CHAPTER 4

Applications to Multivariate Analysis

4.1 Introduction

This chapter deals with the application of a generalized type-1 Dirichlet model in multivariate statistical analysis. A very useful and interesting application of the generalized type-1 Dirichlet density is in obtaining the distribution of certain likelihood ratio statistics in multivariate testing of hypotheses. It is shown that the exact null distribution of the likelihood ratio criterion for testing hypotheses concerning multivariate analysis of variance (MANOVA), multivariate analysis of covariance (MANCOVA), sphericity, canonical correlations, multivariate regression etc in the $p$-variate normal case can be obtained as marginal distributions of a generalized type-1 Dirichlet model. The exact distributions of the likelihood ratio criteria so obtained have a general format for every $p$. A direct and easy method of computation of $p$-values and tables of critical points for certain values of $p$ are also presented. Various types of properties and relations involving hypergeometric series are also established. The materials presented in this chapter is based on the papers Thomas and Thannippura (2007 [14], [15]). Now we shall consider some of the basic notations and properties of elementary functions which we will use later on in the coming sections. The derivations and proofs of these results can be seen from Mathai (1993).
(i) \( B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \).

(ii) For a non-negative integer \( r \),

\[
(a)_r = (a + r - 1)(a + r - 2) \ldots (a) = \frac{\Gamma(a + r)}{\Gamma(a)}; \quad (a)_0 = 1, a \neq 0,
\]

when \( \Gamma(a) \) is defined.

(iii) For \(|z| < 1\),

\[
(1 - z)^{-a} = \sum_{r=0}^{\infty} \frac{(a)_r}{r!} z^r = {}_1F_0(a; ; z).
\]

(iv) \( \Gamma(z)\Gamma(z + \frac{1}{2}) = \pi^{\frac{1}{2}} 2^{1-2z} \Gamma(2z). \)

(v) For \(|z| < 1\),

\[
C_{\alpha, \beta}^{\delta_1, \delta_2} \left( z \middle| (\alpha_1, \beta_1 - 1, \gamma_2 + \delta_2 - 1) \right)
= \frac{z^{\gamma_2 - 1} (1 - z)^{\delta_1 + \delta_2 - 1}}{\Gamma(\delta_1 + \delta_2)} \ 2F_1(\gamma_2 + \delta_2 - \gamma_1; \delta_1; \delta_1 + \delta_2; 1 - z),
\]

where

\[
pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_q)_r} \frac{z^r}{r!}.
\]

(vi) \( C_{\alpha, \beta}^{\delta_1, \delta_2} \left( z \middle| (\alpha - 1, \alpha - \frac{1}{2}) \right) = 2^{\beta - 1} C_{\alpha, \beta}^{\delta_1, \delta_2} \left( z \middle| (\alpha - 1, \alpha - \frac{1}{2}) \right) \).

(vii) For \(|z| < 1, \beta > -1\),

\[
C_{\alpha, \beta}^{\delta_1, \delta_2} \left( z \middle| (\alpha + \beta + 1) \right) = \frac{z^\alpha}{\Gamma(\beta + 1)}.
\]

4.2 Distribution of the Likelihood Ratio Statistic

Currently, most of the testing procedures for testing hypotheses on the parameters of one or more multivariate Gaussian populations are in terms of likelihood ratio statistics or \( \Lambda \)-criteria. We usually use \( \Lambda \)-criteria for testing hypotheses in regression analysis, MANOVA, MANCOVA, canonical correlations, in the problem of testing independence on the subvectors in multivariate normal case and so on. It goes without saying that
the construction of confidence region or simultaneous confidence intervals also make
use of the null distribution of the likelihood ratio criterion. The details of these applica-
tions can be seen from standard textbooks on multivariate analysis, see for example,
Anderson (2003) and Rencher (1998). The exact null distribution of the likelihood ratio
statistic for testing hypotheses on the parameters of multivariate normal and other dis-
tributions has been investigated by several authors, see for example, Schatzoff (1966),
Pillai and Gupta (1969), Mathai (1971) and Coelho (1998). All the results have been ob-
tained in terms of series expansions or in terms of complicated expressions and still the
density function or the cumulative distribution function of the statistic is not expressed
in a tractable form. The present study overcomes this difficulty fully and gives a very
general format for its density function and cumulative distribution function.

Let us denote a Wishart distribution on \( p \)-variates, covariance matrix \( \Sigma > 0 \) and
with \( n \) degrees of freedom by \( W_p(\Sigma, n) \). In all of the above mentioned multivariate tests
the likelihood ratio statistic turns out to be of the form

\[
U = \frac{|G|}{|G + H|}
\]

(4.8)

where \( G \) is distributed according to \( W_p(\Sigma, n) \) and \( H \) is distributed according to \( W_p(\Sigma, m) \).
\( G \) may be called the residual sum of squares and product matrix and \( H \) may be called
the sum of squares and product matrix due to deviation from the hypothesis. Then \( G \)
and \( H \) are independently distributed when the population is normal.

We shall denote the criterion (4.8) as \( U_{p,m,n} \) where \( p \) is the dimension, \( m \) and \( n \)
denote the numbers of degrees of freedom of the Wishart matrices \( H \) and \( G \) respectively.
We determine appropriate values for \( m \) and \( n \) depending on the hypothesis. Let
us consider some of the testing problems where we use the criterion (4.8). Consider the
testing of hypotheses about regression coefficients in multivariate linear regression. Let
\( x_1, \ldots, x_N \) be a set of \( N \) independent observations, \( x_\alpha \) being drawn from \( N_p(\beta z_\alpha, \Sigma) \).
Ordinarily the vectors \( z_\alpha \) are known vectors with \( q \) components and the \( p \times p \) matrix
\( \Sigma \) and the \( p \times q \) matrix \( \beta \) are unknown. We assume \( N \geq p + q \) and the rank of
\( Z = (z_1, \ldots, z_N) \) is \( q \). Suppose we partition \( \beta = (\beta_1, \beta_2) \) so that \( \beta_1 \) has \( q_1 \) columns.
Then the likelihood ratio criterion for testing the hypothesis

\[
H : \beta_1 = \beta_1^0
\]

where \( \beta_1^0 \) is a given matrix, is in terms of \( U_{p,m,n} \) with \( n = N - q \) and \( m = q_1 \). Many
hypotheses can be put in the form of hypotheses concerning regression coefficients.
For example, $U_{p,m,n}$ can be used for testing the hypothesis that the means of, say, $q$ normal distributions with a common covariance matrix are equal. In this case $n = (N_1 + \cdots + N_q) - 1$ and $m = q - 1$. As another application of the statistic (4.8) let us consider MANOVA. Here we have a set of $p$-dimensional independent and normally distributed random vectors $Y_{ij}$, $i = 1, \ldots, r$, $j = 1, \ldots, c$ with $E(Y_{ij}) = \mu + \lambda_i + \nu_j$ and covariance matrix $\Sigma$. Now $U_{p,m,n}$ has $n = (r - 1)(c - 1)$ and $m = c - 1$. Again, the null distribution of the $\Lambda$-criterion for testing the independence of $q$ subvectors with $p_1, \ldots, p_q$ components of a multivariate normal random vector has the distribution of $\prod_{i=2}^q V_i$, where $V_i$'s are independent each having the distribution of $U_{p_i,\bar{p}_i,N-1-\bar{p}_i}$, where $\bar{p}_i = p_1 + \cdots + p_i - 1$. The moment structure of $U_{p,m,n}$ is available in the literature and it is given in the following theorem:

