CHAPTER 3

CONSTRUCTION OF REGULAR GRAPH DESIGNS AND ITS GRAPHICAL REPRESENTATION

3.1 Introduction

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3.1 INTRODUCTION

Block designs are extensively used in many fields of research activities. A wide range of balanced incomplete block (BIB) designs and partially balanced incomplete block (PBIB) designs are available in the literature. However, these designs are restricted to equi-replicate and equi-block sizes. This may pose practical problems in many possible experimental situations [Pearce (1964), Calinski (1971) and Hedayat and Federer (1974)].

With the introduction of efficiency-balanced designs through the work of Calinski (1971), Puri and Nigam (1975) and others, the concept of balance has undergone a change. Das and Ghosh (1985) showed that the efficiency balanced
designs are reinforced incomplete block designs. Another new class of incom-
plete block designs, termed as partially efficiency balanced (PEB) designs was
developed by Puri and Nigam (1977), which can be made available in varying
replications and unequal block sizes. For efficiency balanced designs, the effi-
ciency in terms of effective information obtained on every contrast is the same.
But in certain situations all contrasts may not be of equal interest and it may be
desired to estimate some of the contrasts with more efficiency than others. Such
situations occur in bio-assays and in factorial experiments.

In this chapter, the construction of Regular Graph (RG) designs obtained
from BIB designs, whose parameters satisfy certain conditions on the availability
of RG designs, is discussed. The RG designs are constructed by augmenting a new
treatment in each of the \( b \) blocks of the BIB design, and augmenting \( (r - \lambda + 1) \)
blocks which contain all the \( v \) treatments of the BIB design only. Further, it is
verified that these regular graph (RG) designs belong to a particular class of par-
tially efficiency balanced (PEB) designs with two efficiency classes. This shows
that a particular class of PEB designs with two efficiency classes is also regular
graph designs.

RG designs are also obtained by adding two RG designs having the same
treatments and same size of blocks. Further, a sufficient condition for the exis-
tence of a RG design as the sum of two RG designs, is established.

Also, it is demonstrated that a RG design can always be obtained from a BIB
design if \( v > 4, k > 2 \) and \( \lambda = 2 \), when a particular treatment is deleted along
with all those blocks in which that particular treatment is present.

Again, the adjacency graph of RG designs is given, and thereby some new RG
designs are obtained.
3.2 HISTORICAL REVIEW

Regular graph (RG) designs were introduced by Mitchell and John (1976) as a type of approximation to balanced designs. They conjectured that if a RG design exists for a given set of parameters, then it is optimal. They listed optimal RG designs (or the duals of them) for 209 parameter sets in the practical range ($v \leq 12$) and ($r \leq 10$). Cheng (1980) developed a method to show that a great majority of these listed designs are indeed $E$-optimal over all the designs with the same values of $v, b$ and $k$. Jacroux (1980) proposed several sufficient conditions for the existence of an $E$-optimal RG design within various classes of proper block designs. Wallis (1996) looked into regular graph designs using the concepts of imbalance and asymmetry. Kreher et al. (1996) investigated resolvable RG designs with block size 4, and determined the parameters for which such designs exist, for $v \leq 16$.

3.3 PRELIMINARY RESULTS

For the evaluation of the eigen values of the $M_o$-matrix, the following Lemma (Mukerjee and Kageyama, 1990) is useful.

**Lemma 3.3.1**(Mukerjee and Kageyama)

Let $u, s_1, s_2, ..., s_u$ be positive integers, and consider the $s \times s$ matrix

$$A = \begin{pmatrix}
a_1I_{s_1} + b_{11}J_{s_1s_1} & b_{12}J_{s_1s_2} & \ldots & b_{1u}J_{s_1s_u} \\
b_{21}J_{s_2s_1} & a_2I_{s_2} + b_{22}J_{s_2s_2} & \ldots & b_{2u}J_{s_2s_u} \\
\vdots & \vdots & \ddots & \vdots \\
b_{u1}J_{s_us_1} & b_{u2}J_{s_us_2} & \ldots & a_uI_{s_u} + b_{uu}J_{s_us_u}
\end{pmatrix}$$

where $s = s_1 + s_2 + ... + s_u$ and $u \times u$ matrix $B = (b_{ij})$ is symmetric. Then the
eigen values of $A$ are $a_i$ with multiplicity $s_i - 1 (1 \leq i \leq u)$ and $\mu_1^*, \ldots, \mu_u^*$, where $\mu_1^*, \ldots, \mu_u^*$ are the eigen values of $\Delta = D_a + D_s^{1/2}BD_s^{1/2}$, $D_a = \text{diag}(a_1, \ldots, a_u)$, $D_s = \text{diag}(s_1, s_2, \ldots, s_u)$, $D_s^{1/2} = \text{diag} (s_1^{1/2}, s_2^{1/2}, \ldots, s_u^{1/2})$.

