x) \mid f(x) = 1\}. \]
\[ = 1. \]

Similarly, \( f(\mu)(-1) = 1/2 \).

Obviously \( f(\mu) \) is a fuzzy group on \( H \).
1.2.13 Example

In example 1.2.12, define $\nu$ on $H$ by $\nu(1) = 1, \nu(-1) = 1/2$.

$$f^{-1}(\nu)(x) = \nu(f(x)) = \begin{cases} 1, & \text{when } x \text{ is even} \\ 1/2, & \text{when } x \text{ is odd.} \end{cases}$$

1). If $x$ and $y$ are even, then $x + y$ is even. Then $f^{-1}(\nu)(x + y) = 1$, $f^{-1}(\nu)(x) = 1$, $f^{-1}(\nu)(y) = 1$

$$\therefore f^{-1}(\nu)(x + y) = 1 = 1 \land 1 = f^{-1}(\nu)(x) \land f^{-1}(\nu)(y).$$

2). If $x$ is even and $y$ is odd, $x + y$ is odd. $f^{-1}(\nu)(x + y) = 1/2$, $f^{-1}(\nu)(x) = 1$, $f^{-1}(\nu)(y) = 1/2$

$$\therefore f^{-1}(\nu)(x + y) = 1/2 = f^{-1}(\nu)(x) \land f^{-1}(\nu)(y).$$

3). If $x$ is odd and $y$ is odd, $x + y$ is even. Then $f^{-1}(\nu)(x + y) = 1$, $f^{-1}(\nu)(x) = 1/2$, $f^{-1}(\nu)(y) = 1/2$

$$\therefore f^{-1}(\nu)(x + y) > f^{-1}(\nu)(x) \land f^{-1}(\nu)(y).$$

Hence, $\forall x, y \in G, f^{-1}(\nu)(x + y) \geq f^{-1}(\nu)(x) \land f^{-1}(\nu)(y)$.

$$\therefore f^{-1}(\nu) \text{ is a fuzzy group on } G.$$

$\square$
1.3 Fuzzy homomorphism between fuzzy groups

1.3.1 Definition [12]

Let $G$ be a group and $M$ be a vector space over a field $K$. A linear representation of $G$ with representation space $M$ is a homomorphism of $G$ into $GL(M)$, where $GL(M)$ is the group of units in $Hom_K(M, M)$ and is called the general linear group.

1.3.2 Definition [27]

Let $G$ and $G'$ be groups and let $\mu$ and $\nu$ be fuzzy groups on $G$ and $G'$ respectively. Let $f$ be a group homomorphism of $G$ onto $G'$. Then $f$ is called a weak fuzzy homomorphism of $\mu$ into $\nu$ if $f(\mu) \subseteq \nu$. We say that $\mu$ is weak fuzzy homomorphic to $\nu$ and we write $\mu \sim \nu$.

The homomorphism $f$ is a fuzzy homomorphism of $\mu$ onto $\nu$ if $f(\mu) = \nu$. We say that $\mu$ is fuzzy homomorphic to $\nu$ and we write $\mu \approx \nu$. We also write $f : \mu \approx \nu$ as abbreviation for $f$ is a fuzzy homomorphism of $\mu$ onto $\nu$.

Let $f : G \rightarrow G'$ be an isomorphism. Then $f$ is a weak fuzzy isomorphism if $f(\mu) \subseteq \nu$ and $f$ is a fuzzy isomorphism if $f(\mu) = \nu$. The notation $f : \mu \cong \nu$ is used as short for $f$ is a fuzzy isomorphism of $\mu$ onto $\nu$. 

1.3.3 Example

Let $G = R - \{0\}$ and $G' = \{1, -1\}$, the group operation, in both cases, being usual multiplication. Define $f : G \to G'$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is positive} \\ -1, & \text{if } x \text{ is negative}. \end{cases}$$

Then $f$ is a group homomorphism.

Let $\mu$ be a fuzzy group on $G$ defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in R^+, \text{ set of positive reals} \\ 1/2, & \text{if } x \in R^-, \text{ set of negative reals}. \end{cases}$$

Let $\nu$ be the fuzzy group on $G'$ defined by $\nu(1) = 1$ and $\nu(-1) = 1/2$.

$$f(\mu)(1) = \vee\{\mu(x) | x \in f^{-1}(1)\} = 1$$

$$f(\mu)(-1) = \vee\{\mu(x) | x \in f^{-1}(-1)\} = 1/2.$$ 

$\therefore f(\mu) = \nu$. Hence $f$ is a fuzzy homomorphism. \hfill $\square$

1.3.4 Definition [43]

Let $G$ be a group, $M$ be a vector space over $K$ and $T : G \to GL(M)$ be a representation. Let $\mu$ be a fuzzy group on $G$ and $\nu$ be a fuzzy group on $T(G)$. Then the representation $T$ is a fuzzy representation if $T$ is a fuzzy homomorphism of $\mu$ onto $\nu$. 

1.3.5 Example

Let $G = (\mathbb{Z}, +)$ and $M$ be a vector space over $R$. Let $T : G \rightarrow GL(M)$ be defined by $T(x) = T_x$ where $T_x(m) = xm$, for $x \in G$ and $m \in M$. Then $T$ is a representation. Now, define $\mu$ on $G$ by

$$\mu(x) = \begin{cases} 
1, & \text{if } x \text{ is even} \\
1/2, & \text{if } x \text{ is odd.}
\end{cases}$$

Then $\mu$ is a fuzzy group on $G$.

