CHAPTER 2

PRELIMINARY CONCEPTS

2.1 Introduction
2.2 Definitions

2.1 INTRODUCTION

Some definitions and known results are included in this chapter, which are required for subsequent development of the thesis. All undefined concepts and notations used here are standard by now.

2.2 DEFINITIONS

In this section, most of the concepts and notations used in the thesis are defined.

Definition 2.2.1: Contrasts

Let \( t_i (i = 1, 2, ..., v) \) denote \( v \) treatment means. The linear function \( C = \sum_{i=1}^{v} l_i t_i \) is called a contrast of treatment means, if \( \sum_{i=1}^{v} l_i = 0 \), where \( l_i \)’s are given numbers. In other words, a contrast is a linear combination of treatment means such that the sum of the coefficients is zero.

Definition 2.2.2: Orthogonal contrasts

Two contrasts \( C_1 = \sum_{i=1}^{v} l_i t_i \) and \( C_2 = \sum_{i=1}^{v} m_i t_i \) are said to be orthogonal if \( \sum_i l_i m_i = 0 \). In other words, two contrasts are orthogonal if the sum of the products of the coefficients of the corresponding treatment means is zero.
Definition 2.2.3: Block Design

A block design is an arrangement of \( v \) treatments into \( b \) blocks of sizes \( k_1, k_2, ..., k_b \) respectively such that the \( i^{th} \) treatment occurs in \( r_i \) blocks and the \( i^{th} \) and \( j^{th} \) treatments occur together in \( \lambda_{ij} \) blocks.

Definition 2.2.4: Binary Design

If for the incidence matrix of a design \( N = (n_{ij}) \), \( n_{ij} = 0 \) or \( 1 \), then the design is called a binary design. i.e. each treatment occurs in each block at most once in a binary design.

Definition 2.2.5: Connected Design

A block design is said to be connected if all elementary treatment contrasts are estimable. For a connected design \( \text{Rank} \ (C_d) = v - 1 \).

Definition 2.2.6: Complementary Design

A design \( \bar{d} \) with parameters \( v_1, b_1, r_1, k_1, \lambda_1 \) is said the complementary design of a design \( d \) with parameters \( v, b, r, k, \lambda \), if \( \bar{d} \) is obtained by taking in the \( j^{th} \) block of \( \bar{d} \) all those treatments which do not occur in the \( j^{th} \) block of \( d \), \( j = 1, 2, ..., b \).

Obviously,
\[
  v_1 = v, b_1 = b, r_1 = b - r, k_1 = v - k, \lambda_1 = b - 2r + \lambda
\]

Definition 2.2.7: Dual Design

The dual of an incomplete block design \( d(v, b, k, r) \) is obtained by interchanging the blocks and the treatments. The parameters of the dual design \( d^* \) are \( v^* = b, b^* = v, r^* = k, \) and \( k^* = r \). Thus, the dual of design with incidence matrix \( N \), has incidence matrix \( N^* \) and concurrence matrix \( N'N \).

Definition 2.2.8: Incidence Matrix

The matrix \( N = (n_{ij}) : (i = 1, 2, ..., v; j = 1, 2, ..., b) \) associated with any
design, where \( n_{ij} \) is the number of times the \( i^{th} \) treatment occurs in the \( j^{th} \) block is called the incidence matrix of the design.

**Definition 2.2.9: Concurrence Matrix**

The concurrence matrix of a design is defined as the product \( NN' \) where \( N' \) is the transpose of the incidence matrix \( N \).

**Definition 2.2.10: C-matrix**

The \( C \)-matrix of a design \( d \) is the coefficient matrix of the reduced normal equations for estimating the treatment effects, and it is obtained as

\[
C_d = \text{diag } (r_1, r_2, ..., r_v) - (N_d \text{ diag } (1/k_1, 1/k_2, ..., 1/k_b) N_d')
\]

**Definition 2.2.11: Incomplete Block Design**

If the block size \( k \) is smaller than \( v \), the number of treatments, such a block design is called an incomplete block design.