3.4 CONSTRUCTION OF A RG DESIGN USING A BIB DESIGN WITH $b = 2r - \lambda + 1$

In this section, the construction of regular graph designs which are obtained using BIB designs, is mainly discussed. These designs are also PEB designs with two efficiency classes.

Theorem 3.4.1

Let $d(v, b, k, r, \lambda)$ be a BIB design and its parameters satisfy the relation $b = 2r - \lambda + 1$. Let $N_d$ be the incidence matrix of the BIB design $d$, and $N_{d+}$ be the incidence matrix of the augmented design $d_+$. The incidence matrix $N_{d+}$ defined as:

$$N_{d+} = \begin{pmatrix} N_{v \times b} & J_v \times (r-\lambda+1) \\ - & - & - & - & - & - \\ J_{1 \times b} & 0_{1 \times (r-\lambda+1)} \end{pmatrix}$$

gives a regular graph (RG) design with parameters $v_+ = v + 1$, $b_+ = b + (r - \lambda + 1)$, $r_+ = (2r - \lambda + 1, \ldots, 2r - \lambda + 1, b)$ and $k_+ = (k + 1, k + 1, \ldots, k + 1, v, \ldots, v)$.

Proof

Let $N_d$ be the incidence matrix of a BIB design $d(v, b, k, r, \lambda)$, and $N_{d+}$ the incidence matrix of the augmented design $d_+$. The concurrence matrix $N_{d+}N_{d+}^\prime$ of the design $d_+$ is given by:
Now, \( N_{d+N_{d+}} = \begin{pmatrix} (r - \lambda)I_v + (r + 1)J_{vv} & rJ_{v1} \\ \vdots & \vdots \\ rJ_{1v} & b \end{pmatrix} \)

(i) The differences of the diagonal elements of the concurrence matrix \( N_{d+N_{d+}} \) are:

\[
(r - \lambda) + (r + 1) - b = 2r - \lambda + 1 - b \\
= 0 \quad (\text{since } b = 2r - \lambda + 1)
\]

(ii) The differences of the off-diagonal elements of the concurrence matrix \( N_{d+N_{d+}} \) are:

\[
(r + 1) - r = 1
\]

That is, this class of designs is RG designs, because one can find that all the diagonal elements of its concurrence matrix are equal and all the off-diagonal elements of its concurrence matrix are differing by at most one. In fact, the off-diagonal elements of \( N_{d+N_{d+}} \) take only values \( r \) and \( r + 1 \).

This completes the proof.

**Corollary 3.4.1**

The class of regular graph (RG) designs constructed using Theorem 3.4.1 is in fact PEB designs with two associate classes.

**Proof**

This Corollary is proved by using the \( M_v \)-matrix of the design \( d_+ \), and for that, the \( C \)-matrix and the \( M \)-matrix of the design \( d_+ \) are obtained.
The $C$-matrix of the design $d_+$ is obtained from the expression:

$$C = \text{diag}(r_1, r_2, \ldots, r_{v+1}) - (N_{d+} \ \text{diag} \ (1/k_1, 1/k_2, \ldots, 1/k_{b+(r-\lambda+1)}) \ N_{d+}' \ )$$

where $r_1, r_2, \ldots$ are the treatment replications, $k_1, k_2, \ldots$ are the block sizes, and is of the form:

$$C = \begin{pmatrix}
[(2r - \lambda + 1) - \left(\frac{r-\lambda}{k+1}\right)]I_v - \left(\frac{\lambda}{k+1} + \frac{r-\lambda+1}{v}\right)J_{vv} & -\frac{r}{k+1}J_{v \times 1} \\
- & - \\
-\frac{r}{k+1}J_{1 \times v} & b - \frac{b}{k+1}
\end{pmatrix}$$

The $M$-matrix of the design $d_+$ is obtained from the expression:

$$M = \text{diag}(1/r_1, 1/r_2, \ldots, 1/r_{v+1}) \ (N_{d+} \ \text{diag} \ (1/k_1, 1/k_2, \ldots, 1/k_{b+(r-\lambda+1)}) \ N_{d+}' \ )$$

and is of the form:

$$M = \begin{pmatrix}
\frac{r-\lambda}{(k+1)b}I_v + \frac{1}{b}\left(\frac{\lambda}{k+1} + \frac{r-\lambda+1}{v}\right)J_{vv} & \left(\frac{r}{k+1}\right)\left(\frac{1}{b}\right)J_{v \times 1} \\
- & - \\
\frac{r}{b(k+1)}J_{1 \times v} & \frac{1}{k+1}
\end{pmatrix}$$

where $b = 2r - \lambda + 1$.

