CHAPTER III

THEORY AND METHODOLOGY
3.1 The public as well as the Governments have become increasingly cautious about environmental problems. While privately owned firms are most likely to be efficient with regard to input and output productivity, there are several reasons why they would not be as efficient from an environment standpoint (Tyteca, 1996). To address environmental pollution and to control it regulatory instruments have been developed and implemented. To gauge the impact of regularity measures on environment, environmental performance indicators are to be constructed which give useful information to the decision makers to identify the leaders and the followers. Indicators can serve to indicate the potential for pollution prevention in a production unit.

Among several methods to measure environmental efficiency, the non-parametric method based upon the Data Envelopment Analysis (DEA) is prominent due to its merits.

1. It imposes standardization on efficiency scores, since each score lies between 0 and 1
2. The derivation of efficiency scores or indicators can be obtained solving linear or non-linear programming problems
3. There is no need to assign weights for desirable and undesirable outputs, rather these weights are self generated, and
4. The explicit reference to the best practice.

To measure environmental efficiency Fare et al. (1996) divide the outputs of a production unit into desirable and undesirable. Further, the inputs and good outputs are strongly disposable which means costlessly disposed off, while undesirable outputs are weakly disposable, which means a raise in bad outputs leads to either reduction in good outputs or increase in inputs.

\[ x \in \mathbb{R}^n_+ , \ u \in \mathbb{R}^r_+ \ and \ w \in \mathbb{R}^k_+ \]

are input, good output and bad output respectively. The production possibility set may be expressed as,
\[ S = \{(x, u, w) : x \text{ can produce } u \text{ and } w\} \]

(i). We assume inputs are freely disposable.

\[ x' \geq x, \quad (x, u, w) \in S \Rightarrow (x', u, w) \in S \]

(ii). We also assume desirable outputs are freely disposable

\[ u' \leq u, \quad (x, u, w) \in S \Rightarrow (x, u', w) \in S \]

(iii). Undesirable outputs are weakly disposable

\[ (x, u, w) \in S \Rightarrow (x, 0, 0, w) \in S \quad \text{where } 0 \leq \theta \leq 1 \]

To understand disposability of desirable and undesirable outputs consider the output level set \( P(x) \).

\[ P(x) = \{(u, w) : (x, u, w) \in S\} \]

\[ (\bar{w}, \bar{u}) \in P(x), \quad u \leq \bar{u} \Rightarrow (\bar{w}, u) \in P(x) \]

Desirable output is freely disposed off. \( (\bar{w}, \bar{u}) \in P(x), \quad w \leq \bar{w} \Rightarrow (w, \bar{u}) \in P(x) \)

The undesirable output \( w \) attains zero provided that the desirable output \( u \) also attains zero.
The piecewise linear technology may be expressed as follows:

\[ P(x) = \left\{ (u, w) : \sum_{j=1}^{s} \lambda_j x_{ij} \leq x_i, \sum_{r=1}^{s} \lambda_j u_{ir} \geq u_i, \sum_{r=1}^{s} \lambda_j w_{ij} = w_i, \lambda_j \geq 0 \right\} \tag{3.1.1} \]

\[ i = 1, 2, \ldots, m; \; r = 1, 2, \ldots, s; \; k = 1, 2, \ldots, q \]

3.2 R.W. Shephard’s input distance function may be expressed as,

\[ [D_i(u, w, x)]^{-1} = \text{Min} \lambda \]

subject to

\[ \sum_{j=1}^{s} \lambda_j x_{ij} \leq \lambda x_{o} \]

\[ \sum_{r=1}^{s} \lambda_j u_{ir} \geq u_{o} \]

\[ \sum_{r=1}^{s} \lambda_j w_{ij} = w_{o} \]

\[ \lambda_j \geq 0 \tag{3.2.1} \]

The input distance function \( D_i(u, w, x) \) gives potential reduction in inputs.

\[ D_i(u, w, x) \geq 1 \]

\( D_i(u, w, x) = 1 \Rightarrow \) Input reduction is not possible and the production unit is pure technical efficient.

The input distance function is linear homogenous in inputs.

\[ [D_i(u, w, x)]^{-1} = \text{Min} \left\{ \lambda : \sum_{j=1}^{s} \lambda_j x_{ij} \leq \lambda x_{o}, \sum_{r=1}^{s} \lambda_j u_{ir} \leq u_{o}, \sum_{r=1}^{s} \lambda_j w_{ij} = w_{o} \right\} \]

\[ [D_i(u, w, \theta x)]^{-1} = \text{Min} \left\{ \lambda : \sum_{j=1}^{s} \lambda_j x_{ij} \leq \lambda \theta x_{o}, \sum_{r=1}^{s} \lambda_j u_{ir} \leq u_{o}, \sum_{r=1}^{s} \lambda_j w_{ij} = w_{o} \right\} \]

\[ = \frac{1}{\theta} \text{Min} \left\{ \delta : \sum_{j=1}^{s} \delta_j x_{ij} \leq \delta x_{o}, \sum_{r=1}^{s} \delta_j u_{ir} \leq u_{o}, \sum_{r=1}^{s} \delta_j w_{ij} = w_{o} \right\} \]

\[ [D_i(u, w, \theta x)]^{-1} = \frac{1}{\theta} [D_i(u, w, x)]^{-1} \]

\[ [D_i(u, w, \theta x)] = \theta D_i(u, w, x) \]
The input distance function is linear homogenous in input vector. If production of undesirable inputs has neutral effect on resource use, that is it impacts each input to the same degree then the input distance function can be expressed of the form,

\[
[D_i(u, w, x)] = \tilde{D}_i(u, \tilde{R}(w)x) = \tilde{R}(w)\tilde{D}_i(u, x)
\]

The expression \(R(w)\) measures environmental efficiency. If we write,

\[
[D_i(u, w, x)]^{-1} = \lambda(\text{PTE})
\]

\[
[\tilde{D}_i(u, x)]^{-1} = \lambda(\text{OTE})
\]

Also we have,

\[
[D_i(u, w, x)]^{-1} = [\tilde{R}(w)]^{-1} [\tilde{D}_i(u, x)]^{-1}
\]

\[
\tilde{R}(w) = \frac{[\tilde{D}_i(u, x)]^{-1}}{[D_i(u, w, x)]^{-1}}
\]

\[
\lambda(\text{EE}) = \frac{\lambda(\text{OTE})}{\lambda(\text{PTE})}
\]

where \(\lambda(\text{EE})\) measures environmental efficiency \(\lambda(\text{OTE}), \lambda(\text{PTE})\) and \(\lambda(\text{EE})\) are Farrell's overall technical efficiency, pure technical efficiency and environmental efficiency.

We have, \(0 \leq \lambda(\text{OTE}), \lambda(\text{PTE}), \lambda(\text{EE}) \leq 1\). \(\lambda(\text{OTE})\) is Farrell's overall technical efficiency measure.

\[
\lambda(\text{OTE}) = \text{Min } \lambda
\]

subject to \(\sum \lambda_i x_i \leq \lambda x_a, \ i = 1, 2, \ldots, m\) \hspace{1cm} (3.2.2)

\(\sum \lambda_i u_r \geq u_a, \ r = 1, 2, \ldots, s\)

\(\lambda_i \geq 0\)
Since problem (3.2.1) is more constrained than problem (3.2.2) the optimal solution of problem (3.2.1) is a feasible solution of problem (3.2.2). Consequently, we have the following inequality:

\[ \lambda(\text{OTE}) \leq \lambda(\text{PTE}) \]  \hspace{1cm} (3.2.3)

3.3 UNDESIRABLE OUTPUTS – DIRECTIONAL DISTANCE FUNCTIONS:

The concept of directional distance function was introduced by Chamber et al. (1998). The technology set \( S \) is constituted by efficient and inefficient input and output vectors. An input vector \( x \) and output vector \( u \) are said to be efficient if and only if,

\[ (x, u) \in S, \quad x' \geq x, \quad u' \leq u \quad \text{and} \quad (x, u) \neq (x', u') \Rightarrow (x', u') \in S. \]

The directional distance function may be defined as,

\[ \tilde{D}_s(x, u; g, g_s) = \max \{ \theta : (x - \theta g, u + \theta g_s) \in S \} \]

From the above definition it follows that the directional distance function enquires for output augmentation and input reduction simultaneously.