**Theorem 4.2.1** The $t$-th moment of $U$ for $t > -\frac{1}{2}(n + 1 - p)$ is

$$E(U^t) = \prod_{j=1}^p \frac{\Gamma \left( \frac{1}{2}(n + 1 - j) + t \right) \Gamma \left( \frac{1}{2}(n + m + 1 - j) \right)}{\Gamma \left( \frac{1}{2}(n + 1 - j) \right) \Gamma \left( \frac{1}{2}(n + m + 1 - j) + t \right)}.$$  

(4.9)

We can write (4.9) as

$$E(U^t) = \prod_{j=1}^p E(V_j^t)$$

where $V_j$ has the type-1 beta distribution with the parameters $\left( \frac{1}{2}(n + 1 - j), \frac{1}{2}m \right)$. Hence it follows that the distribution of $U_{p,m,n}$ is that of the product $\prod_{j=1}^p V_j$ where $V_1, \ldots, V_p$ are independent and $V_j$ has the type-1 beta distribution with the parameters $\left( \frac{1}{2}(n + 1 - j), \frac{1}{2}m \right)$.

From Theorem 2.2.1 it follows that when $(x_1, \ldots, x_p)$ has the generalized type-1 Dirichlet density (2.2) then

$$y_1 = \frac{x_1}{x_1 + x_2},$$
$$y_2 = \frac{x_1}{x_1 + x_2 + x_3},$$
$$\vdots$$
$$y_{p-1} = \frac{x_1 + \cdots + x_{p-1}}{x_1 + \cdots + x_p},$$
$$y_p = \frac{x_1 + \cdots + x_p}{x_1 + \cdots + x_p}$$

are independently distributed, and further, $y_j$ has a real type-1 beta density with the parameters $\left( \alpha_1 + \cdots + \alpha_j + \beta_2 + \cdots + \beta_j, \alpha_{j+1} \right)$ for $j = 1, \ldots, p$. Hence we have the
following theorem:

**Theorem 4.2.2** When \((x_1, \ldots, x_p)\) has the generalized type-1 Dirichlet density (2.2) then \(x_1\) is structurally a product of \(p\) independent real type-1 beta random variables and its density can be written in terms of a Meijer’s G-function of the type \(G_{p, p}^{p, 0}(\cdot)\).

**Proof.** From (4.10) by taking the product we see that

\[ x_1 = y_1 y_2 \cdots y_p. \]

Hence we have, for arbitrary \(t\),

\[
E(x_1^t) = \prod_{j=1}^{p} E(y_j^t) = c'_p \prod_{j=1}^{p} \frac{\Gamma(\alpha_1 + \cdots + \alpha_j + \beta_2 + \cdots + \beta_j + t)}{\Gamma(\alpha_1 + \cdots + \alpha_j + \beta_2 + \cdots + \beta_j + t)} \quad (4.11)
\]

where

\[
c'_p = \prod_{j=1}^{p} \frac{\Gamma(\alpha_1 + \cdots + \alpha_{j+1} + \beta_2 + \cdots + \beta_j)}{\Gamma(\alpha_1 + \cdots + \beta_2 + \cdots + \beta_j)}.\]

Treating (4.11) as a Mellin transform of the density of \(x_1\) the density is available by taking the inverse Mellin transform. Denoting the density of \(x_1\) by \(g(x_1)\) we have,

\[
g(x_1) = c'_p x_1^{t-1} \frac{1}{2\pi i} \int_L \frac{\Gamma(\alpha_1 + t)}{\Gamma(\alpha_1 + \alpha_2 + t)} \cdots \times \frac{\Gamma(\alpha_1 + \cdots + \alpha_p + \beta_2 + \cdots + \beta_p + t)}{\Gamma(\alpha_1 + \cdots + \alpha_{p+1} + \beta_2 + \cdots + \beta_p + t)} x_1^{-t} dt = c'_p x_1^{t-1} G_{p, p}^{p, 0} \left[ x_1^{1/2} \alpha_1 + \cdots + \alpha_{p+1} + \beta_2 + \cdots + \beta_p \right] \quad (4.12)
\]

for \(0 < x_1 < 1\) and zero elsewhere. Now let us consider (4.11) and put

\[
\alpha_1 = \frac{n}{2}, \alpha_2 = \alpha_3 = \cdots = \alpha_{p+1} = \frac{m}{2}, \beta_2 = \beta_3 = \cdots = \beta_p = -\left( \frac{m+1}{2} \right). \quad (4.13)
\]

On simplification we have

\[
E(x_1^t) = \prod_{j=1}^{p} \Gamma \left[ \frac{1}{2}(n + 1 - j) + t \right] \Gamma \left[ \frac{1}{2}(n + m + 1 - j) + t \right]
\]

which is the same as (4.9). From the above considerations and since arbitrary moments in this case will determine the density uniquely we can write the following theorem:
Theorem 4.2.3 When \((x_1, \ldots, x_p)\) has the real generalized type-1 Dirichlet density (2.2) with the parameters as given in (4.13), then the distribution of \(U\) in (4.8) and the marginal distribution of \(x_1\) are identical. The density of \(U_{p,m,n}\) is given by

\[
g(x_1) = c_p \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-\cdots-x_{p-1}} \frac{1}{x_1^{n-1} x_2^{m-1} \cdots x_p^{m-1}} (x_1 + x_2)^{-\left(\frac{m+1}{2}\right)} \cdots \times (x_1 + \cdots + x_p)^{-\left(\frac{m+1}{2}\right)} (1 - x_1 - \cdots - x_p)^{m-1} \, dx_2 \cdots dx_2
\]

for \(0 < x_i < 1, i = 1, \ldots, p, 0 < x_1 + \cdots + x_p < 1, n \geq p\) and

\[
c_p = \frac{\prod_{j=1}^{p} \Gamma \left[ \frac{1}{2} (n + m + 1 - j) \right]}{\left[ \Gamma \left( \frac{m}{2} \right) \right]^p \prod_{j=1}^{p} \Gamma \left[ \frac{1}{2} (n + 1 - j) \right]}.
\]