The $M_o$-matrix of the design $d_+$ is given by the following expression:

$$M_o = \text{diag}(1/r_1, 1/r_2, \ldots, 1/r_{v+1}) \ (N_{d+} \ \text{diag} \ (1/k_1, 1/k_2, \ldots, 1/k_{b+(r-\lambda+1)}) \ N_{d+}' \ ) - J \ (r_1, r_2, \ldots, r_{v+1})^{\top}/n$$

where $J$ is a column vector of one’s of order $v + 1$ and $n$ is the total number of units, and is of the form:

$$J = \begin{pmatrix}
\frac{r-\lambda}{(k+1)b}I_v + \frac{1}{b}\left(\frac{\lambda}{k+1} + \frac{r-\lambda+1}{v}\right) - \frac{b}{(v+b)}J_{vv} & \left(\frac{r}{k+1}\right)\left(\frac{1}{b}\right) - \frac{b}{v+b}J_{v \times 1} \\
- & - \\
\left(\frac{r}{b(k+1)} - \frac{b}{v+b}\right)J_{1 \times v} & \frac{1}{k+1} - \frac{b}{v+b}
\end{pmatrix}$$
where $b = 2r - \lambda + 1$

Using Lemma 3.3.1, the eigen values of $M_o$ are $\mu_1 = \frac{r - \lambda}{(k+1)(2r-\lambda+1)}$ with multiplicity $(v-1)$, and $\mu_2 = \text{trace}(M_o) - (v-1)\mu_1$ with multiplicity one. So, the design $d_+$ is partially efficiency balanced (PEB) with efficiency factors $(1 - \mu_1)$ with multiplicity $(v-1)$, and $(1 - \mu_2)$ with multiplicity one.

This completes the proof.

Remarks 3.4.1

Mitchell and John (1976) constructed regular graph designs for equi-replicated and proper block designs. However, he didn’t discuss regular graph designs with unequal block sizes. Hence in this investigation, the construction of RG designs with unequal block sizes is carried out.

3.5 NUMERICAL EXAMPLES

As an application of the above theorem, a numerical example is provided in this section.

Example 3.5.1

Consider a BIB design $d$ with parameters $v = 5$, $b = 10$, $r = 6$, $k = 3$ and $\lambda = 3$ such that $b = 2r - \lambda + 1$, as shown below:

Table 3.5.1: BIB design with blocks shown in columns

$$
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 & 3 & 4 & 3 & 4 & 4 & 5 \\
3 & 4 & 5 & 4 & 5 & 5 & 4 & 5 & 5 & 5
\end{bmatrix}
\end{align*}
$$
Using Theorem 3.4.1, the augmented design $d_+$ with parameters $v_+ = 6$, $b_+ = 14$, $r_+ = 10$ and $k_+ = (4, \ldots, 4, 5, 5, 5, 5, 5)$ is obtained, and its incidence matrix $N_{d+}$ is expressed as:

\[
N_{d+} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

And the concurrence matrix of the design $d_+$ is given by the expression:

\[
N_{d+}N'_{d+} = \begin{pmatrix}
10 & 7 & 7 & 7 & 7 & 6 \\
7 & 10 & 7 & 7 & 7 & 6 \\
7 & 7 & 10 & 7 & 7 & 6 \\
7 & 7 & 7 & 10 & 7 & 6 \\
6 & 6 & 6 & 6 & 6 & 10 \\
\end{pmatrix}
\]

That is, this class of designs is RG designs, because one can find that all the diagonal elements of its concurrence matrix are equal and all the off-diagonal elements of its concurrence matrix $N_{d+}N'_{d+}$ are differing by at most one.

The $C$-matrix of the design $d_+$ is given by:

\[
C = \begin{pmatrix}
7.7000 & -1.5500 & -1.5500 & -1.5500 & -1.5500 & -1.5500 \\
-1.5500 & 7.7000 & -1.5500 & -1.5500 & -1.5500 & -1.5500 \\
-1.5500 & -1.5500 & 7.7000 & -1.5500 & -1.5500 & -1.5500 \\
-1.5500 & -1.5500 & -1.5500 & 7.7000 & -1.5500 & -1.5500 \\
-1.5500 & -1.5500 & -1.5500 & -1.5500 & 7.7000 & -1.5500 \\
-1.5500 & -1.5500 & -1.5500 & -1.5500 & -1.5500 & 7.5000 \\
\end{pmatrix}
\]
The $M$-matrix of the design $d_+$ is given by:

$$
M = \begin{pmatrix}
0.2300 & 0.1550 & 0.1550 & 0.1550 & 0.1550 & 0.1500 \\
0.1550 & 0.2300 & 0.1550 & 0.1550 & 0.1550 & 0.1500 \\
0.1550 & 0.1550 & 0.2300 & 0.1550 & 0.1550 & 0.1500 \\
0.1550 & 0.1550 & 0.1550 & 0.2300 & 0.1550 & 0.1500 \\
0.1550 & 0.1550 & 0.1550 & 0.1550 & 0.2300 & 0.1500 \\
0.1500 & 0.1500 & 0.1500 & 0.1500 & 0.1500 & 0.2500 \\
\end{pmatrix}
$$

