CHAPTER 3

On the energy of signed graphs

In this Chapter, we characterize unicyclic signed graphs with minimal energy. We show that for each positive integer \( n \geq 3 \), there exists a pair of connected, non-cospectral and equienergetic unicyclic signed graphs on \( n \) vertices with one constituent balanced and other constituent unbalanced. It is shown that for each positive integer \( n \geq 4 \), there exists a pair of connected, non-cospectral and equienergetic signed graphs of order \( n \) with both constituents unbalanced.

3.1 Introduction

A signed graph is defined to be a pair \( S = (G, \sigma) \), where \( G = (V, \mathcal{E}) \) is the underlying graph and \( \sigma : \mathcal{E} \rightarrow \{-1, 1\} \) is the signing function. The sets of positive and negative edges of \( S \) are respectively denoted by \( \mathcal{E}^+ \) and \( \mathcal{E}^- \). Thus \( \mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \). Our signed graphs have simple underlying graphs. A signed graph is said to be homogeneous if all of its edges have either positive sign or negative sign and heterogeneous, otherwise. A graph can be considered to be a homogeneous signed graph with each edge positive; thus signed graphs become a generalization of graphs. Throughout this Chapter bold lines denote positive edges and dotted lines denote negative edges. The sign of a signed graph is defined as the product of signs of its edges. A signed graph is said to be positive (respectively, negative) if its sign is positive (respectively, negative) i.e., it contains an even (respectively, odd) number of negative edges. A signed graph is said to be all-positive (respectively, all-negative) if all of its edges are positive (respectively, negative). A signed graph is said to be balanced if each of its cycles is positive and unbalanced, otherwise. We denote by \(-S\) the signed graph obtained by negating each edge of \( S \) and call it the negative of \( S \). We call balanced cycle a positive cycle and an unbalanced cycle a negative cycle and respectively denote them by \( C_n \) and \( C^n \), where \( n \) is number of vertices.

The adjacency matrix of a signed graph \( S \) whose vertices are \( v_1, v_2, \ldots, v_n \) is the \( n \times n \) matrix \( A(S) = (a_{ij}) \), where
\[ a_{ij} = \begin{cases} 
\sigma(v_i, v_j), & \text{if there is an edge from } v_i \text{ to } v_j, \\
0, & \text{otherwise.} 
\end{cases} \]

Clearly, \( A(S) \) is real symmetric and so all its eigenvalues are real. The characteristic polynomial \( |xI - A(S)| \) of the adjacency matrix \( A(S) \) of signed graph \( S \) is called the characteristic polynomial of \( S \) and is denoted by \( \phi_S(x) \). The eigenvalues of \( A(S) \) are called the eigenvalues of \( S \). The set of distinct eigenvalues of \( S \) together with their multiplicities is called the spectrum of \( S \). Let \( S \) be a signed graph of order \( n \) with distinct eigenvalues \( x_1, x_2, \cdots, x_k \) and let their respective multiplicities be \( m_1, m_2, \cdots, m_k \). Then we write the spectrum of \( S \) as \( \text{spec}(S) = \{x_1^{(m_1)}, x_2^{(m_2)}, \cdots, x_k^{(m_k)}\} \).

The following is the coefficient Theorem for signed graphs [1].

**Theorem 3.1.1.** If \( S \) is a signed graph with characteristic polynomial

\[ \phi_S(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \]

then

\[ a_j = \sum_{L \in \mathcal{L}_j} (-1)^{p(L)} 2^{|c(L)|} \prod_{Z \in c(L)} s(Z), \]

for all \( j = 1, 2, \cdots, n \), where \( \mathcal{L}_j \) is the set of all basic figures \( L \) of \( S \) of order \( j \), \( p(L) \) denotes number of components of \( L \), \( c(L) \) denotes the set of all cycles of \( L \) and \( s(Z) \) the sign of cycle \( Z \).

From this result, it is clear that the spectrum of a signed graph remains invariant by changing the signs of non cyclic edges. Here we note that whenever we need to compare the energy of two signed graphs we use \( a_j(S) \) for \( j \)-th coefficient of characteristic polynomial of \( S \) instead of \( a_j \).