![Figure (3.3.1)](image)

The straight line that starts from the origin is a production frontier that admits constant returns to scale. The directional distance function \( \tilde{D}_s = \theta' \) reduces input
from \( x \) to \( (x - \theta g_x) \) and expands output from \( u \) to \( (u + \theta g_u) \). The simultaneous input contraction and output expansion have taken place in the direction of the vector \((g_x, g_u)\).

To find directional efficiency we solve the following linear programming problem:

\[
\tilde{D}_{s(c)} = \text{Max} \theta \\
\text{subject to } \sum_{j=1}^{n} \lambda_j x_{ij} \leq \lambda x_{i0} - \theta g_x, \ i = 1, 2, \ldots, m \\
\sum_{j=1}^{n} \lambda_j u_{ij} \geq u_{i0} + \theta g_u, \ r = 1, 2, \ldots, s \\
\lambda_j \geq 0
\]

where \( \tilde{D}_{s(c)} = D_{s(c)} (x_0, u_0; g_x, g_u) \).

\( s(c) \) production possibility set that admits constant returns to scale.

Problem (3.3.1) evaluates directional efficiency score under constant returns to scale. However, if returns to scale are variable we solve the following problem to assess directional efficiency score.

\[
\tilde{D}_{s(v)} = \text{Max} \theta \\
\text{subject to } \sum_{j=1}^{n} \lambda_j x_{ij} \leq \lambda x_{i0} - \theta g_x, \ i = 1, 2, \ldots, m \\
\sum_{j=1}^{n} \lambda_j u_{ij} \geq u_{i0} + \theta g_u, \ r = 1, 2, \ldots, s \quad \text{(3.3.2)}
\]

\[
\sum_{j=1}^{n} \lambda_j = 1 \\
\lambda_j \geq 0
\]

where \( s(v) \) refers to a production possibility set that admits variable returns to scale.
Fare and Grosskopf (2004) introduced environmental output set as follows:

(i). Let \( u \) be desirable and \( w \) be undesirable output vectors. If outputs are weakly disposable, \((u, w) \in \mathcal{P}(x)\) and \(0 \leq \theta \leq 1\) implies \((\theta u, \theta w) \in \mathcal{P}(x)\).

(ii). Desirable and undesirable outputs are null joint, which means that \((u, w) \in \mathcal{P}(x)\); \(w = 0\) implies that \(u = 0\).

Weak disposability of outputs require at least one of the outputs is bad. If \( u \) is electricity produced by burning coal and sulfur dioxide emissions are \( w \), then weak disposability of outputs require that a 10\% reduction in sulfur dioxide emissions occur with a 10\% reduction in electricity, for a given input vector.

"Alternatively, if regulations restrict emissions of sulfur dioxide, the targets are attained by diverting some of the given input vector to the 'Cleanup of those emissions, which implies that less input would be available for the production of electricity, resulting in a simultaneous decrease in both desirable and undesirable output.

The output oriented directional efficiency can be obtained solving the following linear programming problem:

\[
\tilde{D}_v(x, u, w) = \text{Max } \theta
\]

subject to \[
\sum_{m} \lambda_j x_{jm} \leq \lambda_j x_{j0}, \quad i = 1, 2, \ldots, m
\]

\[
\sum_{m} \lambda_j u_{jm} \geq u_{j0} + \theta g_{j}, \quad r = 1, 2, \ldots, s \quad \text{(3.3.3)}
\]

\[
\sum_{m} \lambda_j w_{kj} = w_{k0} - \theta g_{k}, \quad k = 1, 2, \ldots, q
\]

\[
\lambda_j \geq 0
\]
(i). \( \bar{D}_0(x, u, v; g_u, g_v) = 0 \Rightarrow 100 \text{ percent technical efficiency} \)

\[ > 0 \Rightarrow \text{technical inefficiency.} \]

(ii). If \( g_u = 1 \) and \( g_v = -1 \) we have, \( u_\theta + \theta^* - u_\theta \)

\[ \Leftrightarrow u_\theta + \bar{D}_0(x, u, v; 1, 1) - u_\theta \]

The sum of the first two terms yields potential desirable output of the target production unit. The difference can be interpreted as the net improvement in performance in terms of feasible increases in good-outputs

\[ (w_{\theta^*} - \theta^*) - w_{\theta^*} \]

\[ \Leftrightarrow w_{\theta^*} + \bar{D}_0(x, u, v) - w_{\theta^*} \]

The above expression reveals net improvement in performance in terms of feasible decreases in undesirable outputs.

Fare et al., (1989) introduced a hyperbolic efficiency measure in which inputs are controlled and outputs are expanded simultaneously along hyperbolic path.

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**Figure (3.3.2)**
The straight line that passes through the origin admits constant returns to scale. The curve that falls below the line segment admits increasing returns to scale. At A returns to scale are constant. The curved segment AB admits decreasing returns to scale. The production unit operating at C experiences technical inefficiency. The hyperbolic path connecting C with D is a path that reduces inputs and expands outputs simultaneously. To estimate technical efficiency we solve the following non-linear programming problems:

\[
\begin{align*}
\text{Max } & \lambda \\
\text{subject to } & \sum_{i=1}^{m} \lambda_i x_{i_0} \leq \lambda^{-1} x_{i_0}, \quad i = 1, 2, \ldots, m \\
& \sum_{r=1}^{s} \lambda_r u_{r_0} \geq \lambda u_{r_0}, \quad r = 1, 2, \ldots, s \\
& \lambda_j \geq 0
\end{align*}
\]

The hyperbolic efficiency measurement can be extended to undesirable outputs. Consequently, the following non-linear programming problem:

\[
\begin{align*}
\text{Max } & \lambda \\
\text{subject to } & \sum_{i=1}^{m} \lambda_i x_{i_0} \leq x_{i_0}, \quad i = 1, 2, \ldots, m \\
& \sum_{r=1}^{s} \lambda_r u_{r_0} \geq \lambda u_{r_0}, \quad r = 1, 2, \ldots, s \\
& \sum_{k=1}^{q} \lambda_k w_{k_0} = \lambda^{-1} w_{k_0}, \quad k = 1, 2, \ldots, q \\
& \lambda_j \geq 0
\end{align*}
\]

In the above problem (3.3.5) which is non-linear desirable outputs are expanded and undesirable outputs are contracted simultaneously.
Given the input $x_o$, with one desirable and undesirable output we have the following diagram.

![Diagram](image)

Figure (3.3.3)

$P(x_o)$ is output level set. 'u' and 'w' are desirable and undesirable outputs respectively. In the above figure two production frontiers are found, one the convex frontier and the other Free Disposable Hull (FDH). The producer who operates at A is inefficient. Following hyperbolic path A is connected with C that falls on FDH frontier. A further extension leads to B that falls on the convex frontier.

3.4. **PURE ENVIRONMENTAL EFFICIENCY:**

Given the input and desirable output vectors, if the production possibility set satisfy strong disposability of inputs and desirable outputs, weak disposability of undesirable outputs, pure environmental efficiency can be measured for a production unit solving the following linear programming introduced by D.Tyteca (1996):
Min $\lambda$

subject to $\sum_{j=1}^{n} \lambda_j x_j \leq x_{\theta}$, \hspace{1cm} i = 1, 2, \ldots, m

\[
\sum_{j=1}^{n} \lambda_j u_{\theta} \geq \lambda u_{\theta}, \hspace{1cm} r = 1, 2, \ldots, s \hspace{1cm} \text{(3.4.1)}
\]

\[
\sum_{j=1}^{n} \lambda_j \delta_k = \delta_{\theta}, \hspace{1cm} k = 1, 2, \ldots, q
\]

$\lambda_j \geq 0, \hspace{1cm} j = 1, 2, \ldots, n$

The dual of the above linear programming problem is as follows:

Max $Z = \sum_{i=1}^{s} a_i u_{\theta} - \sum_{i=1}^{m} \beta_i x_{\theta}$

subject to $\sum_{i=1}^{s} a_i u_{\theta} - \sum_{i=1}^{m} \beta_i x_{\theta} - \sum_{i=1}^{s} v_i w_{\theta} \leq 0$

\[
\sum_{k=1}^{q} v_k w_{\theta} = 1, \hspace{1cm} \text{(3.4.2)}
\]

$a_i, \beta_i \geq 0$

$v_k$ are unrestricted for sign.

