Chapter 4

Fuzzy projectivity on quotient $G$-modules

4.1 Introduction

The notion of projectivity of modules was developed by Cartan and Eilemberg [11]. Zahedi and Ameri [57] studied about the fuzzy projectivity and injectivity of modules. Shery Fernandez [45] analysed the fuzzy projectivity of modules. As a continuation of these works, here we introduce the concept of fuzzy projectivity of quotient $G$-modules.

In this chapter we discuss the projectivity and relative projectivity of quotient $G$-modules and the fuzzy projectivity on it.
4.2 Projectivity of $G$-modules

4.2.1 Definition [45]

A $G$-module $M$ is projective, if for any $G$-module $M^*$ and any $G$-submodule $N^*$ of $M^*$, every homomorphism $\phi : M \to M^*/N^*$ can be lifted to a homomorphism $\psi : M \to M^*$. This means that $\exists$ a homomorphism $\psi : M \to M^*$ such that $\pi \circ \psi = \phi$, where $\pi : M^* \to M^*/N^*$ is the canonical homomorphism.

\[
\begin{array}{c}
M \\
\downarrow \phi \\
M^*/N^* \\
\end{array} 
\quad \quad \quad \quad \quad \quad
\begin{array}{c}
M^* \\
\downarrow \pi \\
M^*/N^* \\
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\end{array}
\]

4.2.2 Definition [45]

Let $M$ and $M^*$ be $G$-modules. Then $M$ is $M^*$-projective if for any $G$-submodule $N^*$ of $M^*$, any homomorphism $\phi : M \to M^*/N^*$ can be lifted to a homomorphism $\psi : M \to M^*$.
4.2.3 Proposition [45]

Let $M$ and $M^*$ be $G$-modules such that $M$ is $M^*$-projective. Let $N^*$ be any $G$-submodule of $M^*$. Then $M$ is $N^*$-projective and $M^*/N^*$-projective.

4.2.4 Proposition [45]

Let $M$ be a $G$-module and $N$ be a $G$-submodule of $M$. If $M$ has a fuzzy $G$-module, then the $G$-module $N$ and $M/N$ has fuzzy $G$-modules.

4.2.5 Definition [45]

Let $M$ and $M^*$ be $G$-modules. Let $\mu$ and $\nu$ be fuzzy $G$-modules on $M$ and $M^*$ respectively. Then $\mu$ is $\nu$-projective if

i) $M$ is $M^*$-projective

ii) $\mu(m) \leq \nu(\psi(m)), \forall m \in M$ and $\psi \in \text{Hom}(M, M^*)$.

\[ \begin{array}{c}
M^*/N^* \\
\downarrow \pi \\
M^* \\
\downarrow \mu \\
[0, 1] \\
\end{array} \]

\[ \begin{array}{c}
\phi \\
\downarrow \\
M \\
\downarrow \psi \\
M^* \\
\downarrow \nu \\
[0, 1] \\
\end{array} \]
4.2.6 Remark

Let $\mu$ and $\nu$ be fuzzy $G$-modules on $M$ and $M^*$ respectively and let $\mu$ be $\nu$-projective. Hence $\exists \psi \in Hom(M, M^*)$ such that $\mu(m) \leq \nu(\psi(m))$, $m \in M$. Then we say that $\psi^{-1}$ is an anti-weak fuzzy homomorphism of $\nu$ onto $\mu$.

\begin{center}
\begin{tikzpicture}[description/.style={fill=white,inner sep=2pt},scale=1.2,auto]

\node (M) at (0,0) {$M$};
\node (Mstar) at (2,0) {$M^*$};
\node (Mbar) at (4,0) {$M$};
\node (nu) at (1,1) {$\nu$};
\node (mu) at (1,-1) {$\mu$};
\node (Mstar nu) at (3,1) {$\nu$};
\node (Mstar mu) at (3,-1) {$\mu$};
\node (01) at (0,1) {$[0,1]$};
\node (01 bar) at (4,1) {$[0,1]$};
\node (psi) at (2,0) {$\psi$};
\node (psi^bar) at (2,0) {$\psi^{-1}$};