Remark 4.2.1 On performing the integrations in (4.14) we can express \(g(x_1)\) in terms of the following multiple series:

\[
g(x_1) = c_p \frac{\left[ \Gamma \left( \frac{m}{2} \right) \right]^p}{\Gamma \left( \frac{3m}{2} \right)} x_1^{n-1} (1 - x_1)^{m-1} \sum_{r_1=0}^{\infty} \frac{\left( m + 1 \right) r_1 \left( m \right) r_1}{\left( 2m \right) r_1!} (1 - x_1)^{r_1}
\]

\[
\times \sum_{r_2=0}^{\infty} \frac{\left( m + 1 \right) r_2 \left( 2m \right) r_1 + r_2}{\left( 3m \right) r_1 + r_2!} (1 - x_1)^{r_2}
\]

\[
\times \sum_{r_{p-1}=0}^{\infty} \frac{\left( m + 1 \right) r_{p-1} \left( (p-1)m \right) r_1 + \cdots + r_{p-1}}{\left( pm \right) r_1 + \cdots + r_{p-1}!} (1 - x_1)^{r_{p-1}}.
\]

Remark 4.2.2 On replacing the \(\alpha_j\)'s and \(\beta_j\)'s in (4.12) by (4.13) we can also obtain the exact distribution of \(U_{p,m,n}\) in terms of Meijer’s G-function as follows:

\[
g(u) = c'_p \frac{u^{-1} G_{p,0}^{p,0} \left[ u^{a_1, \ldots, a_p} \right]}{b_1, b_2, \ldots, b_p}
\]

where

\[
b_j = \frac{1}{2} (n + 1 - j) \text{ for } j = 1, \ldots, p, \quad a_j = b_j + \frac{m}{2} \text{ for } j = 1, \ldots, p \quad \text{and} \quad c'_p = \prod_{j=1}^{p} \frac{\Gamma(b_j)}{\Gamma(a_j)}.
\]

The likelihood ratio test procedure rejects the null hypothesis \(H_0\) if the computed value of \(U\) is less than \(u_{p,m,n}(\alpha)\), the \(\alpha\) significant point for \(U_{p,m,n}\). The values of \(u_{p,m,n}(\alpha)\) are tabulated for certain values of \(p, m, n\) and \(\alpha\). When \(p\) or \(m\) is small, there are simpler representations in terms of the \(F\)-distribution. When \(p\) is large, the computation of probabilities and quantiles of \(U\)-distribution involves several steps as stated in the last
section of Chapter 5 in Weerahandi (2004). There are other methods using asymptotic
distribution of the likelihood ratio criterion or using the distribution of some functions of
$U_{p,m,n}$ are also available in the literature.

**Remark 4.2.3** Theorem 4.2.3 enables us to express the distribution of $U_{p,m,n}$ in a sim-
ple manageable form and we can directly obtain the exact $p$-values corresponding to a
computed value $u$ of $U_{p,m,n}$. Hence there is no need to resort to simulation or tabulated
critical points prepared using asymptotic results. The exact $p$-value for a computed
value $u$ of $U_{p,m,n}$ can be obtained very easily by evaluating the following with the help
of Mathematica or Maple:

$$P(U_{p,m,n} \leq u) = \int_0^u g(x_1)dx_1,$$

(4.16)

where $g(x_1)$ is given in (4.14).

**4.2.1 Some special cases**

**Case (i) $p=1$**

From Theorem 4.2.1 we see that the distribution of $U_{1,m,n}$ is type-1 beta with density

$$h(u) = \frac{\Gamma \left[ \frac{1}{2}(n + m) \right]}{\Gamma \left( \frac{1}{2}n \right) \Gamma \left( \frac{1}{2}m \right)} u^{\frac{n}{2}-1}(1 - u)^{\frac{m}{2}-1}, \quad 0 < u < 1$$

and zero elsewhere. Since $[(1 - U_{1,m,n})/U_{1,m,n}](n/m)$ follows F-distribution with $m$
and $n$ degrees of freedom we can also use this result in the computation of $p$-values of
$U_{1,m,n}$.

**Case (ii) $p=2$**

From (4.14) it follows that the marginal density of $x_1$ is

$$g(x_1) = c_2 x_1^{\frac{n}{2}-1} \int_{x_2=0}^{1-x_1} x_2^{\frac{m}{2}-1} (x_1 + x_2)^{-\frac{m+1}{2}}(1 - x_1 - x_2)^{\frac{m}{2}-1}dx_2; \quad 0 < x_1 < 1,$$

where

$$c_2 = \frac{\Gamma \left( \frac{n+m}{2} \right) \Gamma \left( \frac{n+m-1}{2} \right)}{\Gamma \left( \frac{m}{2} \right)^2 \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n-1}{2} \right)}.$$

By applying (4.3) we can write $c_2$ as

$$c_2 = \frac{\Gamma(n + m - 1)}{2^m \Gamma \left( \frac{m}{2} \right)^2 \Gamma(n - 1)}.$$
On writing
\[(x_1 + x_2)^{-\frac{m+1}{2}} = [1 - (1 - x_1 - x_2)]^{-\frac{m+1}{2}} \]
and using (4.2) we have
\[(x_1 + x_2)^{-\frac{m+1}{2}} = \sum_{r=0}^{\infty} \frac{(m+1)_r}{r!} (1 - x_1 - x_2)^r. \]

Now
\[
\int_{x_2=0}^{1-x_1} x_2^{m-1} (x_1 + x_2)^{-\frac{m+1}{2}} (1 - x_1 - x_2)^{\frac{m}{2}-1} dx_2
= \sum_{r=0}^{\infty} \frac{(m+1)_r}{r!} \int_{x_2=0}^{1-x_1} x_2^{m-1} (1 - x_1 - x_2)^{\frac{m}{2}+r-1} dx_2
= \sum_{r=0}^{\infty} \frac{(m+1)_r}{r!} (1 - x_1)^{\frac{m}{2}+r-1} \int_{x_2=0}^{1-x_1} \left[ 1 - \frac{x_2}{1-x_1} \right]^{\frac{m}{2}+r-1} dx_2
= \sum_{r=0}^{\infty} \frac{(m+1)_r}{r!} (1 - x_1)^{m+r-1} \int_{y=0}^{1} y^{\frac{m}{2}-1} (1 - y)^{\frac{m}{2}+r-1} dy
= \sum_{r=0}^{\infty} \frac{(m+1)_r}{r!} (1 - x_1)^{m+r-1} \frac{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m}{2} + r\right)}{\Gamma(m + r)}
= \sum_{r=0}^{\infty} \frac{(m+1)_r}{r!} (1 - x_1)^{m+r-1} \frac{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m}{2} + r\right)}{\Gamma(m)(m)_r} \text{ by using (4.1).}

Thus
\[g(x_1) = \frac{\Gamma(n + m - 1)}{2^m \Gamma(n-1) \Gamma(m)} x_1^{\frac{m}{2}-1} (1 - x_1)^{m-1} \sum_{r=0}^{\infty} \frac{(m)_r (m+1)_r}{(m)_r} \frac{(1 - x_1)^r}{r!}; \ 0 < x_1 < 1. \]

Now by using (4.5) it can be written as
\[g(x_1) = \frac{1}{2^m B(n-1, m)} x_1^{\frac{m}{2}-1} (1 - x_1)^{m-1} \ _2F_1\left(\frac{m}{2}, \frac{m+1}{2}; m; 1 - x_1\right) \quad (4.17) \]
for \(0 < x_1 < 1, n \geq 2\) and zero elsewhere. As an immediate consequence we can write the following remarks.