In ratio form of the above problem is,

Max $Z = \frac{\sum_{i=1}^{s} a_i u_{\theta} - \sum_{i=1}^{m} \beta_i x_{\theta}}{\sum_{i=1}^{s} v_i w_{\theta}}$

\[
\frac{\sum_{i=1}^{s} a_i u_{\theta} - \sum_{i=1}^{m} \beta_i x_{\theta}}{\sum_{i=1}^{s} v_i w_{\theta}} \leq 1
\]

\[
\sum_{k=1}^{q} v_k w_{\theta} \leq 1
\]

$a_i, \beta_i \geq 0$

$v_k$ are unrestricted for sign.
The above problem (3.4.3) can be alternatively written as,

\[
\min \bar{Z} = \frac{\sum_{i=1}^{a} u_{i} w_{l0}}{\sum_{i=1}^{a} a_i u_{i0} - \sum_{i=1}^{b} \beta_i x_{i}}
\]

subject to

\[
\frac{\sum_{i=1}^{a} u_{i} w_{b0}}{\sum_{i=1}^{a} a_i u_{i0} - \sum_{i=1}^{b} \beta_i x_{i}} \geq 1
\]  \hspace{1cm} (3.4.4)

The above problem is the classical formulation of Data Envelopment Analysis (DEA) introduced by Charnes et al., (1978). It minimizes a ratio of undesirable outputs to a weighted sum of desirable outputs and inputs. We view the undesirable outputs as peculiar outputs, which we minimize with respect to the other production factors (Tyteca, 1996). The minus sign appearing in the denominator may be viewed as some net production (Weighted sum of outputs - Weighted sum of inputs) of the production unit whose efficiency is under evaluation.

Into the problems (3.3.3), (3.3.5) and (3.4.1) variable returns to scale can be modeled, augmenting the constraint,

\[
\sum_{j=1}^{s} \lambda_{j} = 1
\]

Single stage DEA models were constructed by Banker and Morey (1986) to incorporate environmental variables. The environmental variables may be categorical or continuous. The approach is to include non-discretionary variables along discretionary inputs and outputs but to restrict the optimization to either inputs or outputs.

When categorical variables are involved the comparison set restricts to producers operating in the same or high categories. This requires that the categories are nested and reduce the size of the comparison set for most producers, thereby reducing the discriminatory power of the model.
In two stage approach, in the first stage DEA problems are solved without including the environmental variables. Taking the efficiency scores as dependent variable a regression equation is fit with environmental variables are explanatory variables. McCarty and Yaisawarng (1993) and Bhattacharyya et al., (1996) used second stage regression residuals to adjust the first stage efficiency scores.

Pastor (1995) proposed a two stage approach. In the first stage he either used input oriented DEA with environmental variables or output oriented DEA with environmental variables. He then replaced either the inputs or the outputs by their radial projections in order to eliminate the effect of environmental variables.

While modeling DEA problems in association with environmental variables there arise three fundamental variables.

1) The influence of each environmental variable shall be known a priori. However this is not difficulty since in most occasions the influence of the environmental variables is well known a priori.

2) A decision making unit shall be compared with such decision making units that are operating in equal and worse environmental conditions. Consequently, those decision making units operating in worse condition are, by definition are considered to be efficient.

![Figure (3.4.1)](image-url)
In the above figure we have nested production frontier. The bottom most piecewise frontier is determined by the decision making units G, H and I.

For decision making unit operating at E, while D and F are operating at equal environment, G, H and I are operating at inferior environment. In this context the model suggested by Ruggiero is worth to be presented as follows:

1) Inputs and outputs are freely disposable.
2) For a given value \((z = z_0)\) of the environmental variable the input and output vectors spin a convex set.
3) For a given value of \((z = z_0)\) the production possibility set is the intersection of all the production possibility sets which contain all the observations \((x_j,u_j), j = 1, 2, \ldots, n\).

There is need to connect environment with production frontier. This can be done as follows:

\[ z_1 \leq z_2 \Rightarrow S(z_1) \subseteq S(z_2) \]

The constraint \(z_1 \leq z_2\) implies that \(z_1\) is an environment no better than \(z_2\). The effect of environment on production can be best understood from the following diagram.
The production units A, B and C enjoy superior environment than D, E, F and G. The DMUs A, B and C can produce more outputs for a given set of inputs. The decision making units D, E, F and G perform their activities in a similar environment. The production unit G is inefficient. If environment is ignored its input technical efficiency is, \( \frac{x_u}{x_0} \). But, given the environment of G its technical efficiency is, \( \frac{x_i}{x_0} \).

Further, the efficient reference set of \( (x_G, u_o) \) when environment is ignored is given by \( (x_A, u_A) \), \( (x_B, u_B) \) and \( (x_C, u_C) \), all are technically efficient. However, if environmental differences are not ignored the reference set is given by, \( (x_0, u_0) \), \( (x_F, u_F) \) and \( (x_E, u_E) \).

\[
\frac{x_u}{x_0} \leq \frac{x_i}{x_0}
\]

Thus, if environment is ignored the true technical efficiency is not only underestimated but the decision making units D, E and F are identified as technically inefficient.

To account for environmental differences the following problem may be solved:

Min \( \theta \)

subject to

\[
\sum_{j=1}^{s} \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1, 2, \ldots, m
\]

\[
\sum_{j=1}^{s} \lambda_j u_{ri} \geq u_{ro}, \quad r = 1, 2, \ldots, s
\]

\[
\sum_{j=1}^{s} \lambda_j = 1
\]

\( \lambda_j \geq 0, \quad \forall j \) such that \( z_j \leq z_0 \)

\( \lambda_j = 0, \quad \forall j \) such that \( z_j > z_0 \)
Equivalently we have,

\[
\begin{align*}
\text{Min } \theta \\
\text{subject to } \sum_{\mu_i} \lambda_j x_{\mu_i} &\leq \theta x_{\mu_0}, & i = 1, 2, \ldots, m \\
\sum_{\mu_i} \lambda_j u_{\mu_i} &\geq u_{\mu_0}, & r = 1, 2, \ldots, s \\
\sum_{\mu_i} \lambda_j & = 1 \\
\text{where } J_0 &= \{j : z_j \leq z_0\}
\end{align*}
\]

(3.4.6)

The above formulation takes into account a single environmental variable which we call as external non-discretionary variable viewed in input perspective.

A methodology proposed by Banker and Morey (1986) handles more than one continuous non-discretionary variables which account for external shocks to the production unit. In the presence of q external non-discretionary factors, to assess input technical efficiency the following linear programming problem is proposed to solve:

\[
\begin{align*}
\text{Min } \theta \\
\text{subject to } \sum_{\mu_i} \lambda_j x_{\mu_i} &\leq \theta x_{\mu_0}, & i = 1, 2, \ldots, m \\
\sum_{\mu_i} \lambda_j u_{\mu_i} &\geq u_{\mu_0}, & r = 1, 2, \ldots, s \\
\sum_{j=1}^{w_{\mu j}} \lambda_j w_{\mu j} &\leq w_{\mu 0}, & k = 1, 2, \ldots, q \\
\end{align*}
\]

(3.4.7)

This model differs from the CCR (1984) model in the sense that the contraction factor \( \theta \) is associated with the discretionary inputs. The above model admits constant returns to scale only.
If a non-discretionary factor is internal for a production unit, in the presence of constant returns to scale, the constraint of $k^{th}$ internal non-discretionary factor may be expresses as,

$$\sum_{j=1}^{s} \lambda_j w_{i,j} \leq \sum_{j=1}^{s} \lambda_j w_{i,0}$$

For internal non-discretionary factors the following specification by Banker and Morey (1986) is solved:

$$\text{Min } \theta$$

subject to

$$\sum_{j=1}^{s} \lambda_j x_{i,j} \leq \theta x_{i,0}, \quad i = 1, 2, ..., m$$

$$\sum_{j=1}^{s} \lambda_j w_{i,j} \leq \sum_{j=1}^{s} \lambda_j w_{i,0}, \quad k = 1, 2, ..., q$$

$$\sum_{j=1}^{r} \lambda_j u_{i,j} \geq u_{i,0}, \quad r = 1, 2, ..., s$$

$$\sum_{j=1}^{s} \lambda_j w_{i,j} \geq \sum_{j=1}^{s} \lambda_j w_{i,0}, \quad k = 1, 2, ..., q$$

$$= w_{i,j} \sum_{j=1}^{s} \lambda_j$$

$$\sum_{j=1}^{s} \lambda_j (w_{i,j} - w_{i,0}) \leq 0$$

The above constraint imposes convexity not on the non-discretionary inputs, but on a new variable reflecting the difference between the non-discretionary level of input ‘i’ of the peers and $DMU_0$ being assessed (Camanho et al., 2009).