\draw[->] (M) to (Mstar) node [midway, above] {$\psi$};
\draw[->] (Mstar) to (Mbar) node [midway, above] {$\psi^{-1}$};
\draw[->] (M) to (Mstar) node [midway, left] {$\mu$};
\draw[->] (M) to (Mbar) node [midway, left] {$\nu$};
\draw[->] (Mstar) to (Mbar) node [midway, left] {$\nu$};
\draw[->] (Mstar) to (M) node [midway, right] {$\mu$};
\end{tikzpicture}
\end{center}

Since $\mu$ is $\nu$-projective,

$$\mu(m) \leq \nu(\psi(m)), \ m \in M, \ \psi \in Hom(M, M^*).$$

$\therefore \psi^{-1}(\nu)(m) \geq \mu(m), \ m \in M.$

$\therefore \psi^{-1}$ is an anti-weak fuzzy homomorphism of $\nu$ onto $\mu$.  \hfill $\square$

4.3 Projectivity on fuzzy $G$-modules

Shery [44] formulated the concept of fuzzy $G$-module projectivity and proved that any finite dimensional $G$-module has a fuzzy $G$-submodules and that completely reducible $G$-modules have fuzzy completely reducible
G-submodules and derived some of its properties. Here we introduce and analyse the concept of projectivity and fuzzy projectivity on quotient \(G\)-modules. We analyse the relative projectivity of two quotient \(G\)-modules too, provided that the \(G\)-modules are relatively projective. Also, if a quotient fuzzy \(G\)-module is fuzzy projective with respect to another fuzzy \(G\)-module, then \(\exists\) an anti-weak fuzzy homomorphism between the quotient fuzzy \(G\)-modules.

### 4.3.1 Proposition

If \(M\) is \(M^*\)-projective, then \(M/N\) is \(M^*\)-projective where \(N\) is a \(G\)-submodule of \(M\).

**Proof.**

\[
\begin{array}{ccc}
M & \xrightarrow{\psi} & M^* \\
| & \downarrow{g} & \downarrow{h} \\
M/N & \xrightarrow{\phi} & \pi \\
| & \downarrow{f} & \\
M^*/N^* & \\
\end{array}
\]

Let \(N\) be a \(G\)-submodule of \(M\) and \(N^*\) be a \(G\)-submodule of \(M^*\). Since \(M\) is \(M^*\)-projective, the homomorphism \(M \to M^*/N^*\) can be lifted to \(\psi : M \to M^*\) such that \(\pi \circ \psi = \phi\) where \(\pi : M^* \to M^*/N^*\) is the canoni-
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cal homomorphism. Let $g : M \to M/N$ be the canonical homomorphism defined by $g(x) = x + N$, $x \in M$.

Now $g : M \to M/N$ and $\psi : M \to M^*$ are homomorphisms. Hence $\exists$ a homomorphism $h : M/N \to M^*$ such that $h \circ g = \psi$. Let $f : M/N \to M^*/N^*$ be the mapping defined by $f(x + N) = \psi(x) + N^*$, $x \in M$. Then $f$ is a homomorphism. Also $\pi \circ h = f$. Hence $f : M/N \to M^*/N^*$ can be lifted to $h : M/N \to M^*$ such that $\pi \circ h = f$.

$\therefore M/N$ is $M^*$-projective. 

4.3.2 Proposition

If $M$ is $M^*$-projective, then $M/N$ is $M^*/N^*$-projective where $N$ and $N^*$ are $G$-submodules of $M$ and $M^*$ respectively.