**Definition 2.2.12: Equi-replicated design**

If all the treatments in a block design is replicated an equal number of times, then it is called an equi-replicated design.

**Definition 2.2.13: Resolvable Designs**

A design is said to be resolvable if its blocks can be divided into subgroups, such that every treatment is replicated exactly once in each of the subgroups. For a resolvable design it is necessary that \( b/r = v/k \) is an integer.

**Definition 2.2.14: \( \alpha \)-Resolvable Designs**

An incomplete block design with parameters \( v, b, r, k \) is said to be \( \alpha \)-resolvable if the blocks can be grouped into \( t \) classes \( S_1, S_2, ..., S_t \) each of \( \beta \) blocks, such that in each class every treatment is repeated \( \alpha \)-times.
Definition 2.2.15: Affine Resolvable Designs

If the number of treatments common between any two blocks belonging to different groups of a resolvable design is constant, such designs are called affine resolvable designs.

Definition 2.2.16: Balanced Incomplete Block (BIB) Design

A balanced incomplete block (BIB) design is an arrangement of \(v\) treatments into \(b\) blocks such that:

(i) each block contains \(k(<v)\) distinct treatments,

(ii) each treatment appears in \(r\) blocks,

(iii) every pair of treatments appears together in \(\lambda\) blocks.

The integers \(v, b, r, k, \lambda\) are called the parameters of the BIB design.

Definition 2.2.17: Symmetrical Balanced Incomplete Block (SBIB) Designs

The class of balanced incomplete block designs in which the number of blocks is equal to the number of treatments is called symmetrical BIB designs.

Let \(d(v, b, k, r, \lambda)\) be a BIB design. For a BIB design \(bk = vr\), and for a symmetrical BIB design \(k = r\).

Definition 2.2.18: Partially Balanced Incomplete Block (PBIB) Design

The definition of PBIB designs is based on the concept of association schemes. An abstract concept defined on \(v\) symbols (treatments) is called a \(m\)-class association scheme \((m \geq 2)\) if the following conditions are satisfied:

(i) Any two treatments \(\alpha\) and \(\beta\) are either \(1^{st}, 2^{nd}, \ldots, \) or \(m^{th}\) associates, the relation of association being symmetrical, i.e., if \(\alpha\) is the \(i\)-th associate of \(\beta\), then so is \(\beta\) of \(\alpha\).
(ii) Given a treatment $\alpha$, the number of treatments that are $i$-th associates of $\alpha$ is $n_i$ for $i = 1, 2, ..., m$, where the number $n_i$ does not depend on the treatment chosen, viz., $\alpha$, and

(iii) Given a pair of treatments $\alpha$ and $\beta$, which are mutually $i$-th associates, the number of treatments which are simultaneously $j$-th associate of $\alpha$ and $k$-th associate of $\beta$ is $P_{jk}^i$, where $P_{jk}^i$ does not depend on the pair of $i$-th associates chosen, viz., $\alpha$ and $\beta$.

Given an association scheme with $m$ classes ($m \geq 2$) we have a partially balanced incomplete block (PBIB) design based on the association scheme, if it is possible to arrange the $v$ treatments in $b$ blocks, such that

(i) each block contains $k(< v)$ distinct treatments;

(ii) each treatment appears in $r$ blocks; and

(iii) if the treatments $\alpha$ and $\beta$ are mutually $i$-th associates in the association scheme, then $\alpha$ and $\beta$ occur together in $\lambda_i$ blocks, where the integer $\lambda_i$ does not depend on the pair $(\alpha, \beta)$ so long as they are mutually $i$-th associates, $i = 1, 2, ..., m$. Further not all $i$’s are equal.

The integers $v, b, r, k, \lambda_i$ are called the parameters of the PBIB design.