If the variable returns to scale constraint $\sum_{j=1}^{s} \lambda_j = 1$ is imposed we have,
Min $\theta$

subject to $\sum_{i=0}^{n} \lambda_i x_{q_i} \leq \theta x_{\theta}$,  \hspace{1cm} i = 1, 2, ..., m

$\sum_{i=0}^{n} \lambda_i u_{q_i} \geq u_{\theta}$ \hspace{2cm} r = 1, 2, ..., s \hspace{2cm} (3.4.9)

$\sum_{j=0}^{q} \lambda_j w_{b_j} \leq w_{\theta}$ \hspace{2cm} k = 1, 2, ..., q

$\sum_{j=0}^{q} \lambda_j = 1$

The above formulation is the BCC (1984) linear programming problem.

Golany and Roll (1993) extended the constant returns to scale Model of Banker and Morey to consider simultaneously non-discretionary inputs and outputs in the frame work of constant returns to scale.

Min $\theta$

subject to $\sum_{i=0}^{n} \lambda_i x_{q_i} \leq \theta x_{\theta}$,  \hspace{1cm} i \in D

$\sum_{i=0}^{n} \lambda_i x_{q_i} \leq \sum_{i=0}^{n} \lambda_i x_{\theta}$,  \hspace{1cm} i \in ND \hspace{2cm} (3.4.10)

$\sum_{i=0}^{n} \lambda_i u_{q_i} \geq u_{\theta}$,  \hspace{2cm} r \in D

$\sum_{i=0}^{n} \lambda_i u_{q_i} \geq \sum_{i=0}^{n} \lambda_i u_{\theta}$,  \hspace{1cm} r \in ND

Where D and ND are sets which include the indices of discretionary and non-discretionary factors. If a non-discretionary factor is external then for such a factor $\sum_{i=0}^{n} \lambda_i = 1$.

Including the internal non-discretionary factors in defining the production possibility set implies that the assumption of convexity also apply to these factors. In practice, the convexity assumption means that the targets for any inefficient unit may be constructed from any set of peer units with better or worst levels of ND factors.
This has been seen as a drawback of the Banker and Morey approach, particularly when the non-discretionary factors are external. Ruggiero (1996) addressed two problems in case of a single external non-discretionary factor.

3.5 NON-DISCRETIONARY FACTORS – CORRECTION OF EFFICIENCY:

If the efficiency scores are to reflect not only the differences in discretionary but also non-discretionary factors which are external, then single stage or multiple stage approaches can be used.

The Ruggiero (1996) approach discussed elsewhere in a single stage approach to solve a DEA model restricting the reference set for each DMU under evaluation to DMUs performing only in equal or more disadvantageous conditions in terms of ND factors.

The non-discretionary factors of Ruggiero problem are not internal to the model, since the convexity condition is not imposed on non-discretionary inputs. This approach is very similar to the approaches that group units according to some categorical factor and assess them in relation to different frontiers. Thus, each DMU is compared with a different peer set.

The Ruggiero (1996) formulation delineates the impact of external ND factors from technical efficiency measure. The Ruggiero model imposes more restrictions on the peer set which leads to more number of technically efficient decision making units, than the approach suggested by Banker and Morey.

![Figure (3.5.1)](image-url)
In the above figure (3.5.1) the line segments GH and HI represent the Banker-Morrey frontier. The DMU operating at J is technically inefficient. The DMUs A, B and C are operating in the same environment as J. The projection of J onto Ruggiero frontier AB-BC gives technical efficiency rating \( \theta (R) \). A further horizontal movement gives the rating \( \theta (BM) \).

\[
\theta (BM) \leq \theta (R)
\]

Thus, it appears that the Ruggiero approach overestimates input technical efficiency.

In the spirit of DEA in Ruggiero approach DMU\(_0\) keeps itself in the best possible light.

If the number of ND factors increases, the chance that a unit being rated efficient will also increase. To accommodate multiple number of ND factors Ruggiero (1998) proposed another model that aggregates the effect of all ND factors into a single factor using multiple linear regression and the emerged single factor is used as he did in his paper (1996).

Ruggiero (2002) proposed an enhanced DEA problem to increase the peer set of a decision making unit. By doing so a certain percentage of DMUs operating in superior environment are augmented to the peer set of DMUs which perform in equal and inferior environment.

Charnes, Cooper and Rhodes (CCR, 1978) proposed an output oriented efficiency problem as a fractional programming problem:

\[
\lambda (CCR) = \text{Min} \sum_{i=1}^{m} \mu_i Y_o
\]

subject to

\[
\sum_{i=1}^{m} \mu_i Y_o \leq 1, \quad j=1,2,\ldots,n
\]

\[
\sum_{i=1}^{m} v_i X_0 = 1
\]

\[ v_i \geq 0, \quad \mu_i \geq 0 \]
In the problem above (3.5.1) there are $n$ production units, each producing similar outputs consuming similar inputs. 'm' is the number of inputs and 's' is the number of outputs. $v_i$ and $u_r$ are unspecified input and output weights. The inputs and outputs in the objective function belongs such production unit whose efficiency is under evaluation. Since there are 'n' production units as many fractional programming problems are solved.

Applying the Charnes, Cooper transformation the ratio problem can be resolved into a linear programming problem. The linear programming problem can be expressed as follows:

$$\theta (CCR) = \text{Min} \sum v_i x_{i\theta}$$

subject to

$$\sum u_r y_{r\theta} = 1$$  \hspace{1cm} \text{(3.5.2)}

$$\sum v_i x_{i\theta} - \sum u_r y_{r\theta} \leq 0, \hspace{0.5cm} j = 1,2,\ldots,n$$

$$v_i \geq 0, \hspace{0.5cm} u_r \geq 0$$

The above problem (3.5.2) is called the multiplier problem. Its dual may be expressed as follows:

$$\theta (CCR) = \text{Max} \theta$$

subject to

$$\sum \lambda_i x_{i\theta} \leq x_{\theta}, \hspace{0.5cm} i = 1,2,\ldots,m$$  \hspace{1cm} \text{(3.5.3)}

$$\sum \lambda_j y_{j\theta} \geq \theta y_{\theta}, \hspace{0.5cm} r = 1,2,\ldots,s$$

$$\lambda_j \geq 0$$

In terms Shephard's (1970) output distance function,

$$D_\theta (x_\theta, y_\theta) = [\theta (CCR)]^\dagger$$

The distance function above satisfies the following important properties:

1. $[D_\theta (x_\theta, y_\theta)]^\dagger = \text{Max} \{\theta : y_\theta \in P (x_\theta)\}$

71
2. \( 0 \leq D_o(x_o, y_o) \leq 1 \)

3. \( D_o(x, \lambda y) = \lambda D_o(x, y) \)

The output distance function is linear homogeneous in \( u \).

\[
\left[ D_o(x, \lambda y) \right]^{-1} = \max \left\{ \theta : \theta \lambda u \in P(x) \right\}
\]

\[
= \frac{1}{\lambda} \max \left\{ \theta : \theta \lambda u \in P(x) \right\}
\]

\[
= \frac{1}{\lambda} \left[ D_o(x, y) \right]^{-1}
\]

\( D_o(x, \lambda y) = \lambda D_o(x, y) \)

4. \( y' \leq y \Rightarrow D_o(x, y) \geq D_o(x, y') \)

\( y' \leq y \Leftrightarrow \theta y' \leq \theta y \)

\[
\max \left\{ \theta : \theta y \in P(x) \right\} \leq \max \left\{ \theta : \theta y' \in P(x) \right\}
\]

\[
\left[ \max \left\{ \theta : \theta y \in P(x) \right\} \right]^{-1} \geq \left[ \max \left\{ \theta : \theta y' \in P(x) \right\} \right]^{-1}
\]

\( D_o(x, y) \geq D_o(x, y') \)

5. \( D_o(x, y) \) is differentiable with components of the output vector.

\( \theta \) (CCR) is the inverse of Shephard's output distance function.