The $M_o$-matrix of the design $d_+$ is obtained as follows:

$$
M_o = \begin{pmatrix}
.0633 & -.0117 & -.0117 & -.0117 & -.0117 & -.0167 \\
-.0117 & .0633 & -.0117 & -.0117 & -.0117 & -.0167 \\
-.0117 & -.0117 & .0633 & -.0117 & -.0117 & -.0167 \\
-.0117 & -.0117 & -.0117 & .0633 & -.0117 & -.0167 \\
-.0117 & -.0117 & -.0117 & -.0117 & .0633 & -.0167 \\
-.0167 & -.0167 & -.0167 & -.0167 & -.0167 & .0833 \\
\end{pmatrix}
$$

The eigen values of $M_o$ are $\mu_1 = 0.075$ with multiplicity 4 ($= v - 1$) and $\mu_2 = 0.100$ with multiplicity one.

So, the design $d_+$ is partially efficiency balanced (PEB) with efficiency factors:

(i) $1 - \mu_1 = 0.925$ with multiplicity $(v - 1) = 4$, and
(ii) $1 - \mu_2 = 0.900$ with multiplicity 1.

Further examples of RG designs with unequal block sizes, constructed using BIB designs are given in the following Table 3.5.2.
Table 3.5.2 Regular Graph Designs with unequal block sizes

<table>
<thead>
<tr>
<th>Parameters of BIB designs</th>
<th>Parameters of Constructed RG designs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$</td>
<td>$b$</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>13</td>
</tr>
</tbody>
</table>

Also, it is found that all the constructed RG designs belong to a particular class of PEB designs with two efficiency classes.

### 3.6 REGULAR GRAPH DESIGNS BY ADDING TWO RG DESIGNS

Another method of obtaining a new RG design is by adding two RG designs. Two designs can be added, if they have the same number of treatments and same size of blocks.

Chang (1997) obtained an $E$-optimal RG design by adding an $E$-optimal RG design with a BIB design. Again, he did not discuss adding the blocks of two RG designs. So in this investigation, RG designs are constructed by adding the blocks of two RG designs with same number of treatments and same size of blocks.
Theorem 3.6.1

Consider a regular graph design $d_1(v, b_1, k, r_1, \lambda_1, \lambda_1')$ and another RG design $d_2(v, b_2, k, r_2, \lambda_2, \lambda_2')$. For a given treatment $\theta$ from the designs $d_1$ and $d_2$, let the number of treatments common between the $i^{th}$ associate of $\theta$ from $d_1$ and $j^{th}$ associate of $\theta$ from $d_2$ be denoted by $M = (m_{ij}); i, j = 1, 2$. If $M = (m_{ij})$ is a constant matrix and one of the elements of $M = (m_{ij})$ is zero, irrespective of the choice of $\theta$, then $d_1 + d_2$ gives a RG design $d_3$. Further, one of the elements of the matrix $M = (m_{ij})$ is zero, is a sufficient condition for the existence of a RG design $d_3$, as the sum of the designs $d_1$ and $d_2$.

Proof

Let any treatment of the design $d_i \ (i = 1, 2, 3)$ has $(n_i, n'_i), \ (i = 1, 2, 3)$ treatments as first and second associates respectively. It may be noted that $(n_i, n'_i), \ (i = 1, 2, 3)$ is a constant for a design and hence, the existence of a RG design implies $M = (m_{ij})$ is a constant irrespective of the choice of $\theta$, and one of the elements of $M = (m_{ij})$ matrix is equal to zero. Hence, the corresponding $(\lambda_i, \lambda_i')$ of $d_1$ and $d_2$ do not get added in $d_3$. So, the design $d_3$ is two associate class with $\lambda_3' = \lambda_3 + 1$. Hence, the resulting design is a RG design. Also $(m_{ij})_{i=j}(\neq 0)$ represents $n_3'$ and $\sum_{i\neq j}(m_{ij})(\neq 0)$ represents $n_3$, which are constants for the design $d_3$.

Example 3.6.1

Consider a RG design $d_1$ with parameters $v = 6, k = 2, r_1 = 3, b_1 = 9, \lambda_1 = 0 \text{ and } \lambda_1' = 1$:
Table 3.6.1: RG design $d_1$ with blocks shown in columns

$$d_1 = \begin{pmatrix} 1 & 3 & 5 & 1 & 3 & 5 & 1 & 3 & 5 \\ 2 & 4 & 6 & 6 & 2 & 4 & 4 & 6 & 2 \end{pmatrix}$$

The first associates (treatments in the first row) and second associates (treatments in the second row) of each of the treatments are shown below:

1 \begin{pmatrix} 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}; 2 \begin{pmatrix} 4 & 6 \\ 1 & 3 & 5 \end{pmatrix}; 3 \begin{pmatrix} 1 & 5 \\ 2 & 4 & 6 \end{pmatrix}; \\
4 \begin{pmatrix} 2 & 6 \\ 1 & 3 & 5 \end{pmatrix}; 5 \begin{pmatrix} 1 & 3 \\ 2 & 4 & 6 \end{pmatrix}; 6 \begin{pmatrix} 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}

Now, consider another RG design $d_2$ with parameters $v = 6$, $k = 2$, $r_2 = 4$, $b_2 = 12$, $\lambda_2 = 0$ and $\lambda_2' = 1$:

Table 3.6.2: RG design $d_2$ with blocks shown in columns

$$d_2 = \begin{pmatrix} 1 & 3 & 4 & 1 & 2 & 5 & 1 & 2 & 3 & 1 & 2 & 4 \\ 2 & 5 & 6 & 3 & 4 & 6 & 5 & 6 & 4 & 6 & 3 & 5 \end{pmatrix}$$

The first and second associates of each of the treatments are shown below:

1 \begin{pmatrix} 4 \\ 2 & 3 & 5 & 6 \end{pmatrix}; 2 \begin{pmatrix} 5 \\ 1 & 3 & 4 & 6 \end{pmatrix}; 3 \begin{pmatrix} 6 \\ 1 & 2 & 4 & 5 \end{pmatrix}; \\
4 \begin{pmatrix} 1 \\ 2 & 3 & 5 & 6 \end{pmatrix}; 5 \begin{pmatrix} 2 \\ 1 & 3 & 4 & 6 \end{pmatrix}; 6 \begin{pmatrix} 3 \\ 1 & 2 & 4 & 5 \end{pmatrix}

For a given treatment $\theta$ from the design $d_1$ and $d_2$, let the number of treatments common between the $i^{th}$ associate of $\theta$ from $d_1$, and $j^{th}$ associate of $\theta$ from $d_2$ be denoted by $M = (m_{ij})$, $i, j = 1, 2$.

Therefore,

$$M = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$
which is a constant irrespective of the choice of $\theta$. The (1,1)th element of $M$ equals to zero, signifies that $\lambda_1$ and $\lambda_2$ corresponding to the first associates of $d_1$ and $d_2$ do not get added in the resulting design $d_3$. Again, $\lambda_1$ corresponding to the first associate of $d_1$, and $\lambda_2'$ corresponding to the second associate of $d_2$ get added in $d_3$. Hence, $\lambda_3 = 0 + 1 = 1$. Also, $\lambda_1'$ corresponding to the second associate of $d_1$, and $\lambda_2'$ corresponding to the second associate of $d_2$, get added in $d_3$. Hence, $\lambda_3' = 1 + 1 = 2$. So, the resulting design $d_3$ is a RG design. Again, $n_3' = (m_{ij})_{i=j} (\neq 0) = 2$ and $n_3 = \sum_{i\neq j} (m_{ij}) (\neq 0) = 2 + 1 = 3$.

i.e. by adding the RG designs $d_1$ and $d_2$, a RG design $d_3$ is obtained with parameters $v = 6$, $k = 2$, $r_3 = 7$, $b_3 = 21$, $\lambda_3 = 1$ and $\lambda_3' = 2$ and is given by:

Table 3.6.3: RG design $d_3$ with blocks shown in columns

\[
\begin{pmatrix}
1 & 3 & 5 & 1 & 3 & 5 & 1 & 3 & 5 & 1 & 3 & 5 & 1 & 3 & 4 & 1 & 2 & 5 & 1 & 2 & 3 & 1 & 2 & 4 \\
2 & 4 & 6 & 6 & 2 & 4 & 4 & 6 & 2 & 2 & 5 & 6 & 3 & 4 & 6 & 5 & 6 & 4 & 6 & 3 & 5
\end{pmatrix}
\]

3.7 CONSTRUCTION OF RG DESIGNS USING DELETION TECHNIQUES

Dey (1986, P. 173) has demonstrated that a RG design can always be obtained from a BIB design if $\lambda = 1$, when a particular treatment is deleted along with all those blocks in which that particular treatment is present, that is to say, retaining only those blocks in which the particular treatment is absent. Here, a new method of constructing RG designs is presented. The method of construction is illustrated below.

3.7.1 Method of construction of the RG designs

Let $D^*$ be a BIB design with parameters $v^*, b^*, r^*, k^*$ and $\lambda^* = 2$, where
$v^* > 4$, $k^* > 2$ and, let $\theta$ be a treatment of the design. By definition, $\theta$ occurs in $r^*$ blocks of $D^*$. If these $r^*$ blocks containing $\theta$ is deleted, a design $D$ in $(v^* - 1)$ treatments and $(b^* - r^*)$ blocks is obtained. Then, the following theorem is proved:

**Theorem 3.7.1**

The design $D$ is a RG design with parameters $v = v^* - 1, b = b^* - r^*, r = r^* - 2, k = k^*, \lambda_1 = 1$ and $\lambda_2 = 2$.