This definition is due to Bose and Nair (1939).

**Definition 2.2.19: Simple PBIB Design**

A PBIB design with two associate classes is said to be simple if either $\lambda_1 \neq 0, \lambda_2 = 0$ or $\lambda_1 = 0, \lambda_2 \neq 0$.

**Definition 2.2.20: Group Divisible (GD) Design**

A binary design $d(v, b, k, r)$ having $v = mn$ treatments is called a Group Divisible (GD) design, if the treatment set can be divided into $m$ groups each consisting of size $n$ treatments, such that a pair of treatments occurs together in
$\lambda_1$ blocks if they belong to the same group, and in $\lambda_2$ blocks if they belong to the different groups.

A GD design is:

(i) Singular (S) if $r - \lambda_1 = 0$,

(ii) Semi-regular (SR) if $r - \lambda_1 > 0$, $rk - v\lambda_2 = 0$,

(iii) Regular (R) if $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$.

Definition 2.2.21: Triangular PBIB Design

A PBIB design with two associate classes is said to be triangular if the number of treatments $v = n(n - 1)/2$ and the association scheme is an array of $n$ rows and $n$ columns such that

(i) the positions in the principal diagonal of the scheme (upper left to lower right) are left blank,

(ii) the $n(n - 1)/2$ positions above the principal diagonal are filled by the treatment numbers $1, 2, ..., n(n - 1)/2$,

(iii) the $n(n - 1)/2$ positions below the diagonal are so filled that the array is symmetrical about the principal diagonal, and

(iv) for any treatment $i$ the first associates are exactly those treatments which lie in the same row as $i$.

Definition 2.2.22: Latin Square Type PBIB Design

Let a square array of $n$ rows and $n$ columns be formed with $n^2$ treatment numbers from 1 to $n^2$, so that two treatments are first associates if they occur in the same row or the same column of the array and second associates otherwise. A design with the above array as association scheme is said to belong to the sub-type $L_2$. 
Definition 2.2.23: Cyclic PBIB Design

An association scheme on \( v \) treatments is called cyclic design if the first associates of the treatment \( i \) are

\[
(i + d_1, i + d_2, ..., i + d_{n_1}) \mod v,
\]

others being second associates of \( i \), where the \( d_i \)'s satisfy the following conditions:

(i) The elements \( d_i \) are all different and \( 0 < d_i < v \) for \( i = 1, 2, ..., n_1 \), and

(ii) Among the \( n_1(n_1 - 1) \) differences \((d_i - d_j)\) reduced mod \( v \), each of the elements \( d_1, d_2, ..., d_{n_1} \) occurs \( \alpha \) times and each of the elements \( e_1, e_2, ..., e_{n_2} \) occurs \( \beta \) times, where \( d_i, (i = 1, 2, ..., n_1) \) and \( e_i, (i = 1, 2, ..., n_2) \) are all distinct non-zero elements of the module of \( v \) elements \( 0, 1, 2, ..., v - 1 \) and \( \alpha \neq \beta \).

Definition 2.2.24: Variance Balanced Design

A block design is said to be variance balanced if it permits the estimation of all estimable normalized treatment contrasts with the same variance.

Definition 2.2.25: Efficiency Balanced (EB) Design

A design \( d(v, b, k, r) \) is said to be efficiency balanced (EB) if for all treatment contrasts \( s'T \)

\[
M_o s = \mu s
\]

where \( \mu \) is the unique nonzero eigen value of \( M_o \) with multiplicity \( (v - 1) \) and

\[
M_o = R^{-1} P - J r' / n, \quad \text{and} \quad P = N K^{-1} N'
\]