The CCR (1978) problem was extended by Banker, Charnes and Cooper (BCC, 1984). The former problem failed to identify output scale differences of production units.
The line segments PQ and QR constitute a variable returns to scale piecewise linear production frontier. Along the line segment PQ returns to scale are increasing. At Q returns to scale are constant. Along the line segment QR returns to scale are decreasing. The supporting hyper plane that passes through P and Q has a negative intercept. The supporting hyper plane that is tangent to the production possibility set possess zero intercept. Thus, we have,

\[ \mu > 0 \Rightarrow \text{Returns to scale are decreasing} \]
\[ \mu = 0 \Rightarrow \text{Returns to scale are constant} \]
\[ \mu < 0 \Rightarrow \text{Returns to scale are increasing} \]

The multiplier version of BCC (1984) problem can be expressed as,

\[
\begin{align*}
\text{Min} & \quad \sum_{i=1}^{n} v_i x_{i0} \\
\text{subject to} & \quad \sum_{i=1}^{n} \mu_i y_{i0} = 1 \\
& \quad \sum_{i=1}^{n} v_i x_{i0} - \sum_{i=1}^{n} \mu_i y_{i0} \leq 0, \quad j = 1,2,\ldots,n \\
& \quad v_i \geq 0, \quad \mu_i \geq 0 \\
\end{align*}
\]  

(3.5.4)

The dual of the linear programming problem (3.5.4) may be expressed as follows:

\[
\begin{align*}
\theta (\text{BCC}) = \text{Max} \theta \\
\text{subject to} & \quad \sum_{j=1}^{s} \lambda_j x_{j0} \leq x_{i0}, \quad i = 1,2,\ldots,m \\
& \quad \sum_{j=1}^{s} \lambda_j y_{j0} \geq \theta y_{i0}, \quad r = 1,2,\ldots,s \\
& \quad \sum_{j=1}^{s} \lambda_j = 1 \\
& \quad \lambda_j \geq 0
\end{align*}
\]  

(3.5.5)

There are as many input constraints as there are inputs. The output constraints are ‘s’ in number. The final constraint is the convexity constant.

73
Problem (4) can be constructed axiomatically.

Let \( x \in \mathbb{R}_{+}^{n}, y \in \mathbb{R}_{+}^{n} \) and further let \( T \) be the production possibility set.

1. \((x_{i}, y_{j}) \in T, \ i = 1, 2, \ldots, n\)

2. \((x, y) \in T, \ \hat{x} \geq x \) and \( \hat{y} \leq y \Rightarrow (\hat{x}, \hat{y}) \in T\)

3. \((x_{i}, y_{j}) \in T \Rightarrow \left( \sum_{i} \lambda_{i} x_{i}, \sum_{i} \lambda_{i} y_{j} \right) \in T, \) where \( \sum_{i} \lambda_{i} = 1, \ \lambda_{j} \geq 0\)

4. \( T \) is the intersection of all the production possibility sets which contain \((x_{i}, y_{j}), j = 1, 2, \ldots, n\).

The production possibility set which satisfies the above axioms may be expressed as,

\[
T = \left\{ (x, u) : \sum_{i} \lambda_{i} x_{i} \leq x, \ \ \sum_{i} \lambda_{i} y_{j} \geq y, \ \ \sum_{i} \lambda_{i} = 1, \ \ \lambda_{j} \geq 0 \right\} \quad (3.5.6)
\]

The output level set of the production possibility set is,

\[
P(x) = \left\{ y : \sum_{i} \lambda_{i} x_{i} \leq x, \ \ \sum_{i} \lambda_{i} y_{j} \geq y, \ \ \sum_{i} \lambda_{i} = 1 \right\} \quad (3.5.7)
\]

Combining to the CCR (1978) and BCC (1984) problem, we have, every feasible solution of the later problem is a feasible solution of the former problem. Consequently, we have,

\[
\theta (\text{CCR}) \geq \theta (\text{BCC})
\]

The ratio \( \frac{\theta (\text{CCR})}{\theta (\text{BCC})} \) measures output scale efficiency.

\[
\frac{\theta (\text{CCR})}{\theta (\text{BCC})} \geq 1
\]
\( \theta (CRR) \) and \( \theta (BCC) \) respectively measure overall and pure technical efficiency. \( \theta (CRR) \) can be multiplicatively decomposed into output scale and pure output technical efficiency.

\[
\theta (CRR) = \left[ \frac{\theta (CRR)}{\theta (BCC)} \right] \theta (BCC)
\]

(3.5.8)

In this study we consider one of several bank risks namely Credit Risk.

3.6 INDIAN COMMERCIAL BANKS – CREDIT RISK:

Financial sector is lifeline of any economy. They play the role of financial intermediaries where basic objectives are collection of deposits and advance them as loans. Every debt instrument is subjected to an element of risk whose consequences can not be predicted accurately as risk is associated with uncertainty. In the case of commercial banks which constitute an important part of financial sectors several aspects of risk are heard, Credit Risk, interest Risk, Liquidity Risk and Insolvency risk are some of them. Control or management of risk of any type is not cost free for any commercial bank. In modern days, in particular after globalization and liberalization commercial banks function in fierce competition, higher the levels of competition greater the institution has to adjust to improve efficiency and to undertake riskier activities. Efficiency and risk have to be examined jointly. It is desirable that a financial institution should not only be efficient but also secure. In banking efficiency measurement literature the inter relationship between efficiency and risk is mostly ignored.

Higher yields require banker to take higher risks. A banker looks for highest yield without compromising the securities of the bank’s funds. Credit risk is an important one associated with loans and investments. Things which affect borrower’s financial position may contribute to credit risk. Credit risk also occur if a borrower capable of meeting payment may choose the course of action not to clear the installments which cover the interest and principal amounts. Credit risk is a factor in all loans, but to varying degrees. Banks should recognize this variability by matching
loan rates to risk. A bank that charges the same rates to all its loans fails to receive adequate compensation for its riskier loans.

 Numerous quantitative techniques emerged to judge the performance of a commercial bank, important one being the technical efficiency measure. While this measure serves as a good indicator of bank’s performance other factors affecting efficiency apart from the classical inputs and outputs should also be considered. One of the most important factors that influence bank’s efficiency is credit risk which originates from internal and external factors. The former and the later may also be called as endogenous and exogenous factors respectively. Government’s invention, Central Bank’s (Reserve Bank of India) directions from time to time, business cycles, competition from rival banks contribute to exogenous non-discretionary factors which lead to credit risk. The bank has no control over these factors, therefore, these are external non-discretionary factors. Poor internal risk control system, ineffective screening of applications, failure to apply checks and balances effectively constitute bad management. These factors are endogenous factors which influence bank’s efficiency from inside.

 In the presence of credit risk there is need to obtain risk adjusted efficiency scores. There are attempts to measure efficiency of commercial banks adjusting them for risk. These studies considered problem loans as an additional input, but failed to decompose the credit risk management efficiency into its sources attributed to exogenous and endogenous risk efficiency. Jose M.Pastor (1999) mentioned that:

 “—those who do try to achieve measurements of efficiency adjusted for risk do so by simply entering the problem loans as an additional input, without discounting the part of the total that is due to exogenous factors, the economic cycle or ‘bad luck’, thus penalizing these firms which, though cautious and good managers of risk, have a high level of problem loans due to external factors (economic cycle)”. Thus, such banks good in risk management may emerge with poor efficiency scores due to the dominant of influence exogenous factors on efficiency. Therefore, there is a need not only to compute risk efficiency scores, but also decompose them into exogenous and endogenous risk efficiency scores. The later gives risk management efficiency scores.
3.7 EVOLUTION OF INDIAN COMMERCIAL BANKS:

The commercial banks operating on Indian soil can be grouped into Public, Private and Foreign sector banks. In 1991 the public and private sector banks faced financial repression which is an outcome of the fiscal policies of the central government and monetary policies of the central bank (Reserve Bank of India). The policies, laws, regulations and poor internal risk controls were all contributed to the financial repression. With its intervention policies the government can intervene into banks business in a number of ways through statutory preemptions, interest regulation and directing credit to the priority sectors. In 1991-1992 it was found that 40 percent of total credit of these banks were diverted to reach the priority sectors such as agriculture, small scale industries, transport operators and the export sector. The interest rate payable by these sectors to the banks was regulated and the regulated rate of interest was much less than the prevailing market rate of interest.