Proof.

\begin{center}
\begin{tikzpicture}
    \node (M) at (0,0) {$M$};
    \node (M*) at (3,0) {$M^*$};
    \node (M/N) at (0,-3) {$M/N$};
    \node (M^*/N^*) at (3,-3) {$M^*/N^*$};
    \node (M^*/Y^*) at (3,-6) {$M^*/Y^*$};
    \node (psi) at (1.5,0) {$\psi$};
    \node (pi) at (-1.5,-3) {$\pi$};
    \node (pi') at (1.5,-3) {$\pi'$};
    \node (phi) at (1.5,-1.5) {$\phi$};
    \node (f) at (-1.5,-4.5) {$f$};
    \node (f') at (0,-4.5) {$f'$};
    \node (pi'') at (1.5,-6) {$\pi''$};
    \draw[->] (M) -- (M*) node[midway,above] {$\psi$};
    \draw[->] (M) -- (M/N) node[midway,left] {$\pi$};
    \draw[->] (M*N) -- (M^*/N^*) node[midway,right] {$\phi$};
    \draw[->] (M/N) -- (M^*/N^*) node[midway,above] {$f$};
    \draw[->] (M^*/N^*) -- (M^*/Y^*) node[midway,above] {$\pi''$};
    \draw[dashed,->] (M) -- (M^*/Y^*) node[midway,left] {$f'$};
\end{tikzpicture}
\end{center}
Since $M$ is $M^*$-projective, $M$ is $M^*/N^*$-projective and $M/N$ is $M^*$-projective. Since $M$ is $M^*$-projective, $\phi : M \rightarrow M^*/N^*$ can be lifted to $\psi : M \rightarrow M^*$ such that $\pi' \circ \psi = \phi$ where $\pi' : M^* \rightarrow M^*/N^*$. Let $Y^*/N^*$ be a $G$-submodule of $M^*/N^*$. Since $M$ is $M^*/N^*$-projective, $\exists \pi'' : M^*/N^* \rightarrow M^*/Y^*$ such that $\pi'' \circ \phi = g$ where $\pi''(y + N^*) = y + Y^*$, $y \in M^*$.

Let $\pi$ be the natural homomorphism from $M$ into $M/N$ defined by $\pi(x) = x + N$, $x \in M$. Since $M$ is $M^*/N^*$-projective, $\exists$ a mapping $f : M/N \rightarrow M^*/N^*$, defined by $f(x + N) = \psi(x) + N^*$ such that $f \circ h = \phi$. Then $f$ is a homomorphism. Let $f' : M/N \rightarrow M^*/N^*$ be defined by $f'(x + N) = \psi(x) + N^*$ where $\psi(x) \in M^*$. Then $f'$ is a homomorphism. Since $M$ is $M^*/N^*$-projective, $f' \circ \pi$ can be lifted to $\phi$ such that $f' \circ \pi = \pi'' \circ \phi$. Hence $\exists$ homomorphisms $f : M/N \rightarrow M^*/N^*$, $f' : M/N \rightarrow M^*/Y^*$ and $\pi'' : M^*/N^* \rightarrow M^*/Y^*$ where $Y^*/N^*$ is a $G$-submodule of $M^*/N^*$.

For $x + N \in M/N$,

$$(\pi'' \circ f)(x + N) = \pi''[f(x + N)]$$

$$= \pi''[\psi(x) + N^*], \psi(x) \in M^*$$

$$= \psi(x) + Y^* \in M^*/Y^*$$

$$= f'(x + N)$$

Hence $\pi'' \circ f = f'$. 

f' can be lifted to f such that π" ∘ f = f'.
∴ M/N is M*/N*-projective.

4.3.3 Definition [45]

A G-module M is said to be self-projective if M is M-projective.

4.3.4 Definition [45]

Two G-modules M and M* are said to be relatively projective if M is M*-projective and M* is M-projective.

4.3.5 Proposition

If M and M* are relatively projective G-modules then M/N and M*/N* are relatively projective quotient G-modules where N and N* are G-submodules of M and M* respectively and N* = Im(N).

Proof. If M is M*-projective, by the above proposition, M/N is M*/N*-projective. Since M and M* are relatively projective G-modules, M* is M-projective. Hence M*/N* is M/N-projective.