where \( N \) is the \( v \times b \) incidence matrix, \( r \) is the \( v \times 1 \) vector of treatment replications, \( k \) is the \( b \times 1 \) vector of block sizes, \( R \) and \( K \) denote the diagonal matrices with diagonal elements as \( r \) and \( k \), and \( R^{-1} \) and \( K^{-1} \) are their inverses, and \( n \) denotes the total number of units.
It follows from Jones (1959) and Calinski (1971) that if there exists a set of \((v - 1)\) linearly independent contrasts \((s_{ij})\) such that \(\rho_i\) of them satisfy the equation

\[ M_o s_{ij} = \mu_i s_{ij}, \quad i = 1, 2, ..., m; j = 1, 2, ..., \rho_i \]

where \(\mu_i\)’s are the distinct eigen values of \(M_o\) with multiplicities \(\rho_i\), then effective information obtained on all the \(\rho_i\) contrasts is \((1 - \mu_i)\).

This definition is given by Puri and Nigam (1975).

**Definition 2.2.26: General Efficiency Balanced Design**

When an incomplete block design is compared against any other design i.e., either completely randomized or randomized block, both having the same number of treatments but not necessarily the same number of replications such that the ratio of the variances of the estimates of any treatment contrast for the two designs is constant, then such an incomplete block design has been called general efficiency balanced design.

This definition is given by Das and Ghosh (1985).

**Definition 2.2.27: Partially Efficiency Balanced (PEB) Design**

A design \(d(v, b, k, r)\) is said to be a PEB design with \(m\)-efficiency classes if

(i) there exists a set of \((v - 1)\) linearly independent contrasts \(s_i, \quad i = 1, 2, ..., m\) such that \(\rho_i\) of them satisfy the equation

\[ M_o s_{ij} = \mu_i s_{ij}, \quad i = 1, 2, ..., m; j = 1, 2, ..., \rho_i \]

so that the efficiency factor associated with every contrast of the \(i^{th}\) class is \((1 - \mu_i)\) where \(\mu_i(i = 1, 2, ..., m)\) are eigen values of \(M_o\) with multiplicities \(\rho_i(\Sigma \rho_i = v - 1)\), and

(ii) there exists mutually orthogonal idempotent matrices \(L_i(i = 1, 2, ..., m)\) of ranks \(\rho_i\) such that
\[ M_o = \sum_{i=1}^{m} \mu_i L_i, \text{ and } \sum_{i=1}^{m} L_i = I - J r'/n \]

where \( M_o = R^{-1} P - J r'/n \), \( P = NK^{-1}N' \), and \( N \) is the \( v \times b \) incidence matrix, \( r \) is the \( v \times 1 \) vector of treatment replications, \( k \) is the \( b \times 1 \) vector of block sizes, \( R \) and \( K \) denote the diagonal matrices with diagonal elements as \( r \) and \( k \), and \( R^{-1} \) and \( K^{-1} \) are their inverses, and \( n \) denotes the total number of units.

The parameters of PEB design with \( m \)-efficiency classes may now be written as \( v, b, r, k, \mu_i, \rho_i, L_i (i = 1, 2, ..., m) \).

This definition was given by Puri and Nigam (1977).

The EB design may be regarded as trivial PEB design with only one efficiency class and with the parameters \( v, b, r, k, \mu, \rho = v - 1, L = (I - J r'/n) \).

**Definition 2.2.28: Simple PEB Design**

A particular class of two-efficiency class PEB designs having \( \mu_1 \neq 0 \) and \( \mu_2 = 0 \) with multiplicities \( \rho_1 \) and \( \rho_2 = v - \rho_1 - 1 \) may be of special interest because of their simple analysis. Such a class of designs is termed as simple PEB or PEB(S). For such designs \( M_o \) is given by:

\[ M_o = \mu_1 L_1 \]

Orthogonally supplemented balanced designs considered by Calinski (1971), affine resolvable PBIB designs, semi-regular and singular group divisible PBIB designs, PBIB designs obtained through partial geometry \( (r, k, t) \), simple lattice designs and linked block designs are all PEB(S). Hence the analysis of all such PBIB designs can be greatly simplified.