Let $\nu$ be the fuzzy group on $T(G)$ defined by $\nu(T_{\text{even}}) = 1$, $\nu(T_{\text{odd}}) = 1/2$

$$T(\mu)(T_{\text{even}}) = \vee\{\mu(x) | x \in T^{-1}(T_{\text{even}})\} = 1$$
$$T(\mu)(T_{\text{odd}}) = \vee\{\mu(x) | x \in T^{-1}(T_{\text{odd}})\} = 1/2.$$

$\therefore T(\mu) = \nu$. Hence $T$ is a fuzzy representation of $\mu$ onto $\nu$. \hfill \Box$

1.4 Normal fuzzy groups

A subgroup $N$ of a group $G$ is said to be a normal subgroup of $G$ if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$. In symbols, we write $N \Delta G$ for “$N$ is a normal subgroup of $G$”. We recall that the factor group (or quotient group) $G/N$ is defined as the set of all cosets. i.e., $G/N = \{xN : x \in G\}$. 

1.4.1 Definition [26]

If $N \triangleleft G$, then the canonical map $\pi : G \rightarrow G/N$ given by $\pi(x) = xN$, $\forall x \in G$ is a homomorphism and it is called the canonical homomorphism or natural homomorphism of $G$ onto $G/N$.

1.4.2 Definition [12]

A normal series for a group $G$ is a chain of subgroups $G = G_1 \supset G_2 \supset \cdots G_r = \{e\}$ in which $G_{i+1} \triangleleft G_i$, $1 \leq i \leq r - 1$. The factors of the normal series are the factor groups $G_1/G_2, \cdots, G_{r-1}/G_r$. We say that $G$ is solvable if $G$ has a normal series in which all the factor groups are abelian.

1.4.3 Definition [27]

Let $\mu$ and $\nu$ be fuzzy groups on $G$ and $\mu \subseteq \nu$. Then $\mu$ is called a normal fuzzy subgroup of $\nu$, denoted by $\mu \triangleleft \nu$, if

$$\mu(xy^{-1}) \geq \mu(y) \land \nu(x), \quad \forall x, y \in G.$$ 

1.4.4 Example

Let

$$G = \{1, -1, i, -i\}$$

$$G_1 = \{1, -1, i, -i\}$$

$$G_2 = \{1, -1\}$$

$$G_3 = \{1\}.$$
Then $G = G_1 \supset G_2 \supset G_3 = \{1\}$.

Also, $G_2 \Delta G_1$, $G_3 \Delta G_2$ and $G_1/G_2$ and $G_2/G_3$ are abelian.

$\therefore G$ is solvable.

Define $\mu$ on $G$ by

$$
\mu(x) = \begin{cases} 
1, & \text{if } x = 1 \\
1/2, & \text{if } x = -1 \\
1/3, & \text{if } x = i \text{ or } -i.
\end{cases}
$$

It can be verified that $\mu$ is a fuzzy group on $G$. Now, define $\mu_1$ on $G$ by

$$
\mu_1(x) = \begin{cases} 
1, & \text{if } x = 1 \\
1/4, & \text{if } x = -1 \\
1/5, & \text{if } x = i \text{ or } -i.
\end{cases}
$$

Then $\mu_1$ is a fuzzy group on $G$ such that $\mu_1 \subseteq \mu$ and

$$
\mu_1(xyx^{-1}) \geq \mu_1(y) \wedge \mu(x), \ \forall \ x, y \in G.
$$

Hence $\mu_1 \Delta \mu$. \hfill $\square$

1.4.5 Definition [27]

Let $\mu \in F(X)$. Then $\mu^*$, the support of $\mu$, is defined as

$$
\mu^* = \{x | x \in X, \mu(x) > 0\}.
$$
1.4.6 Definition [27]

Let $\mu$ be a fuzzy group on $G$ and let $N$ be a normal subgroup of $G$. Then the fuzzy group $\xi$ on $G/N$ defined by

$$\xi([x]) = \vee \{\mu(z) | z \in [x]\}, \text{ for all } x \in G, \text{ where } [x] = xN.$$ 

is called the quotient fuzzy group or factor fuzzy group of $\mu$ relative to the normal subgroup $N$ of $G$ and is denoted by $\mu/N$.

Let $\mu$ and $\nu$ be fuzzy groups on $G$ such that $\mu \Delta \nu$. Then $\mu^*$ is a normal subgroup of $\nu^*$. Clearly $\nu_{\nu^*}$ is a fuzzy group on $\nu^*$. Then the factor fuzzy group of $\nu_{\nu^*}$ relative to $\mu^*$ exists. i.e., $\nu_{\nu^*}/\mu^*$ exists. We denote this factor fuzzy group by $\nu/\mu$ and call it the quotient fuzzy group or factor fuzzy group of $\nu$ relative to $\mu$.

1.4.7 Definition [27]

Let $\mu$ be a fuzzy group on $G$. Then

(i) $\mu$ is abelian if $\mu^*$ is abelian.

(ii) $\mu$ is said to be solvable if $\exists$ fuzzy groups $\mu_i$ of $G$, $i = 1, 2, \ldots, k$ such that $\mu_i \Delta \mu_{i-1}$ and $\mu_{i-1}/\mu_i$ is abelian, $i = 1, 2, \ldots, k$ where $\mu_0 = \mu$ and $\mu_k = e_{\mu(e)}$, where $e_{\mu(e)}$ is the fuzzy set on $G$ defined by $e_{\mu(e)}(x) = \mu(e)$, when $x = e$ and 0 otherwise.
1.4.8 Remark

If $\mu$ and $\nu$ are fuzzy groups on $G$ such that $\mu \subseteq \nu$, then it can be readily seen that $\mu$ is solvable if $\nu$ is solvable.