Now M/N is M*/N*-projective and M*/N* is M/N-projective.

Hence M/N and M*/N* are relatively projective quotient G-modules.
4.3.6 Example

Let $G = \{1, -1\}$, $M = Q(\sqrt{2})$ and $M^* = Q(\sqrt{3})$. Then $M$ and $M^*$ are $G$-modules. Also $Q$ is a proper $G$-submodule of both $Q(\sqrt{2})$ and $Q(\sqrt{3})$. Consider the quotient $G$-modules is $Q(\sqrt{2})/Q$ and $Q(\sqrt{3})/Q$. Let $\phi$ be the homomorphism defined from $Q(\sqrt{2}) \to Q(\sqrt{3})/Q$ by $\phi(a + b\sqrt{2}) = b\sqrt{3} + Q$, $a, b \in Q$. Define $\psi : Q(\sqrt{2}) \to Q(\sqrt{3})$ by $\psi(a + b\sqrt{2}) = a + b\sqrt{3}$, $a, b \in Q$. Then $\psi$ is a homomorphism. Let $\pi$ be the natural homomorphism from $Q(\sqrt{3}) \to Q(\sqrt{3})/Q$.

$$\therefore \quad \pi(a + b\sqrt{3}) = (a + b\sqrt{3}) + Q$$
$$= b\sqrt{3} + Q; \quad a, b \in Q.$$

For $a + b\sqrt{2} \in Q(\sqrt{2})$,

$$\pi(\psi(a + b\sqrt{2})) = \pi(a + b\sqrt{3})$$
$$= b\sqrt{3} + Q$$
$$= \phi(a + b\sqrt{2})$$

$$\therefore \quad \pi \circ \psi = \phi.$$

Hence $\phi : Q(\sqrt{2}) \to Q(\sqrt{3})/Q$ can be lifted to $\psi : Q(\sqrt{2}) \to Q(\sqrt{3})$, and so $Q(\sqrt{2})$ is $Q(\sqrt{3})$-projective.

Now to show that $Q(\sqrt{2})/Q$ is $Q(\sqrt{3})$-projective.
Define $f : Q(\sqrt{2})/Q \to Q(\sqrt{3})/Q$ by $f((a+b\sqrt{2})+Q) = (a+b\sqrt{3})+Q$. i.e., $f(b\sqrt{2} + Q) = b\sqrt{3} + Q$; $a, b \in Q$. Let $\pi' : Q(\sqrt{2}) \to Q(\sqrt{2})/Q$ be the natural homomorphism. That is

$$\pi'(a + b\sqrt{2}) = (a + b\sqrt{2}) + Q, \quad a, b \in Q$$

$$= b\sqrt{2} + Q.$$

Define $h : Q(\sqrt{2})/Q \to Q(\sqrt{3})/Q$ by $h((a+b\sqrt{2})+Q) = a+b\sqrt{3}, a, b \in R$. For $a + b\sqrt{2} + Q \in Q(\sqrt{2})/Q$,

$$(\pi \circ h)((a + b\sqrt{2}) + Q) = \pi[h[(a + b\sqrt{2}) + Q]]$$

$$\pi \circ h(b\sqrt{2} + Q) = \pi(a + b\sqrt{3})$$

$$= b\sqrt{3} + Q$$

$$= f(b\sqrt{2} + Q).$$
\( \therefore \pi \circ h = f \). Hence \( Q(\sqrt{2})/Q \) is \( Q(\sqrt{3}) \)-projective.

### 4.3.7 Example

Let \( G = \{1, -1\} \), \( M = Q(\sqrt{2}) \) and \( M^* = Q(\sqrt{3}) \). Then \( Q(\sqrt{2})/Q \) is \( Q(\sqrt{3})/Q \)-projective.

Let \( Q(\sqrt{3})/Q (= Y^*/N^*) \) be a \( G \)-submodule of \( Q(\sqrt{3})/Q (= M^*/N^*) \).