**Remark 4.2.4** For \(0 < x_i < 1, i = 1, 2, 0 < x_1 + x_2 < 1,\)
\[
\int_{x_2=0}^{1-x_1} x_2^{m-1} (x_1 + x_2)^{-\frac{m+1}{2}} (1 - x_1 - x_2)^{\frac{m}{2}-1} dx_2
= \frac{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m}{2}\right)}{\Gamma(m)} (1 - x_1)^{m-1} \ _2F_1\left(\frac{m}{2}, \frac{m+1}{2}; m; 1 - x_1\right).
\]
Remark 4.2.5 Since \( \int_0^1 g(x_1)dx_1 = 1 \), then from (4.17) it follows that

\[
\int_0^1 x^{\alpha-1}(1-x)^{m-1} 2F_1 \left( \frac{m}{2}, \frac{m+1}{2}; m; 1-x \right) dx = 2^m B(n-1, m).
\]

Alternatively, we can obtain the density (4.17) from (4.15) in terms of Meijer’s G-function as follows:

\[
g(u) = c'_2 u^{-1} G^{2,0}_{2,2} \left( u \Big|^{a_1,a_2}_{b_1,b_2} \right)
\]

where

\[
a_1 = \frac{1}{2}(n + m), \quad a_2 = \frac{1}{2}(n + m - 1), \quad b_1 = \frac{1}{2}n, \quad b_2 = \frac{1}{2}(n - 1)
\]

and

\[
c'_2 = \frac{\Gamma \left( \frac{1}{2}(n + m) \right) \Gamma \left( \frac{1}{2}(n + m - 1) \right)}{\Gamma \left( \frac{1}{2}n \right) \Gamma \left( \frac{1}{2}(n - 1) \right)} = \frac{\Gamma(n + m - 1)}{2^m \Gamma(n - 1)}.
\]

Now \( G^{2,0}_{2,2} \left( u \Big|^{a_1,a_2}_{b_1,b_2} \right) \) is of the form \( c'_2 G^{2,0}_{2,2} \left( u \Big|^{\gamma_1+\delta_1-1, \gamma_2+\delta_2-1}_{\gamma_1-1, \gamma_2-1} \right) \) with \( \gamma_1 - 1 = \frac{1}{2}(n - 1) \), \( \gamma_2 - 1 = \frac{1}{2}n \), \( \delta_1 = \delta_2 = \frac{1}{2}m \). Hence we can use the result given in (4.4) and write the density function as

\[
g(u) = \frac{\Gamma(n + m - 1)}{2^m \Gamma(n - 1)} u^{-1} \frac{\Gamma(1-u)^{m-1}}{\Gamma(m)} 2F_1 \left( \frac{m+1}{2}, \frac{m}{2}; m; 1-u \right), 0 < u < 1
\]

\[
= \frac{1}{2^m B(n-1, m)} u^{\frac{\gamma_1}{2}} (1-u)^{m-1} 2F_1 \left( \frac{m}{2}, \frac{m+1}{2}; m; 1-u \right), \quad (4.18)
\]

for \( 0 < u < 1 \) and it is the same as (4.17).

If we use the form \( G^{2,0}_{2,2} \left( u \Big|^{a+\beta-1, a+\beta-1}_{a-1, a-1} \right) \) with \( a = \frac{1}{2}(n + 1) \) and \( \beta = \frac{1}{2}m \) instead of \( G^{2,0}_{2,2} \left( u \Big|^{\gamma_1+\delta_1-1, \gamma_2+\delta_2-1}_{\gamma_1-1, \gamma_2-1} \right) \) we obtain a different form of the density function as given below.

\[
g(u) = c'_2 u^{-1} G^{2,0}_{2,2} \left( u \Big|^{a+\beta-1, a+\beta-1}_{a-1, a-1} \right)
\]

where \( a = \frac{1}{2}(n + 1), \beta = \frac{1}{2}m \) and \( c'_2 = \frac{\Gamma(n+m-1)}{2^m \Gamma(n-1)} \). Now apply (4.6) and obtain \( g(u) \) as

\[
g(u) = c'_2 u^{-1} 2^m B(n-1, m) G^{1,0}_{1,1} \left( u^\frac{1}{2}\big|^{n+m-1}_{n-1} \right).
\]

Again, by using (4.7) we can write \( g(u) \) as

\[
g(u) = c'_2 u^{-1} 2^{m-1} \frac{(u^\frac{1}{2})^{n-1}(1-u^\frac{1}{2})^{m-1}}{\Gamma(m)} , 0 < u < 1.
\]
That is,
\[ g(u) = \frac{1}{2B(n-1, m)}(u^{\frac{1}{2}})^{n-3}(1 - u^{\frac{1}{2}})^{m-1}, \quad 0 < u < 1, \quad (4.19) \]
and zero elsewhere. Hence the density of \( U_{2,m,n} \) is identical with (4.18) and (4.19).

Since \([(1 - \sqrt{U_{2,m,n}})/(\sqrt{U_{2,m,n}}][(n-1)/m]]\) follows F-distribution with \(2m\) and \(2(n-1)\) degrees of freedom we can also use F-distribution for computation of \( p \)-values of \( U_{2,m,n} \).

Since the density function is unique, (4.18) and (4.19) must be equal. Therefore we obtain the following relation which we shall write as a remark.

**Remark 4.2.6**

\[ _2F_1\left(\frac{m}{2}, \frac{m+1}{2}; m; 1-u\right) = 2^{m-1}u^{-\frac{1}{2}}[1 + u^{\frac{1}{2}}]^{-(m-1)} , \quad 0 < u < 1. \]

**Case (iii) \( p=4 \)**

We can write the required density function using (4.15) as
\[ g(u) = c'_4u^{-1}G_{4,4}^{4,0}\left(u^{a_1,a_2,a_3,a_4}_{b_1,b_2,b_3,b_4}\right). \]

Note that
\[ G_{4,4}^{4,0}\left(u^{a_1,a_2,a_3,a_4}_{b_1,b_2,b_3,b_4}\right) = (2\pi i)^{-1}\int_L \phi(s)u^{-s}ds \]

where
\[ \phi(s) = \prod_{j=1}^4 \frac{\Gamma(b_j + s)}{\Gamma(a_j + s)} = \prod_{j=1}^4 \frac{\Gamma\left[\frac{1}{2}(n + 1 - j) + s\right]}{\Gamma\left[\frac{1}{2}(m + m + 1 - j) + s\right]}. \]