The cash reserve ratio (CRR) was about 5 percent in 1960s and 1970s reached a record level of 15 percent. The legal minimum and maximum levels of CRR are 3 and 15 percent. The upper limit of SLR was 40%, the prevailing SLR in Feb 1992 was found to be 38.5%. The Indian Commercial Banks were left with 14.5 percent of bank credit left at their disposal to perform their own business.

The countries which faced financial repression were suggested the policy recommendations that the real interest rates shall be allowed market determined, the levels of CRR and SLR shall be decreased approximately to facilitate the banks to improve their own business and to move towards a stable financial system and reduce directed credit to the priority sector.

However, all forms of government intervention do not induce repression policies. For example, when market failures occur the intervention policies of the government may strengthen banks position in financial market.
To get away from the evil effects of financial repression, Narasimham Committee (I) suggested several recommendations:

(i) Entry deregulation
(ii) deregulation of interest rates
(iii) raise the equity in the Capital markets upto 49%
(iv) gradual reduction of CRR and SLR
(v) increase in capital adequacy to a minimum of 8%
(vi) asset classification and income recognition
(vii) gradual reduction of priority sector to lending and
(viii) Institutional reforms to improve managerial efficiency.

During the financial year 1992-93 the public sector banks share in Deposits, Investments, Advances and Total Assets were 87.9, 85.9, 89.3 and 87.2 respectively. Thus, public sector banks were found to dominate all the financial institutions.

Prior to the implementation of Narasimham Committees recommendations, the Indian Commercial Banks in particular the public sector banks suffered due to the presence of Non-Performing Assets (NPA). An asset or account of borrower can be classified by a bank as standard, doubtful or loss asset according to the norms suggested by the Reserve Bank of India. NPA doesn’t yield any income either in the form of principal or interest. Growing NPAs reduce lizable-funds leading to slow credit growth which serves as a barrier to India’s economic growth and development. NPAs affect the profitability and net worth of the bank.

NPA requires provisions which reduce overall profits and share holders value. RBI imposed provision norms against asset classification, which ranged from 0.25% to 100% from standard and loss assets respectively.

The anti-repression measures, namely:

(a) Entry deregulation leads to increased competition integrated with a considerable amount of risk. During the financial year 1992-93 the market
share in Deposits, Investments, Advances and Total assets of public sector banks were 87.5, 85.9, 89.3 and 87.2 respectively. Owing to the entry of more private and foreign sector banks the market share of public sector banks in Deposits, Investments, Advances and total assets were reduced to 74.9, 73.1, 72.9 and 72.3 percent respectively.

(b) Deregulation of interest rates leads to a situation where they are market determined due to the interaction of demand and supply factors of funds.

(c) To increase their capital base to meet the risk that arises from time to time due to financial deregulation several public sector banks were allowed to invite private participation raising upto 49 percent of equity in the capital market.

(d) Gradual reduction in statutory preemptions prevents under supply of credit and the commercial banks can explore new avenues to improve their income generating capabilities.

While the 1st stage reforms were in progress the second stage reforms were implemented in the year 1998 by the policy makers. The Reserve Bank of India issued guidelines for management of credit, market and operational risks. A risk management system should continuously measure, monitor and control all risks such as credit, market and operational risks. Credit risk mostly refer to loans and advances.

Briefly, the evaluation of Indian Banking System is as follows:

- Planning era began in 1980
- Imperial Bank of India was nationalized and renamed as State Bank of India (SBI) in 1955
- Indian Commercial Banks supplied credit to urban affluent people, Industry, Trade and commerce during 1947-1967
- Regulated and control measures were introduced to shape the banking system as an instrument for rapid economic development in 1960
- Social control on banks administered for more equitable and purposeful distribution of bank credit in 1967
- Banks with deposits exceeding Rs. 20 billion were nationalized in 1967
- RBI imposed 1:4 license rule in 1977.
- 3000 banks existed in rural locations with credit and savings rose from 1.5 and 3 percent to 15 percent in 1990.
- Narasimham Committee (I) submitted its reports and recommendations in 1991.
- Narasimham Committee recommendations were implemented in 1992.
- Narasimham Committee (II) recommendations implemented from 1998 onwards.

The most important of all the risks faced by a commercial bank is credit risk. ‘Credit Risk is most simply defined as the potential that a bank borrower or counterparty will fail to meet its obligations in accordance with agreed terms’ (Basel Committee report, Sept. 2000). The effective credit risk management is very much necessary for the survival and long run success of a commercial bank. Credit risk is found to exist in the entire portfolio of a financial institution such as a commercial bank. ‘For most banks loans are the largest and most obvious source of credit risk. Other sources of credit risk exist throughout the activities of a bank, including in the banking book and in the trading book and both on and off the balance sheet (Basel, 2000)’.

The other sources of credit risk other than loans are acceptances, inter bank transactions, trade financing, foreign exchange transactions, financial futures, swaps, bonds, equities, options and in the extension of commitments and guarantees and the settlement of transactions.

3.8 John Ruggiero (1998) proposed a DEA model that captures exogenous environment governing public sector institutions. He claimed that, as should be the case with every DEA problem, to calculate technical efficiency of any production unit governed by external environment should be compared with it the production units operating in equal and inferior environment.
The credit risk of a commercial bank is measured, measuring the quantum of non-performing assets. NPAs are classified into standard, doubtful and loss assets. Commercial banks allocate provisions to cover loan losses. Existences of non-performing assets deplete other outputs, since NPAs require provisions to compensate. Total NPAs per unit of advances serves as an undesirable output. Since this undesirable output is not costlessly disposed we consider it as weakly disposed output.

Larger ratio of NPAs to advances reflects inferior environment, such production units whose undesirable output is larger than the undesirable output of the production unit whose efficiency is under evaluation, are assumed to operate under inferior environment. The production units which form reference technology of the targeted production unit have the index set as follows:

\[ J_0 = \{ j : u_j \geq u_0 \} \]  \hspace{1cm} (3.8.1)

The DEA problem that captures exogenous environment may be expressed as follows:

\[
\begin{align*}
\theta (BCC, JR) = & \text{Max } \theta \\
\text{subject to } & \sum_{\eta \in k} \lambda_{\eta} x_{\eta, \theta} \leq x_{\theta}, \quad i = 1, 2, \ldots, m \\\n& \sum_{\eta \in k} \lambda_{\eta} y_{\eta, \theta} \geq y_{\theta}, \quad r = 1, 2, \ldots, s \hspace{1cm} (3.8.2) \\
& \sum_{\eta \in k} \lambda_{\eta} = 1, \quad J_0 = \{ j : u_j \geq u_0 \} \\
& \lambda_{\eta} \geq 0, \quad \forall \eta \in J_0
\end{align*}
\]

In the above formulation \( J_0 \) identifies the production units in the reference technology which changes from one production unit to another.

**Theorem (1):** \( \theta (BCC) \geq \theta (BCC, JR) \)

**Proof:** Let \( \lambda_j \geq 0, \forall j \in \{ j : j \in J_0 \} \)

\( \hat{\lambda}_j = 0, \forall j \in \{ j : j \in J_0 \} \) and \( \hat{\theta} \) be a feasible solution of (3.8.2)

81
This is a feasible solution of (3.5.5) also. All the feasible solutions of (3.8.2) form subset of feasible solutions of (3.5.5). This implies,

$$
\theta (BCC) \geq \theta (BCC, JR)
$$

Banker and Morey (1986) initiated a DEA model that accounts for exogenous non-discretionary inputs and outputs. The DEA problem with exogenous non-discretionary inputs may be expressed as,

$$
\theta (BCC, BM) = \text{Min } \lambda
$$

subject to

$$
\sum_{i=1}^{m} \lambda_i x_{iq} \leq x_{i0}, \quad i = 1, 2, \ldots, m
$$

$$
\sum_{r=1}^{s} \lambda_r y_{rq} \geq y_{r0}, \quad r = 1, 2, \ldots, s \quad \text{(3.8.3)}
$$

$$
\sum_{j=1}^{q} \lambda_j u_{j0} \leq u_{j0}, \quad p = 1, 2, \ldots, q
$$

$$
\sum_{j=1}^{q} \lambda_j = 1
$$

$u_p$ are undesirable outputs.