Since \( Q(\sqrt{2}) \) is \( Q(\sqrt{3})/Q \)-projective, \( \exists \pi'' : Q(\sqrt{3})/Q \to Q(\sqrt{3}) \) such that \( \pi'' \circ \phi = g \) where \( \pi''(b\sqrt{3} + Q) = b\sqrt{3}, b \in Q \). Let \( \pi \) be the natural homomorphism from \( Q(\sqrt{2}) \to Q(\sqrt{2})/Q \), defined by \( \pi(a + b\sqrt{2}) = b\sqrt{2} + Q, a, b \in Q \). Since \( Q(\sqrt{2}) \) is \( Q(\sqrt{3})/Q \)-projective, \( \exists \) a mapping \( f : Q(\sqrt{2})/Q \to Q(\sqrt{3}) \), defined by \( f(b\sqrt{2} + Q) = b\sqrt{3} + Q \) such that
$f \circ \pi = \phi$. Then $f$ is a homomorphism. For $a + b\sqrt{2} \in Q(\sqrt{2})$, 

$$(f \circ \pi)(a + b\sqrt{2}) = f[\pi[(a + b\sqrt{2})]]$$

$$= f(b\sqrt{2} + Q)$$

$$= b\sqrt{3} + Q$$

$$= \phi(a + b\sqrt{2}).$$

Let $f' : Q(\sqrt{2})/Q \rightarrow Q(\sqrt{3})$ be defined by $f'(b\sqrt{2} + Q) = b\sqrt{3}, b \in Q$. Then $f'$ is a homomorphism. Since $Q(\sqrt{2})$ is $Q(\sqrt{3})/Q$-projective, $f' \circ \pi$ can be lifted to $\phi$ such that $f' \circ \pi = \pi'' \circ \phi$.

$$(\pi'' \circ f)(b\sqrt{2} + Q) = \pi''[f(b\sqrt{2} + Q)]$$

$$= \pi''(b\sqrt{3} + Q)$$

$$= b\sqrt{3}$$

$$= f'(b\sqrt{2} + Q).$$

$\therefore \pi'' \circ f = f'$. Hence $Q(\sqrt{2})/Q$ is $Q(\sqrt{3})/Q$-projective.

If $Y^*/N^* = Q/Q = Q$ is the $G$-submodule of $M^*/N^* = Q(\sqrt{3})/Q$ under consideration, then $\pi'' : Q(\sqrt{3})/Q \rightarrow Q(\sqrt{3})/Q/Q = Q(\sqrt{3})/Q$ is the identity mapping and hence the result is true. ■
4.3.8 Proposition

Let $\mu$ and $\nu$ be fuzzy $G$-modules on the $G$-modules $M$ and $M^*$-respectively such that $\mu$ is $\nu$-projective. Then $\eta : M/N \to [0, 1]$ is a fuzzy $G$-module on $M/N$, $N$ is a $G$-submodule of $M$, defined by $\eta(m + N) = \mu(m)$, $m \in M$ and $\eta$ is $\nu$-projective.

Proof. Given that $\mu$ is $\nu$-projective. \[ M \] is $M^*$-projective and $\mu(m) \leq \nu(\psi(m))$, $\forall m \in M$ and $\psi \in \text{Hom}(M, M^*)$. Since $M$ is $M^*$-projective, $M/N$ is $M^*$-projective.