Now on combining the two consecutive gammas in the numerator and denominator of \( \phi(s) \) by using (4.3) we obtain
\[ \phi(s) = 2^{2m}\frac{\Gamma(n - 1 + 2s)\Gamma(n - 3 + 2s)}{\Gamma(n + m - 1 + 2s)\Gamma(n + m - 3 + 2s)}. \]

On putting \( 2s = s' \) the above Mellin-Barnes type integral becomes
\[ G_{4,4}^{4,0}\left(u^{a_1,a_2,a_3,a_4}_{b_1,b_2,b_3,b_4}\right) = 2^{2m-1}(2\pi i)^{-1}\int_L \frac{\Gamma(n - 1 + s')\Gamma(n - 3 + s')}{\Gamma(n + m - 1 + s')\Gamma(n + m - 3 + s')}\left(u^{\frac{1}{2}}\right)^{-s'}ds' \]
\[ = 2^{2m-1}G_{2,2}^{2,0}\left(u^{\frac{1}{2}}\left|\begin{array}{c} n+m-1, n+m-3 \\ n-1, n-3 \end{array}\right.\right). \]
On using (4.4) with $\gamma_1 = n - 2, \gamma_2 = n, \delta_1 = \delta_2 = m$ and simplifying $c'_4$ using (4.3) we obtain the final form of the density function as

$$g(u) = \frac{\Gamma(n + m - 1)\Gamma(n + m - 3)}{2\Gamma(n - 1)\Gamma(n - 3)\Gamma(2m)} (u^{\frac{1}{2}})^{n-3}(1 - u^{\frac{1}{2}})^{2m-1} \times 2F_1(m + 2, m; 2m; 1 - u^{\frac{1}{2}})$$

(4.20)

for $0 < u < 1, n \geq 4$ and zero elsewhere. Hence the density of $U_{4,m,n}$ is given by (4.20). Since (4.20) is a density we obtain the following relation:

**Remark 4.2.7**

$$\int_0^1 (u^{\frac{1}{2}})^{n-3}(1 - u^{\frac{1}{2}})^{2m-1} 2F_1(m + 2, m; 2m; 1 - u^{\frac{1}{2}})du = \frac{2\Gamma(n - 1)\Gamma(n - 3)\Gamma(2m)}{\Gamma(n + m - 1)\Gamma(n + m - 3)}.$$  

**Remark 4.2.8** In some cases the deduction of density function using (4.15) may be quite tedious. For example, the case when $p = 3$ requires evaluation of residues at poles of orders one and two. Hence in such cases we may end up with a density involving psi and gamma functions. When $p > 3$ it can involve zeta functions also. But for practical purposes we need only to compute the values of cumulative distribution function of $U_{p,m,n}$ and this can be done very easily by using (4.16) for any $p$.

### 4.2.2 Computations

When $p = 2$ we have seen that the density function of $U_{2,m,n}$ is of the form (4.19) and hence we can replace $g(x_1)$ in (4.16) by (4.19). Then putting $u^{\frac{1}{2}} = v$ we can write (4.16) as

$$P(U_{2,m,n} \leq u) = \frac{1}{B(n - 1, m)} \int_0^{\sqrt{u}} v^{(n-1)-1}(1 - v)^{m-1}dv.$$  

(4.21)

Therefore, the computation of the above probability is the same as the evaluation of cumulative distribution function of a type-1 beta random variable with the parameters $(n - 1, m)$ and it can be done very easily using MS Excel. Tabulated values are also available from incomplete beta tables.

When $p \geq 3$, the computation of $p$-value for an observed value $u$ of $U_{p,m,n}$ can be performed with widely used software packages such as Mathematica and Maple. For example, let us compute (4.16) for an observed value $u = 0.121$ of $U_{4,6,17}$ using
gives the output value 0.0503688. Note that 0.121 is an entry in the table of lower critical values of likelihood ratio criterion $U_{p,m,n}$ corresponding to $\alpha = 0.05$, $p = 4$, $m = 6$ and $n = 17$ (Table, B4, Rencher (1998)). We can also evaluate (4.16) for the above mentioned case by using (4.20) instead of (4.14).

### Table 4.1: Lower Critical Values of $U_{p,m,n}$ for $\alpha = 0.05$ and $p = 2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.025000</td>
<td>0.064113</td>
<td>0.092879</td>
<td>0.137500</td>
<td>0.186989</td>
<td>0.244666</td>
<td>0.311243</td>
<td>0.387820</td>
<td>0.475437</td>
</tr>
<tr>
<td>3</td>
<td>0.050000</td>
<td>0.083325</td>
<td>0.118752</td>
<td>0.158125</td>
<td>0.204479</td>
<td>0.259466</td>
<td>0.325437</td>
<td>0.392500</td>
<td>0.461011</td>
</tr>
<tr>
<td>4</td>
<td>0.065789</td>
<td>0.094466</td>
<td>0.128281</td>
<td>0.170352</td>
<td>0.218479</td>
<td>0.272750</td>
<td>0.334000</td>
<td>0.399091</td>
<td>0.466589</td>
</tr>
<tr>
<td>5</td>
<td>0.069703</td>
<td>0.090536</td>
<td>0.125875</td>
<td>0.170352</td>
<td>0.217281</td>
<td>0.269237</td>
<td>0.329281</td>
<td>0.391455</td>
<td>0.459281</td>
</tr>
<tr>
<td>6</td>
<td>0.067287</td>
<td>0.088408</td>
<td>0.127462</td>
<td>0.169389</td>
<td>0.217281</td>
<td>0.268408</td>
<td>0.328056</td>
<td>0.389479</td>
<td>0.455437</td>
</tr>
<tr>
<td>7</td>
<td>0.069200</td>
<td>0.088712</td>
<td>0.125000</td>
<td>0.168408</td>
<td>0.217281</td>
<td>0.268408</td>
<td>0.328056</td>
<td>0.389479</td>
<td>0.455437</td>
</tr>
<tr>
<td>8</td>
<td>0.070099</td>
<td>0.085003</td>
<td>0.125938</td>
<td>0.168408</td>
<td>0.217281</td>
<td>0.268408</td>
<td>0.328056</td>
<td>0.389479</td>
<td>0.455437</td>
</tr>
<tr>
<td>9</td>
<td>0.070390</td>
<td>0.086136</td>
<td>0.127096</td>
<td>0.169389</td>
<td>0.217281</td>
<td>0.268408</td>
<td>0.328056</td>
<td>0.389479</td>
<td>0.455437</td>
</tr>
</tbody>
</table>

*The corresponding table value given in Rencher (1998) differs slightly from the exact value shown in this table.*
For $\alpha = 0.05$ and $p = 2$, the lower critical values computed from the representation (4.21) by using MS Excel is given in Table 4.1. Since the percentage points for $\alpha = 0.05$ and $p = 1(1)8$ are available in Table B4 of Rencher (1998) and the values are found to be generally correct to the decimal places they have given we are not reproducing the entire table values here.