**Proof**

Let $B_{1}^*, B_{2}^*, ..., B_{r^*}^*$ be the blocks in $D^*$ containing the treatment $\theta$. Let the same blocks, after deleting $\theta$ be $B_1, B_2, ..., B_{r^*}$. Then, since $\lambda^* = 2$, no two blocks of $D^*$ have more than two treatments in common. Hence, no two blocks in $B_1, B_2, ..., B_{r^*}$ have more than one treatment in common. So, all possible pairs of treatments in each of the blocks $B_1, B_2, ..., B_{r^*}$ occur only once in $D$, so $\lambda_1 = 1$. And, all other pairs of treatments in $(v^* - 1)$ occur twice in $D$. So, $\lambda_2 = 2$ for all these pairs of treatments. The expressions for other parameters are obvious.

**Example 3.7.1**

Consider the BIB design with parameters $v^* = 6, b^* = 10, r^* = 5, k^* = 3$ and $\lambda^* = 2$. A solution for this design is given below:

<table>
<thead>
<tr>
<th>Table 3.7.1: BIB design with blocks shown in rows</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1, 3); (1, 2, 4); (2, 3, 0); (3, 4, 1); (4, 0, 2);</td>
</tr>
<tr>
<td>(5, 0, 1); (5, 1, 2); (5, 2, 3); (5, 3, 4); (5, 4, 0).</td>
</tr>
</tbody>
</table>

Deleting the blocks containing the treatment numbered 5, the solution of a RG design is obtained, with parameters $v = 5, b = 5, r = 3, k = 3, \lambda_1 = 1$ and $\lambda_2 = 2,$
whose blocks are shown below:

Table 3.7.2: RG design with blocks shown in rows

(0, 1, 3); (1, 2, 4); (2, 3, 0); (3, 4, 1); (4, 0, 2).

Results

In the present investigation, the concept of Dey (1986) is extended from $\lambda = 1$ to $\lambda = 2$. Here, it is shown that the existence of a BIB design with parameters $v > 4$, $k > 2$ and $\lambda = 2$ implies the existence of a RG design, by deleting a particular treatment and those blocks in which that particular treatment is present. Most of the RG designs obtained are a type of PBIB design with two associate classes. That is, these designs are GD type or Triangular type or Latin Square type or Cyclic type etc. These PBIB designs exist in the literature. In Mitchell and John (1976), optimal RG designs are given in Table 1 for the cases where $v \leq 12$ and $r \leq 10$ only. But, by using the above method, RG designs are obtained for the following five more cases also: $(v, k, r, b) = (15, 6, 4, 10), (15, 4, 8, 30), (20, 5, 8, 32), (27, 7, 7, 27)$ and $(36, 9, 7, 28)$.

Further examples of RG designs constructed using BIB designs are given in the following Table 3.7.3:
Table 3.7.3 showing the parameters of BIB designs and RG designs

<table>
<thead>
<tr>
<th>Parameters of BIB designs</th>
<th>Parameters of Constructed RG designs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$  $b$  $r$  $k$  $\lambda$</td>
<td>$v'$  $b'$  $r'$  $k'$  $\lambda_1'$  $\lambda_2'$</td>
</tr>
<tr>
<td>6  10  5  3  2</td>
<td>5  5  3  3  1  2</td>
</tr>
<tr>
<td>7  7  4  4  2</td>
<td>6  3  2  4  1  2</td>
</tr>
<tr>
<td>7  14  6  3  2</td>
<td>6  8  4  3  1  2</td>
</tr>
<tr>
<td>9  24  8  3  2</td>
<td>8  16  6  3  1  2</td>
</tr>
<tr>
<td>10  15  6  4  2</td>
<td>9  9  4  4  1  2</td>
</tr>
<tr>
<td>10  30  9  3  2</td>
<td>9  21  7  3  1  2</td>
</tr>
<tr>
<td>11  11  5  5  2</td>
<td>10  6  3  5  1  2</td>
</tr>
<tr>
<td>13  26  8  4  2</td>
<td>12  18  6  4  1  2</td>
</tr>
<tr>
<td>16  16  6  6  2</td>
<td>15  10  4  6  1  2</td>
</tr>
<tr>
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<td>15  30  8  4  1  2</td>
</tr>
<tr>
<td>21  42  10  5  2</td>
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</tr>
<tr>
<td>28  36  9  7  2</td>
<td>27  27  7  7  1  2</td>
</tr>
<tr>
<td>37  37  9  9  2</td>
<td>36  28  7  9  1  2</td>
</tr>
</tbody>
</table>