Now it remains to prove that $\eta(m + N) \leq \nu(h(m + N))$, $\forall m + N \in M/N$
and $h : M/N \to M^*$. For $m + N \in M/N$,

$$\eta(m + N) = \mu(m) \quad \text{given}$$

$$\leq \nu(\psi(m)), \quad \therefore \mu \text{ is } \nu\text{-projective}$$

$$= \nu\{(h \circ \pi)(m)\}$$

$$= \nu\{h[\pi(m)]\}$$

$$\eta(m + N) \leq \nu\{h(m + N)\}, \quad \therefore \pi \text{ is the canonical homomorphism from } M \text{ into } M/N.$$

$\therefore \eta$ is $\nu$-projective. \hfill \Box

### 4.3.9 Proposition

Let $\mu$ and $\nu$ be fuzzy $G$-modules on $M$ and $M^*$ such that $\mu$ is $\nu$-projective. Then the fuzzy $G$-modules $\eta$ and $\eta^*$ on $M/N$ and $M^*/N^*$ result defined by $\eta(m + N) = \mu(m), \forall m \in M$ and $\eta^*(m^* + N^*) = \nu(m^*), \forall m^* \in M^*$. Then $\eta$ is $\eta^*$-projective.

**Proof.** Given that $\mu$ is $\nu$-projective, $\therefore M$ is $M^*$-projective and

$$\mu(m) \leq \nu(\psi(m)), \forall m \in M \text{ and } \psi \in \text{Hom}(M, M^*).$$
Define $f : M/N \to M^*/N^*$ by $f(m + N) = \psi(m) + N^*$. Then $f$ is a homomorphism. By Proposition 4.3.2, we know that $M/N$ is $M^*/N^*$-projective.

Now, it remains to prove that $\eta(m + N) \leq \eta^*(f(m + N))$, $\forall m + N \in M/N, f \in \text{Hom}(M/N, M^*/N^*)$.

For $m + N \in M/N$,

\[
\eta(m + N) = \mu(m) \\
\leq \nu(\psi(m)) \quad m \in M \\
= \eta^*(\psi(m) + N^*) \\
= \eta^*(f(m + N)).
\]
\[ \therefore \eta(m+N) \leq \eta^*(f(m+N)), \ m+N \in M/N \text{ and } f \in \text{Hom}(M/N, M^*/N^*). \]
\[ \therefore \eta \text{ is } \eta^* \text{-projective.} \]

4.3.10 Example

Let \( G = \{1, -1\}, M = \mathbb{Q}(\sqrt{2}) \) and \( M^* = \mathbb{Q}(\sqrt{3}) \). Let \( \phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})/\mathbb{Q} \) be the homomorphism defined by \( \phi(a+b\sqrt{2}) = (a+b\sqrt{3})+\mathbb{Q}, a, b \in \mathbb{Q} \) and \( \pi : \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{3})/\mathbb{Q} \) be the canonical homomorphism so that \( \pi(a+b\sqrt{3}) = (a+b\sqrt{3})+\mathbb{Q} \). Then \( \phi \) can be lifted to \( \psi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3}) \) given by \( \psi(a+b\sqrt{2}) = a+b\sqrt{3}, a, b \in \mathbb{Q} \), such that \( \pi \circ \psi = \phi \).

\[ \therefore \mathbb{Q}(\sqrt{2}) \text{ is } \mathbb{Q}(\sqrt{3}) \text{-projective.} \]
Now, define $\mu : Q(\sqrt{2}) \to [0, 1]$ by

$$\mu(a + b\sqrt{2}) = \begin{cases} 1/4, & \text{if } a = b = 0 \\ 1/5, & \text{if } a \neq 0, b = 0 \\ 1/6, & \text{if } b \neq 0 \end{cases}$$

and $\nu : Q(\sqrt{3}) \to [0, 1]$ by

$$\nu(c + d\sqrt{3}) = \begin{cases} 1, & \text{if } c = d = 0 \\ 1/2, & \text{if } c \neq 0, d = 0 \\ 1/3, & \text{if } d \neq 0 \end{cases}$$

$\therefore \mu$ is $\nu$-projective.