**Remark 4.2.9** Since the distributions of $U_{p,m,n}$ and $U_{m,p,n+m-p}$ are same we can use this fact in computations of the $p$-value especially when $m < p$.

### 4.3 Distribution of $\Lambda$-Criterion for Sphericity Test

In most of the univariate statistical analyses usually we assume that the random variables under consideration are independent and have a common variance. Here we consider a test of these assumptions based on repeated sets of observations.

Suppose that the $p \times 1$ vector $X$ is distributed according to $N_p(\mu, \Sigma)$, $\Sigma > 0$. Consider the problem of testing the hypothesis

$$H_0 : \Sigma = \gamma I, \gamma > 0$$

based on a random sample $X_1, \ldots, X_n$ from $N_p(\mu, \Sigma)$. Note that the density function of $X$ is constant on ellipsoids

$$(x - \mu)'\Sigma^{-1}(x - \mu) = c$$

for every positive value of $c$ in a $p$-dimensional Euclidean space. When the $p$-components of $X$ are independent and have the same variance or when $\Sigma = \gamma I$ the ellipsoids become spheres. The density $n(x \mid \mu, \gamma I)$ is known as a spherical normal density and the test corresponding to the hypothesis (4.22) is called sphericity test. Mauchly (1940) derived the likelihood ratio criterion for the sphericity test and it is given by

$$\Lambda = \frac{L(\gamma I, \hat{\mu})}{L(\hat{\Sigma}, \hat{\mu})} = \left[ \frac{|\frac{1}{n} A|}{\left( \frac{1}{np} \text{tr} A \right)^{\frac{1}{2}}} \right]^{\frac{1}{2}} ,$$

where $\hat{\mu} = \bar{x}, \quad \hat{\gamma} = \frac{1}{np} \text{tr} A, \quad \hat{\Sigma} = \frac{1}{n} A$ and $A = \Sigma^n_{\alpha=1} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'$. Consider

$$\tilde{\Lambda} = \Lambda^{\frac{1}{2}} = \left( \frac{1}{\text{tr} A} \right)^{\frac{1}{2}} = \left( \frac{\prod_{i=1}^{p} \lambda_i^\frac{1}{2} \left( \frac{1}{p} \sum_{i=1}^{p} \lambda_i \right)^{\frac{1}{2}}}{\left( \frac{1}{p} \sum_{i=1}^{p} \lambda_i \right)^{\frac{1}{2}}} \right)^{\frac{1}{2}} ,$$

(4.23)

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $A$. Thus $\tilde{\Lambda}$ compares the geometric and arith-
metic means of the eigenvalues of $A$. If they coincide then $A$ has the structure as in the null hypothesis (4.22). The exact distribution of $\hat{\Lambda}$ for some values of $p$ has been investigated by some authors, namely, Consul (1967), Nagarsenker and Pillai (1973), Mathai (1977), Korin (1968) among others. It can be shown that the exact null distribution of $\hat{\Lambda}$ can be characterized as a product of $(p - 1)$ independent type-1 beta variables. Let us present this result as a theorem and the proof of which can be seen from Bilodeau and Brenner (1999).

**Theorem 4.3.1** The $h$-th moment of $\hat{\Lambda}$ for $h > -\frac{1}{2}(n - p)$ is

$$E(\hat{\Lambda}^h) = \prod_{i=1}^{p-1} \frac{\Gamma[\frac{1}{2}(n-1-i) + h] \Gamma[\frac{1}{2}(n-1) + \frac{i}{p}]}{\Gamma[\frac{1}{2}(n-1-i)] \Gamma[\frac{1}{2}(n-1) + \frac{i}{p} + h]}$$

That is, $\hat{\Lambda} \sim \prod_{i=1}^{p-1}$ type-1 beta $[\frac{1}{2}(n-1-i), i(\frac{1}{2} + \frac{1}{p})]$.

We shall denote the statistic $\hat{\Lambda}$ in (4.24) by $\hat{\Lambda}_n$. The easiest method available for computation of exact $p$-value corresponding to an observed value $\bar{\lambda}$ of $\hat{\Lambda}_n$ is Monte Carlo simulation. The method involves the following steps.

(i) Generate a large number, say $N = 10000$, random numbers from each of the independent $(p - 1)$ type-1 beta random variables $B_1, \ldots, B_{p-1}$.

(ii) Compute $N$ random numbers from the $\hat{\Lambda}_n$ random variable using the formula $\hat{\Lambda}_n = B_1B_2 \cdots B_{p-1}$.

(iii) Estimate the probability $Pr(\hat{\Lambda}_n \leq \bar{\lambda})$ by the fraction $\hat{\Lambda}_n$ random numbers that are less than or equal to $\bar{\lambda}$.

From the general theory of likelihood ratio test it follows that the asymptotic null distribution of

$$-2 \ln \Lambda \overset{d}{\to} \chi^2_f, \quad f = \frac{1}{2}p(p + 1) - 1$$

and this result is used in practical situations.

The above procedure can be applied for testing a general form of the hypothesis given in (4.22). Suppose that the hypothesis to be tested is $\psi = \gamma\psi_0$ where $\gamma > 0$ and $\psi_0$ is a given matrix, based on a random sample $Y_1, \ldots, Y_n$ from $N_p(\nu, \psi)$. Let $C$ be a matrix such that $C\psi_0C' = I$. Now using the transformation $X^*_\alpha = CY_\alpha$ for $\alpha = 1, \ldots, n$ the above problem is transformed into the problem of testing hypothesis $\Sigma^* = \gamma I$ based on observations $X_1^*, \ldots, X_n^*$ from $N_p(\mu^*, \Sigma^*)$ where $\mu^* = C\nu$ and $\Sigma^* = C\psi C'$. 

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Theorem 4.3.2 When $(x_1, \ldots, x_{p-1})$ has a generalized type-1 Dirichlet model of (2.2) then $x_1$ is structurally a product of $p - 1$ independent real type-1 beta random variables and the $h$-th moment of $x_1$ for $h > -(\alpha_1 + \cdots + \alpha_{p-1} + \beta_2 + \cdots + \beta_{p-1})$ is

$$E(x_1^h) = c'_{p-1} \prod_{i=1}^{p-1} \frac{\Gamma(\alpha_1 + \cdots + \alpha_i + \beta_2 + \cdots + \beta_i + h)}{\Gamma(\alpha_1 + \cdots + \alpha_i + \beta_2 + \cdots + \beta_i + h)}$$

(4.25)

where

$$c'_{p-1} = \prod_{i=1}^{p-1} \frac{\Gamma(\alpha_1 + \cdots + \alpha_i + \beta_2 + \cdots + \beta_i)}{\Gamma(\alpha_1 + \cdots + \alpha_i + \beta_2 + \cdots + \beta_i)}.$$

The result follows from Theorem 4.2.2 by considering a generalized type-1 Dirichlet model in $p - 1$ variables.