Golany and Roll (1993) modeled exogenous inputs/outputs into constant returns to scale DEA problem formulation as follows:

$$
\theta (CCR, GR) = \text{Max } \theta
$$

subject to

$$
\sum_{i=1}^{m} \lambda_i x_{iq} \leq x_{i0}, \quad i \in I
$$

$$
\sum_{i=1}^{m} \lambda_i x_{iq} \leq x_{i0} \sum_{j=1}^{q} \lambda_j, \quad i \in I, \quad \text{(3.8.4)}
$$

$$
\sum_{r=1}^{s} \lambda_r y_{rq} \geq y_{r0}, \quad r \in R
$$

$$
\sum_{r=1}^{s} \lambda_r y_{rq} \geq \sum_{r=1}^{s} \lambda_r y_{r0}, \quad r \in R
$$
\[ x_i, \ i \in I_0 \Rightarrow \text{Inputs are exogenous non-discretionary} \]
\[ x_i, \ i \in I_1 \Rightarrow \text{Inputs are endogenous non-discretionary} \]
\[ u_r, \ r \in R_0 \Rightarrow \text{Outputs are exogenous non-discretionary} \]
\[ u_r, \ r \in R_1 \Rightarrow \text{Outputs are endogenous non-discretionary} \]

If the convexity condition \[ \sum_{j=1}^{n} \lambda_j = 1 \] is imposed all non-discretionary factors become endogenous.

Invoking the Banker and Morey and Golany and Roll formulations, we formulate the following linear programming problem:

\[
\theta (BCC, JR, BM) = \max \quad \theta \\
\text{subject to} \quad \sum_{j \in I_0} \lambda_j x_i \leq x_{i0}, \quad i = 1, 2, \ldots, m \\
\sum_{j \in I_0} \lambda_j y_{r0} \geq \theta y_{r0}, \quad r = 1, 2, \ldots, s \quad (3.8.5) \\
\sum_{j \in I_0} \lambda_j u_j \leq u_0 \\
\sum_{j \in I_0} \lambda_j = 1
\]

In the above model NPAs are considered to play the role of input which is costlessly disposed off. But in reality provisions are shown to cover NPAs. Therefore, NPAs are weakly disposed off. In this study NPAs are treated as an undesirable output which is weakly disposed off.

Fare et al., (1996) formulation of weak disposability, combined with problem (3.8.5) is expressed as follows:
\( \theta(\text{BCC, JR, BM, WD}) = \max \; \theta \)

subject to

\[ \sum_{j \in I_s} \lambda_j x_{ij} \leq x_{i0} \]

\[ \sum_{j \in I_s} \lambda_j y_{ij} \geq \alpha \theta y_{i0} \]

\[ \sum_{j \in I_s} \lambda_j u_j = \alpha u_0 \]

\[ \sum_{j \in I_s} \lambda_j = 1 \]

\[ \alpha \geq 0, \; \lambda_j \geq 0 \]

Problem (3.8.6) is a non-linear programming problem.

**Theorem (2):** \( \theta(\text{BCC, JR}) \geq \theta(\text{BCC, JR, BM, WD}) \)

**Proof:**

\[ \sum_{j \in I_s} \lambda_j y_{ij} \geq \alpha \theta y_{i0} \geq \theta y_{i0} \]

Since \( \alpha \geq 1, \; \alpha \theta \geq \theta \)

\[ \Rightarrow \sum_{j \in I_s} \lambda_j y_{ij} \geq \theta y_{i0} \]

The set of all feasible solutions of \( \theta(\text{BCC, JR, BM, WD}) \) is a subset of the set of all feasible solutions of \( \theta(\text{BCC, JR}) \), which implies that,

\[ \theta(\text{BCC, JR}) \geq \theta(\text{BCC, JR, BM, WD}) \]

The non-linear programming problem (3.8.6) can be transformed into a linear programming problem. Dividing both sides of the constraints with \( \alpha \) we get,
\[ \theta (\text{BCC, JR, BM, WD}) = \max \theta \]

subject to
\[ \sum_{\mu \in s} \frac{\lambda_j}{\alpha} x_{i,j} \leq \alpha^{-1} x_{i,0} \]
\[ \sum_{\mu \in s} \frac{\lambda_j}{\alpha} y_{r} \geq \theta y_{r,0} \]
\[ \sum_{\mu \in s} \frac{\lambda_j}{\alpha} u_j = u_0 \]
\[ \sum_{\mu \in s} \frac{\lambda_j}{\alpha} = \alpha^{-1} \]
\[ \alpha \geq 0, \lambda_j \geq 0 \]

\[ \theta (\text{BCC, JR, BM, WD}) = \max \theta \]

subject to
\[ \sum_{\mu \in s} \delta_j x_{i,j} \leq \beta x_{i,0}, \quad i = 1,2,\ldots,m \]
\[ \sum_{\mu \in s} \delta_j y_{r} \geq \theta y_{r,0}, \quad r = 1,2,\ldots,s \]
\[ \sum_{\mu \in s} \delta_j u_j = u_0 \]
\[ \sum_{\mu \in s} \delta_j = \beta \]
\[ \delta_j \geq 0, \quad 0 \leq \beta \leq 1 \]

The technical efficiency measures can be combined and expressed as follows:

\[ \theta (\text{CCR}) \geq \theta (\text{BCC}) \geq \theta (\text{BCC, JR}) \geq \theta (\text{BCC, JR, BM, WD}) \]

3.9 EXOGENOUS CREDIT RISK EFFICIENCY:

Let
\[ \theta^{-1} (\text{BCC, JR, BM, WD}) = D_0(x, y, u) \quad (3.9.1) \]

Following Fare et al., (2004) if the production of undesirable output has neutral effect on other desirable outputs, the output distance function may be expressed as follows:

\[ D_{e}(x, y, u) = D_{e}(x, f(u), y) \]
Since the output distance function is linear homogenous in desirable output,

\[ D_o(x, y; u) = f(u) \tilde{D}_o(x, y) \]

The expression \( f(u) \) measures endogenous credit risk efficiency.

\[
f(u) = \frac{D_o(x, y; u)}{\tilde{D}_o(x, y)} = \left[ \frac{\tilde{D}_o(x, y)}{D_o(x, y; u)} \right]^{-1} - \frac{\theta(BCC, JR)}{\theta(BCC, JR, BM, WD)} \geq 1
\]

\[ f(u) \geq 1 \]

\[f(u)\]

\[u\]

\[y\]

Figure (3.9.1)

In the above figure (3.9.1), \( P(x_o) \) is an output set constituted by all the desired and undesired outputs which can be produced consuming the input vector \( x_o \). The desired and undesired outputs are null joint. \( P(x_o) \) satisfies weak disposability of undesirable outputs \( (u) \) and strong disposability of desirable output \( (y) \). It also assumes inputs are strongly disposable.

The producer who operates at \( A \) is inefficient, expanding his desirable outputs, which is possible through expansion of his undesirable outputs. At ‘A’ in the above figure, the scaling factor \( f(u) = 1 \). The function \( f(u) \geq 1 \) increases monotonically with \( u \).
At \( u = u_0 \), \( f(u_0) = 1 \). Any attempt to expand \( u_0 \) is accompanied by expansion of \( u_0 \) also. Reduction of NPAs per unit of advances is followed by depletion of reduction of other desirable outputs. Therefore, an increase in NPA hurts all the desirable outputs which implies that NPAs are not costlessly disposed off.

\[
\frac{\theta \text{(CCR)}}{\theta \text{(BCC)}} \text{ measures output scale efficiency as proved by BCC (1984).}
\]

\[
\frac{\theta \text{(CCR)}}{\theta \text{(BCC)}} \geq 1
\]

Such commercial banks for which the above ratio attains unit value do not experience loss of outputs due to scale inefficiency.

b) The output efficiency measure \( \theta \text{(BCC)} \) ignores non-discretionary environmental factors that are exogenous to influence NPAs. The efficiency measure that delineates exogenous environmental differences from \( \theta \text{(BCC)} \) is \( \theta \text{(BCC, JR)} \). We define the exogenous credit risk efficiency of a commercial bank as follows:

\[
\text{Exogenous Credit Risk Efficiency} : \frac{\theta \text{(BCC)}}{\theta \text{(BCC, JR)}}
\]

c) The credit risk efficiency attributed to endogenous non-discretionary factors is,

\[
\text{Endogenous Credit Risk Efficiency:} \frac{\theta \text{(BCC, JR)}}{\theta \text{(BCC, JR, BM, WD)}}.
\]