This definition is due to Puri and Nigam (1977).
Definition 2.2.29: Efficiency of a Block Design

The efficiency $E$ of a block design is the average variance of all pairs of treatment differences, divided into the (hypothetical) minimum, which would be achieved by a randomized block design, if it existed.

For calculating the efficiency $E$, the variance-covariance matrix $\Omega$ for treatment parameters is first defined, as given by Tocher (1952), where

$$\Omega^{-1} = R - (1/k)NN' + (r^2/bk)J$$

Here $N$ is the $v \times b$ incidence matrix of the design and $J$ is a $v \times v$ matrix of one’s. Let the eigen values of $\Omega^{-1}$ be $(\gamma_1, \gamma_2, ..., \gamma_v)$. Now, the efficiency is given by:

$$E = (v - 1)C_2/(r - C_2)$$

where $C_2 = (\Sigma_i \gamma_i^{-1})^{-1}$

where $v$ denotes the number of treatments and $r$ the number of replications of the treatments.

Definition 2.2.30: Regular Graph (RG) Design

If a design is binary, and if all the diagonal elements of its concurrence matrix are equal and all the off-diagonal elements of its concurrence matrix are differing by at most one, then the design is called a regular graph design.

This definition is due to John and Mitchell (1977).

Definition 2.2.31: Semi-Regular Graph (SRG) Design

If a design is binary, and if all the diagonal elements and off-diagonal elements of its concurrence matrix are differing by at most one, then the design is called a semi-regular graph design.

This definition is due to Jacroux (1985).
Definition 2.2.32: Balanced Treatment Incomplete Block (BTIB) Design

A connected block design $d(v+1, b, k)$ having $v+1$ treatments arranged in $b$ blocks of size $k$, binary in the test treatments, is called a balanced treatment incomplete block (BTIB) design with parameters $r_o$, $r$, $\lambda_0$ and $\lambda_1$ if:

(i) $\lambda_{01} = ... = \lambda_{0v} = \lambda_0$

(ii) $\lambda_{12} = \lambda_{13} = ... = \lambda_{v-1,v} = \lambda_1$

where $\lambda_{ij} = \sum_{p=1}^{b} n_{ip} n_{jp}; i, j = 0, 1, 2, ..., v.$

This definition is due to Bechhofer and Tamhane (1981).

Definition 2.2.33: Group Divisible Treatment (GDT) Design

A connected block design $d(v+1, b, k)$ having $v+1$ treatments arranged in $b$ blocks of size $k$, binary in the test treatments, is called a GDT design with parameters $m$, $n$, $\lambda_0$, $\lambda_1$ and $\lambda_2$, if the treatments $0, 1, ..., v$ can be partitioned into $m+1$ disjoint groups $V_0, V_1, ..., V_m$ of sizes $v_0, v_1, ..., v_m$ such that the following conditions hold:

(i) $V_0 = \{0\}$

(ii) $v_1 = ... = v_m = n$

(iii) $\lambda_{0j} = \lambda_0$ for $j = 1, ..., v$.

(iv) For $p, q \in V_i (p \neq q; i = 1, ..., m)$, $\lambda_{pq} = \lambda_1$.

(vi) For $p \in V_i$, $q \in V_j (i, j = 1, ..., m; i \neq j)$, $\lambda_{pq} = \lambda_2$

This definition is due to Jacroux (1989).

Definition 2.2.34: Step S-type Design

Let the design $d(v+1, b, k; t, s)$ be a BTIB design, where integers $t \in \{0, 1, ..., k-1\}$ and $s \in \{0, 1, ..., b-1\}$, such that
(i) \( n_{ij} \in \{0, 1\}, \ i = 1, 2, ..., v; \ j = 1, ..., b; \)

(ii) \( n_{01} = ... = n_{0b} = t + 1; \)

(iii) \( n_{0,s+1} = ... = n_{0b} = t \)

when \( s > 0 \), the BTIB \((v, b, k; t, s)\) is called a Step \((S)\) type \((r_o > bt \text{ and } r_o < bt)\) design.