Remark 4.3.1 On treating (4.25) as a Mellin transform of the density of $x_1$ we can evaluate the density function of $x_1$ in terms of a Meijer’s G-function of the type $G_{p-1,0}^{p-1,p-1}(\cdot)$.

Denoting the density of $x_1$ by $g(x_1)$ we have

$$g(x_1) = c'_{p-1}x_1^{-1}G_{p-1,0}^{p-1,p-1}\left[x_1^{\alpha_1+\alpha_2+\cdots+\alpha_{p-1}+\beta_2+\cdots+\beta_{p-1}}\mid x_1^{\alpha_1,\alpha_2,\ldots,\alpha_{p-1},\beta_2,\ldots,\beta_{p-1}}\right]$$

(4.26)

for $0 < x_1 < 1$ and zero elsewhere.

Now let us consider (4.25) and put

$$\alpha_1 = \frac{n - 2}{2}, \quad \alpha_i = (i - 1)\left(\frac{1}{2} + \frac{1}{p}\right) \quad \text{for } i = 2, 3, \ldots, p,$$

$$\beta_i = -\left(\frac{i}{2} + \frac{i - 1}{p}\right) \quad \text{for } i = 2, 3, \ldots, p - 1. \quad (4.27)$$

On simplification we have

$$E(x_1^h) = \prod_{i=1}^{p-1} \frac{\Gamma\left[\frac{1}{2}(n - 1 - i) + h\right]}{\Gamma\left[\frac{1}{2}(n - 1 - i)\right]} \frac{\Gamma\left[\frac{1}{2}(n - 1) + \frac{i}{p} + h\right]}{\Gamma\left[\frac{1}{2}(n - 1) + \frac{i}{p}\right]},$$

which is the same as (4.24). From the above considerations and since arbitrary moments in this case will determine the density uniquely we can write the following theorem:

Theorem 4.3.3 When $(x_1, \ldots, x_{p-1})$ has the real generalized type-1 Dirichlet density (2.2) with the parameters as given in (4.27), then the distribution of $\tilde{\Lambda}$ in (4.23) and the marginal distribution of $x_1$ are identical. Hence the density of $\tilde{\Lambda}_n$ is given by the density
of $x_1$, namely,

$$g(x_1) = c_{p-1} \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-\cdots-x_{p-2}} \frac{n-2-1}{x_1^{\frac{1}{2}+\frac{1}{p}} - 1} \times \left( \frac{2}{x_3^{\frac{3}{2}+\frac{1}{p}} - 1} \cdots x_{p-1}^{\frac{p-1}{2}+\frac{1}{p}} \right) \times (x_1 + x_2 + \cdots) \left( x_1 + \cdots + x_{p-1} \right) \left( x_1 + \cdots + x_{p-1} \right) \cdots x_1^{\frac{2}{2}+\frac{1}{p}} \cdots x_{p-1}^{\frac{p-1}{2}+\frac{1}{p}} \cdots x_2^{\frac{1}{2}+\frac{1}{p}} d(x_{p-1} \cdots dx_2) \quad (4.28)$$

for $0 < x_i < 1$, $i = 1, \ldots, p-1$, $0 < x_1 + \cdots + x_{p-1} < 1$, $n > p$ and

$$c_{p-1} = \prod_{i=1}^{p-1} \frac{\Gamma \left( \frac{n}{2} \frac{1}{2} + \frac{1}{p} \right)}{\Gamma \left[ i \left( \frac{1}{2} + \frac{1}{p} \right) \right] \Gamma \left[ \frac{1}{2} (n - i - 1) \right]}.$$

**Remark 4.3.2** From the literature it is known that the exact $p$-values of $\tilde{\Lambda}_n$ are computed either based on simulation or evaluating some approximate expressions involving special functions. In most of the practical situations the asymptotic distribution of the likelihood ratio criterion is used. In view of Theorem 4.3.3 we can directly obtain the exact $p$-value corresponding to an observed value $\tilde{\lambda}$ of $\tilde{\Lambda}_n$. Hence there is no need to depend on simulation or asymptotic distribution of the criterion. The exact $p$-value for a computed value $\tilde{\lambda}$ of $\tilde{\Lambda}_n$ can be obtained very easily by evaluating the following:

$$P(\tilde{\Lambda}_n \leq \tilde{\lambda}) = \int_0^{\tilde{\lambda}} g(x_1) dx_1, \quad (4.29)$$

where $g(x_1)$ is given in (4.28).

### 4.3.1 Special case when $p=3$

When $p = 2$ we see that the distribution of $\tilde{\Lambda}_n$ is type-1 beta with the parameters $(\frac{n-2}{2}, 1)$. Now let us derive the exact density function of $\tilde{\Lambda}_n$ when $p = 3$.

From (4.28) we can write the density of $\tilde{\Lambda}_n$ when $p = 3$ as

$$g(x_1) = c_2 x_1^{\frac{n-4}{2}} \int_{x_2=0}^{1-x_1} x_2^{-\frac{1}{3}} (x_1 + x_2)^{-\frac{4}{3}} (1 - x_1 - x_2)^{\frac{2}{3}} dx_2; \quad 0 < x_1 < 1,$$

where

$$c_2 = \frac{\Gamma \left( \frac{n-1}{2} + \frac{1}{3} \right) \Gamma \left( \frac{n-1}{2} + \frac{2}{3} \right)}{\Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{3} \right) \Gamma \left( \frac{n-2}{2} \right) \Gamma \left( \frac{n-3}{2} \right)}.$$
On writing

\[(x_1 + x_2)^{-\frac{4}{3}} = [1 - (1 - x_1 - x_2)]^{-\frac{4}{3}}\]

and using (4.2) we have

\[(x_1 + x_2)^{-\frac{4}{3}} = \sum_{r=0}^{\infty} \frac{(\frac{4}{3})_r}{r!} (1 - x_1 - x_2)^r.\]

Now

\[
\int_{x_2=0}^{1-x_1} x_2^{-\frac{1}{6}} (x_1 + x_2)^{-\frac{4}{3}} (1 - x_1 - x_2)^{\frac{2}{3}} dx_2
\]

\[
= \sum_{r=0}^{\infty} \frac{(\frac{4}{3})_r}{r!} \int_{x_2=0}^{1-x_1} x_2^{-\frac{1}{6}} (1 - x_1 - x_2)^{\frac{2}{3}+r} dx_2
\]

\[
= \sum_{r=0}^{\infty} \frac{(\frac{4}{3})_r}{r!} (1 - x_1)^{\frac{2}{3}+r} \int_{y=0}^{1} y^{\frac{5}{3}-1} (1 - y)^{\frac{2}{3}+r-1} dy
\]

\[
= \sum_{r=0}^{\infty} \frac{(\frac{4}{3})_r}{r!} (1 - x_1)^{\frac{2}{3}+r} \frac{\Gamma(\frac{5}{6})\Gamma(\frac{5}{3}+r)}{\Gamma(\frac{5}{2}+r)} \text{ by using (4.1).}
\]

\[
= \frac{\Gamma(\frac{5}{6})\Gamma(\frac{5}{3})}{\Gamma(\frac{5}{2})} (1 - x_1)^{\frac{2}{3}} \sum_{r=0}^{\infty} \frac{(\frac{4}{3})_r (\frac{5}{3})_r (1 - x_1)^r}{r!}
\]

\[
= \frac{\Gamma(\frac{5}{6})\Gamma(\frac{5}{3})}{\Gamma(\frac{5}{2})} (1 - x_1)^{\frac{2}{3}} \text{ }_2F_1 \left( \frac{4}{3}, \frac{5}{3}; \frac{5}{2}; 1 - x_1 \right) \text{ by using (4.5).}
\]