**4.3.11 Example**

Let $G = \{1, -1\}$, $M = Q(\sqrt{2})$, $M^* = Q(\sqrt{3})$. Define $\mu$ and $\nu$ on $M$ and $M^*$ as in the above example. Define $\eta$ on $Q(\sqrt{2})/Q$ by

$$\eta((a + b\sqrt{2}) + Q) = \mu(a + b\sqrt{2}); \quad a, b \in Q.$$  

Since $Q(\sqrt{2})$ is $Q(\sqrt{3})$-projective, by example 4.3.6, $Q(\sqrt{2})/Q$ is $Q(\sqrt{3})$-projective. Let $h : Q(\sqrt{2})/Q \to Q(\sqrt{3})$ be the mapping

$$h((a + b\sqrt{2}) + Q) = \psi(a + b\sqrt{2}) = a + b\sqrt{3}.$$
Let $\pi : Q(\sqrt{2}) \to Q(\sqrt{2})/Q$ be the canonical homomorphism defined by $\pi(a + b\sqrt{2}) = (a + b\sqrt{2} + Q$. Now, it remains to prove that

$$\eta((a + b\sqrt{2}) + Q) \leq \nu(h(a + b\sqrt{2}) + Q).$$

For $(a + b\sqrt{2}) + Q \in Q(\sqrt{2})/Q$,

$$\eta((a + b\sqrt{2}) + Q) = \mu(a + b\sqrt{2})$$

$$\leq \nu(\psi(a + b\sqrt{2})), \quad \because \mu \text{ is } \nu\text{-projective}$$

$$= \nu((h \circ \pi)(a + b\sqrt{2}))$$

$$= \nu(h[\pi(a + b\sqrt{2})])$$

$$\leq \nu(h(a + b\sqrt{2}) + Q))$$

$\because \eta$ is $\nu\text{-projective.}$

4.3.12 Example

Let $G = \{1, -1\}, M = Q(\sqrt{2}), M^* = Q(\sqrt{3})$. Let $\mu$ and $\nu$ be fuzzy $G$-modules on $M$ and $M^*$ as defined in the above example. Then $\mu$ is $\nu$-projective. Define $\eta : Q(\sqrt{2})/Q \to [0, 1]$ by

$$\eta((a + b\sqrt{2}) + Q) = \mu(a + b\sqrt{2})$$

and $\eta^* : Q(\sqrt{3})/Q \to [0, 1]$ by

$$\eta^*((a + b\sqrt{3}) + Q) = \nu(a + b\sqrt{3}); a, b \in Q.$$

Since $\mu$ is $\nu$-projective,

$$\mu(a + b\sqrt{2}) \leq \nu(\psi(a + b\sqrt{2})), \quad \forall \psi \in Hom(M, M^*).$$
Define $f : Q(\sqrt{2})/Q \to Q(\sqrt{3})/Q$ by $f((a+b\sqrt{2})+Q) = (a+b\sqrt{3})+Q$. Then $f$ is a homomorphism. By example 4.3.7, we know that $Q(\sqrt{2})/Q$ is $Q(\sqrt{3})/Q$-projective. Now, it remains to prove that $\eta((a+b\sqrt{2})+Q) \leq \eta^*(f((a+b\sqrt{2})+Q))$.

For $(a+b\sqrt{2})+Q \in Q\sqrt{2}/Q$,

$$
\eta((a+b\sqrt{2})+Q) = \mu(a+b\sqrt{2}) \\
\leq \nu(\psi(a+b\sqrt{2})), \quad \because \mu \text{ is } \nu \text{-projective} \\
= \eta^*(\psi(a+b\sqrt{2})+Q) \\
= \eta^*(f((a+b\sqrt{2})+Q)) \\
\Rightarrow \eta((a+b\sqrt{2})+Q) \leq \nu^*(f((a+b\sqrt{2})+Q)).
$$

\therefore $\eta$ is $\nu$-projective. \qed

### 4.3.13 Remark

It may be recalled that when $\mu$ is $\nu$-projective, $\psi^{-1}$ is an anti-weak fuzzy homomorphism of $\nu$ and $\mu$. In a similar manner, since $\eta$ is $\eta^*$-projective, we can say that $f$ is an anti-weak fuzzy homomorphism of $\eta^*$ onto $\eta$. \qed