If we refer to \( \theta \text{(CCR)} \) as overall productive efficiency, it can be multiplicatively decomposed and expressed as follows:

\[
\theta \text{(CCR)} = \left[ \frac{\theta \text{(CCR)}}{\theta \text{(BCC)}} \right] \left[ \frac{\theta \text{(BCC)}}{\theta \text{(BCC, JR)}} \right] \left[ \frac{\theta \text{(BCC, JR)}}{\theta \text{(BCC, JR, BM, WD)}} \right] \left[ \theta \text{(BCC, JR, BM, WD)} \right]
\]

We call \( \theta \text{(BCC, JR, BM, WD)} \) as pure technical efficiency.
3.10 MIXED ENVIRONMENTAL INDEX:

Referring to BCC we look for simultaneous adjustments of desirable and undesirable outputs. For this we solve the following optimization problem that appears to be a non-linear programming problem:

\[
\begin{align*}
\text{Max } & \frac{\theta}{\lambda} \\
\text{subject to } & \sum_{j \in J_0} \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, 2, \ldots, m \\
& \sum_{j \in J_0} \lambda_j y_{ij} \geq \alpha_0 y_{i0}, \quad r = 1, 2, \ldots, s \\
& \sum_{j \in J_0} \lambda_j u_j = \alpha u_0 \\
& \sum_{j \in J_0} \lambda_j = 1 \\
& \lambda_j \geq 0
\end{align*}
\]

(3.10.1)

The problem above with non-linear objective function can be transformed into a linear programming problem:

Let \( \frac{\theta}{\lambda} = \rho \)

Divide each parameter with \( \lambda \) throughout all the constraints.

\[
\begin{align*}
\text{Max } & \rho \\
\text{subject to } & \sum_{j \in J_0} \left( \frac{\lambda_j}{\lambda} \right) x_{ij} \leq \frac{1}{\lambda} x_{i0} \\
& \sum_{j \in J_0} \left( \frac{\lambda_j}{\lambda} \right) y_{ij} \geq \alpha_0 \frac{1}{\lambda} y_{i0} \\
& \sum_{j \in J_0} \left( \frac{\lambda_j}{\lambda} \right) u_j = \alpha u_0 \\
& \sum_{j \in J_0} \left( \frac{\lambda_j}{\lambda} \right) = 1 \\
& \lambda_j \geq 0
\end{align*}
\]

(3.10.2)
Further dividing throughout with $\alpha$, we have:

$$\begin{align*}
\text{Max } \rho \\
\text{subject to } \sum_{j \in J_0} \frac{\lambda_j}{\lambda \alpha} x^{\delta}_{j} &\leq \frac{1}{\lambda \alpha} x^{\delta}_{s} \\
\sum_{j \in J_0} \frac{\lambda_j}{\lambda \alpha} y_{j} &\geq \rho y_{i0} \\
\sum_{j \in J_0} \frac{\lambda_j}{\lambda \alpha} u_{j} &= u_{c} \\
\sum_{j \in J_0} \frac{\lambda_j}{\lambda \alpha} &= \frac{1}{\lambda \alpha} \quad \cdots (3.10.3)
\end{align*}$$

Let $\left(\frac{\lambda_j}{\lambda \alpha}\right) = \delta_j$, the above problem can be expressed as follows:

$$\begin{align*}
\rho^* = \text{Max } \rho \\
\text{subject to } \sum_{j \in J_0} \delta_j x_{j} &\leq \delta x_{i0} \\
\sum_{j \in J_0} \delta_j y_{j} &\geq \theta y_{i0} \\
\sum_{j \in J_0} \delta_j u_{j} &= u_{0} \\
\sum_{j \in J_0} \delta_j &= \delta \quad \cdots (3.10.4)
\end{align*}$$

The DEA optimization problem (3.10.4) is a linear programming problem. Let $(\rho^*, \delta^*_j, \delta^*, \theta^*)$ be an optimal solution.

$\theta^*$ measures desirable outputs expansion.

$\lambda^*$ = $\frac{\theta^*}{\rho^*}$ measures undesirable outputs reduction.

$$\rho^* = \frac{\theta^*}{\lambda^*}$$

$\frac{\theta^* y_{i0}}{\lambda^* u_{0}}$ measures potential desirable output per unit of potential undesirable output produced.
\( \lambda^* \) may be viewed as a measure of credit risk management. If \( u_o \) represents NPAs per unit of loans and advances, by effective credit risk management current NPAs would have been reduced by \((1 - \lambda)\) percent. We call the ratio \( \frac{\theta^*}{\lambda^*} \) as credit risk management efficiency.

\[
\frac{\theta^* y_o}{\lambda^* u_o} \geq \frac{\theta^* y_o}{u_o}
\]

If credit risk is left unmanaged potential desirable output per unit of undesirable output is seen smaller than potential desirable output per unit of undesirable output in the case when credit risk is effectively managed. The ratio \( \frac{\theta}{\lambda} \) captures credit risk.

\[
\frac{\theta^*}{\lambda^*} = \theta (\text{BCC, JR, BM, WD, CRM}) \geq \theta (\text{BCC, JR, BM, WD})
\]

The ratio \( \frac{\theta (\text{BCC, JR, BM, WD, CRM})}{\theta (\text{BCC, JR, BM, WD})} \) measures Credit Risk Management Efficiency. Larger is this ratio poorer is credit risk management. CRM refers to credit risk management.

**Theorem (3):** \( \theta (\text{BCC, JR, BM, WD, CRM}) \geq \theta (\text{BCC, JR, BM, WD}) \)

**Proof:** Consider the following fractional programming problem:

\[
\text{Max } \left( \frac{\theta}{\lambda} \right)
\]

subject to

\[
\sum_{j \in J_o} \lambda_j x_{j} \leq x_o
\]

\[
\sum_{j \in J_o} \lambda_j y_{j} \geq y_o
\] \hspace{1cm} (3.10.5)

\[
\sum_{j \in J_o} \lambda_j u_{j} = \lambda u_o
\]

\[
\sum_{j \in J_o} \lambda_j = 1
\]

\[
\lambda_j \geq 0
\]

Since \( \lambda \leq 1 \) we find \( t \) such that, \( \lambda t = 1 \).
Max \(\frac{t \theta}{t \lambda}\)

subject to

\[\sum_{\mu_i} (h_i t) x_i \leq t x_{\mu_0}\]

\[\sum_{\mu_i} (\delta_i t) y_i \geq (t \theta) y_\mu\] \quad --- (3.10.6)

\[\sum_{\mu_i} (\delta_i t) u_i = (t \lambda) u_0\]

\[\sum_{\mu_i} (\delta_i t) = t\]

Equivalently, we have,

Max \(\hat{\theta}\)

subject to

\[\sum_{\mu_i} \delta_i x_i \leq t x_{\mu_0}\]

\[\sum_{\mu_i} \delta_i y_i \geq \hat{\theta} y_\mu\] \quad --- (3.10.7)

\[\sum_{\mu_i} \delta_i u_i = u_0\]

\[\sum_{\mu_i} \delta_i = t \geq 1\]

Consider the following linear programming problem:

Max \(\theta\)

subject to

\[\sum_{\mu_i} \delta_i x_i \leq x_{\mu_0}\]

\[\sum_{\mu_i} \delta_i y_i \geq \theta y_\mu\] \quad --- (3.10.8)

\[\sum_{\mu_i} \delta_i u_i = u_0\]

\[\sum_{\mu_i} \delta_i = 1\]
Let us consider the above two (3.10.7) and (3.10.8) linear programming problems:

$$\sum_{\mu \in \lambda} \delta_{\mu} x_{\mu} \leq x_{\lambda}, \quad \text{since } t \geq 1.$$ 

Every feasible solution of (3.10.8) is a feasible solution of (3.10.7).

$$\text{Max } \theta \leq \text{Max } \bar{\theta}$$

$$\theta(\text{BCC, JR, BM, WD}) \leq \theta(\text{BCC, JR, BM, WD, CRM})$$

In view of the inequality above the multiplicative decomposition can be extended by one more step.

$$\theta(\text{CCR}) = \frac{\theta(\text{CCR})}{\theta(\text{BCC})} \times \frac{\theta(\text{BCC})}{\theta(\text{BCC, JR})} \times \frac{\theta(\text{BCC, JR})}{\theta(\text{BCC, JR, BM, WD, CRM})}$$

$$\frac{\theta(\text{BCC, JR, BM, WD, CRM})}{\theta(\text{BCC, JR, BM, WD})} [\theta(\text{BCC, JR, BM, WD})]$$