3.8 GRAPHICAL REPRESENTATION OF RG DESIGNS

A RG design can have an associated adjacency graph for its concurrence matrix, whose vertex set consists of the treatments of the design, with two distinct vertices are joined by an edge precisely when they are contained in $\lambda_2$ blocks of the design (Mitchell and John, 1976). For the designs considered, each row of the concurrence matrix of the design contains $n_1\lambda_1's$ and $n_2\lambda_2's$ and $r$ on the diagonal. Thus,

$$NN' = (r - \lambda_1)I + \lambda_1J + T$$  \hspace{0.5cm} (1)

where $J$ is a $v \times v$ matrix of 1’s and $T$ is a symmetric $v \times v$ matrix of 0’s and
1’s with \( n_2 \) 1’s in each row (or column) and 0’s along the diagonal. Here \( T \) is the adjacency matrix of a regular graph \( G_{v,t} \) of degree \( t \), where \( t = n_2 \). \( G_{v,t} \) has \( v \) vertices, and vertex \( i \) is connected to vertex \( j \) if and only if \( T_{ij} = 1 \). Each vertex is therefore joined to \( t = n_2 \) other vertices, hence by definition \( G_{v,t} \) is a regular graph of degree \( t \). It is evident that the set of all graphs of degree \( t \) with \( v \) vertices generates the set of all concurrence matrices in every case for which:

\[
 r(k - 1) = m(v - 1) \pm t
\]

where \( m \) is an integer. Using this property of the regular graph design, new designs are constructed.

**Example 3.8.1**

Now, the design given in Example 3.5.1 with parameters \( v_+ = 6 \), \( b_+ = 14 \), \( r_+ = 10 \), \( k_+ = (4, 4, 4, 5, 5, 5) \), \( \lambda_1+ = 6 \) and \( \lambda_2+ = 7 \) having the following blocks, is considered: \((1, 2, 3, 6), (1, 2, 4, 6), (1, 2, 5, 6), (1, 3, 4, 6), (1, 3, 5, 6), (1, 4, 5, 6), (2, 3, 4, 6), (2, 3, 5, 6), (2, 4, 5, 6), (3, 4, 5, 6), (1, 2, 3, 4, 5), (1, 2, 3, 4, 5), (1, 2, 3, 4, 5), (1, 2, 3, 4, 5) \) and \((1, 2, 3, 4, 5) \).

The concurrence matrix is:

\[
N_{d+}N_{d+}' = \begin{pmatrix}
10 & 7 & 7 & 7 & 7 & 6 \\
7 & 10 & 7 & 7 & 7 & 6 \\
7 & 7 & 10 & 7 & 7 & 6 \\
7 & 7 & 7 & 10 & 7 & 6 \\
7 & 7 & 7 & 7 & 10 & 6 \\
6 & 6 & 6 & 6 & 6 & 10 \\
\end{pmatrix}
\]

\[
= (r - \lambda_1)I + \lambda_1J + T
\]

\[
= (10 - 6)I + 6J + T
\]

\[
= 4I + 6J + T \quad \text{(using relation (1))}
\]

where,
The associated adjacency graph of the design $d_+$ is:

\[
T = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Graph 3.8.1 (Adjacency Graph)

which is a disconnected graph with two components.

Remarks 3.8.1

This family of constructed RG designs has a disconnected graph with two components; one, a complete graph of degree $(v - 1)$, and the other, an isolated vertex.

Example 3.8.2

An optimal design from Mitchell and John (1976) in Table 1, with parameters $v = 6$, $k = 2$, $r = 3$, $b = 9$, $\lambda_1 = 0$, $\lambda_2 = 1$, $C_2 = 0.30$, and $E = 0.556$ (Ref. P. SR6), is considered.
The corresponding design is given by:

Table 3.8.1: RG design with blocks shown in columns

\[
d_1 = \begin{pmatrix}
1 & 3 & 5 & 1 & 3 & 5 & 1 & 3 & 5 \\
2 & 4 & 6 & 6 & 2 & 4 & 4 & 6 & 2
\end{pmatrix}
\]

The Concurrence matrix is:

\[
NN' = \begin{pmatrix}
3 & 1 & 0 & 1 & 0 & 1 \\
1 & 3 & 1 & 0 & 1 & 0 \\
0 & 1 & 3 & 1 & 0 & 1 \\
1 & 0 & 1 & 3 & 1 & 0 \\
0 & 1 & 0 & 1 & 3 & 1 \\
1 & 0 & 1 & 0 & 1 & 3
\end{pmatrix}
= 3I + T \quad \text{(using relation (1))}
\]

Where the matrix T is given by:

\[
T = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

And, \( r(k - 1) = m(v - 1) \pm t \), \( m \) an integer (by using relation (2))
that is, \( 3(2-1) = m(6 - 1) \pm 2 \)
by solving, \( m = 1/5 \) or 1
Since \( m \) is defined as an integer, \( m = 1. \)

The associated regular graph of the design \( d_1 \) is:
Another regular graph of degree 3 is given by:

The corresponding $T$ matrix is given by:

$$
T = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
$$

Which gives the concurrence matrix given by:
\[ NN' = \begin{pmatrix} 3 & 0 & 1 & 1 & 1 & 0 \\
0 & 3 & 0 & 1 & 1 & 1 \\
1 & 0 & 3 & 0 & 1 & 1 \\
1 & 1 & 0 & 3 & 0 & 1 \\
1 & 1 & 1 & 0 & 3 & 0 \\
0 & 1 & 1 & 1 & 0 & 3 \end{pmatrix} \]

The corresponding design is given by:

Table 3.8.2: RG design with blocks shown in columns

\[ d_2 = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
3 & 4 & 5 & 4 & 5 & 6 & 5 & 6 & 6 \end{pmatrix} \]

To find the Efficiency of the design, \( \Omega^{-1} \) is calculated, using the relation:

\[ \Omega^{-1} = R - (1/k)NN' + (r^2/bk)J \]

\[ \Omega^{-1} = \begin{pmatrix} 2 & .5 & 0 & 0 & 0 & .5 \\
.5 & 2 & .5 & 0 & 0 & 0 \\
0 & .5 & 2 & .5 & 0 & 0 \\
0 & 0 & .5 & 2 & .5 & 0 \\
0 & 0 & 0 & .5 & 2 & .5 \\
.5 & 0 & 0 & 0 & .5 & 2 \end{pmatrix} \]

The eigen values of \( \Omega^{-1} \) are:

\( \left( 1.0, 1.5, 1.5, 2.5, 2.5, 3.0 \right) \)

Now, the optimality criterion \( C_2 = (\Sigma \gamma_i^{-1})^{-1} \), where \( \gamma_1, \gamma_2, \ldots, \gamma_v \) denote the eigenvalues of \( \Omega^{-1} \).

i.e. \( C_2 = 1/3.4667 = 0.2885 \).

And the conventional efficiency is given by:

\[ E = (v - 1)C_2/(r - C_2) = 0.532 \] (which is less than that given by Mitchell and
Now, the case with degree, \( t = 4 \), is considered.

One regular graph of degree 4 is given by:

![Graph 3.8.4 (Regular Graph)](image)

The corresponding \( T \) matrix is given by:

\[
T = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

Which gives the concurrence matrix given by:

\[
NN' = \begin{pmatrix}
4 & 0 & 1 & 1 & 1 & 1 \\
0 & 4 & 1 & 1 & 1 & 1 \\
1 & 1 & 4 & 0 & 1 & 1 \\
1 & 1 & 0 & 4 & 1 & 1 \\
1 & 1 & 1 & 1 & 4 & 0 \\
1 & 1 & 1 & 1 & 0 & 4
\end{pmatrix}
\]

The corresponding design is given by:
Table 3.8.3: RG design with blocks shown in columns

\[
d_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 5 & 6 & 5 & 6 \end{pmatrix}
\]

To find the Efficiency of the design, \( \Omega^{-1} \) is first calculated, using the relation:

\[
\Omega^{-1} = R - (1/k)NN' + (r^2/bk)J
\]

\[
\Omega^{-1} = \begin{pmatrix} 2.6667 & 0.66667 & 0.16667 & 0.16667 & 0.16667 & 0.16667 \\ 0.66667 & 2.6667 & 0.16667 & 0.16667 & 0.16667 & 0.16667 \\ 0.16667 & 0.16667 & 2.6667 & 0.66667 & 0.16667 & 0.16667 \\ 0.16667 & 0.16667 & 0.66667 & 2.6667 & 0.16667 & 0.16667 \\ 0.16667 & 0.16667 & 0.16667 & 0.16667 & 2.6667 & 0.66667 \\ 0.16667 & 0.16667 & 0.16667 & 0.16667 & 0.66667 & 2.6667 \end{pmatrix}
\]

The eigenvalues of \( \Omega^{-1} \) are:

\[
\left( \begin{array}{cccccc} 2.0 & 2.0 & 2.0 & 3.0 & 3.0 & 4.0 \end{array} \right)
\]

Now, the optimality criterion \( C_2 = (\Sigma \gamma_i^{-1})^{-1} \), where \( (\gamma_1, \gamma_2, ..., \gamma_v) \) denote the eigen values of \( \Omega^{-1} \).

i.e. \( C_2 = 1/2.4167 = 0.4138 \).

And the conventional efficiency is given by:

\[
E = (v - 1)C_2/(r - C_2) = 0.577 \text{(which is same as that given by Mitchell and John, 1976)}.
\]

Similarly, by using adjacency graphs, new designs in the class of RG designs can be constructed.\(^1\)

\(^1\)A paper based on this chapter and chapter 7, entitled “A new class of E-optimal regular graph designs” is accepted for publication and is expected to appear in 2007, in the Journal “Utilitas Mathematica”, published from Manitoba, Canada.
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