Thus

\[g(x_1) = \frac{\Gamma \left( \frac{n-1}{2} + \frac{1}{3} \right) \Gamma \left( \frac{n-1}{2} + \frac{2}{3} \right)}{\Gamma \left( \frac{n-2}{2} \right) \Gamma \left( \frac{n-3}{2} \right) \Gamma(\frac{5}{2})} x_1^{\frac{n-4}{2}} (1 - x_1)^{\frac{2}{3}} \text{ }_2F_1 \left( \frac{4}{3}, \frac{5}{3}; \frac{5}{2}; 1 - x_1 \right) \text{ for } 0 < x_1 < 1, n > 3 \text{ and zero elsewhere.}
\]

Alternatively, the above density function can be obtained from (4.26) in terms of Meijer’s G-function as follows:

\[g(z) = c_2 z^{-1} G^{2,0}_{2,2} \left( z \middle| a_1, a_2 \right) \]
where

\[ a_1 = \alpha_1 + \alpha_2 = \frac{n}{2} + \frac{1}{3}, \quad a_2 = \alpha_1 + \alpha_2 + \alpha_3 + \beta_2 = \frac{n}{2} + \frac{2}{3} \]

\[ b_1 = \frac{1}{2}(n - 2), \quad b_2 = \alpha_1 + \alpha_2 + \beta_2 = \frac{1}{2}(n - 3) \]

and

\[ c'_j = \prod_{j=1}^{2} \frac{\Gamma(a_j)}{\Gamma(b_j)}. \]

Now \( G_{2,2}^{2,0} \left[ z^{a_1, a_2} \right] \) in (4.31) is of the form \( G_{2,2}^{2,0} \left[ z^{\gamma_1 + \delta_1 - 1, \gamma_2 + \delta_2 - 1} \right] \) where

\[ \gamma_1 - 1 = \frac{1}{2}(n - 3), \quad \gamma_2 - 1 = \frac{1}{2}(n - 2), \quad \delta_1 = \frac{4}{3} \text{ and } \delta_2 = \frac{7}{6}. \]

By using (4.4) we can write (4.31) as

\[
g(z) = \frac{\Gamma \left( \frac{n-1}{2} + \frac{1}{3} \right) \Gamma \left( \frac{n-1}{2} + \frac{2}{3} \right)}{\Gamma \left( \frac{n-2}{2} \right) \Gamma \left( \frac{n-3}{2} \right)} z^{-1} \frac{z^{n-2} (1 - z)^{\frac{3}{2}}}{\Gamma \left( \frac{5}{2} \right)} \text{ _2F_1} \left( \frac{5}{3}, \frac{4}{3}, \frac{5}{2}; 1 - z \right) \]

\[
= \frac{\Gamma \left( \frac{n-1}{2} + \frac{1}{3} \right) \Gamma \left( \frac{n-1}{2} + \frac{2}{3} \right)}{\Gamma \left( \frac{n-2}{2} \right) \Gamma \left( \frac{n-3}{2} \right) \Gamma \left( \frac{5}{2} \right)} z^{n-4} (1 - z)^{\frac{3}{2}} \text{ _2F_1} \left( \frac{5}{3}, \frac{4}{3}, \frac{5}{2}; 1 - z \right)
\]

for \( 0 < z < 1, \ n > 3 \) and zero elsewhere, which is the same as (4.30).

### 4.3.2 Computations

When \( p = 2 \), \( \tilde{\Lambda}_n \) has a type-1 beta distribution with parameters \( \left( \frac{n-2}{2}, 1 \right) \). In this case the computation of (4.29) is the same as the evaluation of an incomplete beta integral.

When \( p \geq 3 \), the \( p \)-values can be computed using Mathematica. As an illustration, let us evaluate the \( p \)-value of an observed value \( \tilde{\lambda} = 0.2133 \) of \( \tilde{\Lambda} \) when \( n = 10 \) and \( p = 3 \). The Mathematica input

\[
n := 10 \\
p := 3 \\
c = N[\prod_{i=1}^{p-1} \Gamma(n-1)/2 + i/p) / (\Gamma(1/2 + 1/p) \Gamma((n-1)/2))] \\
d = N[\int_0^{0.2133} \int_0^{1-x} x^{(n-4)/2} y^{(1/2+1/p)-1} (y-x)^{-2/(2+1/p)} (1-x-y)^{(p-1)(1/2+1/p)-1} dy dx] \\
e = c * d
\]

gives the output value 0.0372024.

For \( \alpha = 0.05 \) and \( \alpha = 0.01 \), and \( p = 3 \) the percentage points computed from the representation given in (4.28) by using Mathematica is given in Table 4.2. Since
the percentage points for $\alpha = 0.05$ and $\alpha = 0.01$, and $p = 4(1)10$ are available in Nagarsenker and Pillai (1973) and their results are found to be generally correct to the decimal places that they have given we are not reproducing the entire table values here. A comparison of our values with that of Nagarsenker and Pillai (1973) for $\alpha = 0.05$ and $p = 4$ is also made in Table 4.2. Our values are correct up to five decimal places. For the case $p = 2$ incomplete beta tables are readily available or the $p$-value can be easily computed using MS Excel.

**Table 4.2: Percentage points of $\hat{\Lambda}_n$**

<table>
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<tr>
<th>n</th>
<th>$p=3$</th>
<th>$p=4$</th>
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<tr>
<td></td>
<td>$\alpha = 0.05$</td>
<td>$\alpha = 0.01$</td>
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<tr>
<td>4</td>
<td>0.0040420</td>
<td>0.00415770</td>
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<td>5</td>
<td>0.012679</td>
<td>0.0123669</td>
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<td>6</td>
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<td>7</td>
<td>0.090921</td>
<td>0.037466</td>
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<tr>
<td>8</td>
<td>0.14026</td>
<td>0.068152</td>
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<tr>
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<td>0.23564</td>
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<td>11</td>
<td>0.27876</td>
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<td>12</td>
<td>0.31836</td>
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<td>13</td>
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<td>0.24309</td>
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<td>14</td>
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</table>

* deviates considerably from our value and that may be due to typographical error

**Note:**
The author prepared papers [14] and [15] using the materials given in this chapter.
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