This definition is due to Cheng et al. (1988).

**Definition 2.2.35: R-type Design**

A design \(d(v + 1, b, k; t, s)\) is called a \(R\)-type (Rectangular Type: \( r_o = bt \)) design when \( s = 0 \). In \( R\)-type design, control treatment is replicated same number of times in all the blocks.

**Definition 2.2.36: Factorial Experiment**

Experiments in which the effects (main effects and interactions) of more than one factor each at two or more levels are studied together, are called factorial experiments.

**Definition 2.2.37: Graph**

A graph \( G = (V, E) \) consists of a set of objects \( V = (v_1, v_2, ...) \) called vertices; and another set \( E = (e_1, e_2, ...) \), whose elements are called edges, such that each edge \( e_k \) is identified with an unordered pair \((v_i, v_j)\) of vertices. The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

**Definition 2.2.38: Degree of a Graph**

The number of edges incident on a vertex \( v_i \), is called the degree \( d(v_i) \) of the vertex \( v_i \).
Definition 2.2.39: Connected Graph

A graph \( G \) is said to be connected if there is at least one path between every pair of vertices in \( G \); otherwise \( G \) is disconnected. A disconnected graph consists of two or more connected sub-graphs. Each of these sub-graphs is called a component.

Definition 2.2.40: Isolated Vertex

A vertex having no incident edge is called an isolated vertex. In other words, isolated vertices are vertices with zero degree.

Definition 2.2.41: Bipartite Graph

A graph \( G \) is called a bipartite if its vertex set \( V \) can be decomposed into two disjoint subsets \( V_1 \) and \( V_2 \) such that every edge in \( G \) joins a vertex in \( V_1 \) with a vertex in \( V_2 \).

Definition 2.2.42: Regular Graph

A graph \( G \) in which all vertices are of equal degree is called a regular graph.

Definition 2.2.43: Incidence Graph of a design

A block design can be represented by a bipartite graph \( K_{v,b} \) whose vertices are treatments and blocks; where \( v \) is the number of treatments and \( b \) the number of blocks. An edge joins two vertices if one is a treatment and the other a block such that the treatment appears in the block. The diagram obtained is called Incidence graph or Levi graph of the design.

Definition 2.2.44: Isomorphic Graph

Two graphs \( G \) and \( G' \) are said to be isomorphic if there is one-to-one correspondence between their vertices and their edges such that the incidence relationship is preserved. In other words, suppose that edge \( e \) is incident on vertices
\(v_1\) and \(v_2\) in \(G\); then the corresponding edge \(e'\) in \(G'\) must be incident on the vertices \(v_1'\) and \(v_2'\) that correspond to \(v_1\) and \(v_2\), respectively.

**Definition 2.2.45: A-optimality**

Let \(D\) be the class of all available designs and \(T_{v \times 1}\) is the vector of treatment effects. Let \(\eta = LT\) contains all treatment contrasts, and \((\hat{\eta}_d)\) be the BLUE of \(\eta\) using design \(d\) with \(V(\hat{\eta}_d) = V_d\). A design \(d^*\) in \(D\) is said to be \(A\)-optimal in \(D\) iff \(tr(V_d) \leq tr(V_{d^*})\) for any design \(d \in D\).

**Definition 2.2.46: D-optimality**

A design \(d^*\) in \(D\) is said to be \(D\)-optimal in \(D\) iff \(det(V_d) \leq det(V_{d^*})\) for any design \(d \in D\).

**Definition 2.2.47: E-optimality**

A design \(d^*\) in \(D\) is said to be \(E\)-optimal in \(D\) if maximum characteristic root of \(V_{d^*}\) is less than or equal to maximum characteristic root of \(V_d\) for any design \(d \in D\).
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