The spectral criterion for balance of signed graphs given by Acharya [1] is as follows.
Theorem 3.1.2. A signed graph is balanced if and only if it is cospectral with the underlying unsigned graph.

The Cartesian product of two signed graphs \( S_1 = (V_1, E_1, \sigma_1) \) and \( S_2 = (V_2, E_2, \sigma_2) \) denoted by \( S_1 \times S_2 \) is the signed graph \( (V_1 \times V_2, E, \sigma) \), where the edge set is that of the Cartesian product of underlying unsigned graphs and the sign function is defined by

\[
\sigma((u_i, v_j), (u_k, v_l)) = \begin{cases} 
\sigma_1(u_i, u_k), & \text{if } j = l, \\
\sigma_2(v_j, v_l), & \text{if } i = k.
\end{cases}
\]

The Kronecker product of two signed graphs \( S_1 = (V_1, E_1, \sigma_1) \) and \( S_2 = (V_2, E_2, \sigma_2) \) denoted by \( S_1 \otimes S_2 \) is the signed graph \( (V_1 \times V_2, E, \sigma) \), where edge set is that of the Kronecker product of underlying unsigned graphs and the sign function is defined by

\[
\sigma((u_i, v_j), (u_k, v_l)) = \sigma_1(u_i, u_k) \sigma_2(v_j, v_l).
\]

Let \( S \) be a signed graph with vertex set \( V \). Switching \( S \) by set \( X \subset V \), means reversing the signs of all edges between \( X \) and its complement.

Another way to define switching is by means of a function \( \theta : V \to \{+1, -1\} \), called a switching function. Switching \( S \) by \( \theta \) means changing \( \sigma \) to \( \sigma^\theta \) defined by

\[
\sigma^\theta(u, v) = \theta(u)\sigma(u, v)\theta(v).
\]

We denote switched graph by \( S^\theta \). Two signed graphs are said to be switching equivalent if one can be obtained from the other by switching. Switching equivalence is an equivalence relation on the signings of a fixed graph. An equivalence class is called a switching class. A switching class of \( S \) is denoted by \([S]\). If \( S' \) is isomorphic to a switching of \( S \), we say \( S \) and \( S' \) are switching isomorphic.

The concept of energy was extended to signed graphs by Germina, Hameed and Zaslavsky \[23\]. They defined the energy of a signed graph \( S \) to be the sum of absolute values of eigenvalues of \( S \).

A connected signed graph of order \( n \) is said to be unicyclic if the number of its edges is also \( n \). The girth of signed graph is the length of its smallest cycle and
we denote it by \( g \). Let \( S^g_n, n \geq g \geq 3 \) (respectively, \( S^g_u \)) denote the balanced (respectively, unbalanced) unicyclic signed graph of order \( n \) obtained by identifying the root vertex of a signed star on \( n - g + 1 \) vertices with a vertex of a positive (respectively, negative) cycle of order \( g \) (see Fig. 3.1) and let \( S(n, g) \) denote the set of all unicyclic signed graphs with \( n \) vertices and girth \( g \leq n \). For unicyclic graphs with minimal energy see [38, 53]. Caprossi et al. [15] posed the following conjecture based upon the results attained with the computer system AutoGraphix.

**Conjecture 3.1.3.** Among all connected graphs \( G \) with \( n \geq 6 \) vertices and \( n - 1 \leq m \leq 2(n - 2) \) edges, the graph with minimum energy are stars with \( m - n + 1 \) additional edges all connected to the same vertex for \( m \leq n + \left\lfloor \frac{(n-7)}{2} \right\rfloor \), and bipartite graphs with two vertices on one side, one of which is connected to all vertices on the other side, otherwise.

For \( m = n - 1 \) and \( m = 2(n - 2) \) Caprossi et al. [15] proved the conjecture. The following result of Hou [38] proves conjecture for \( m = n \).

**Theorem 3.1.4.** Let \( G \) be a unicyclic graph with \( n \geq 6 \) vertices and \( G \neq \mathcal{S}^3_n \). Then \( E(\mathcal{S}^3_n) < E(G) \), where \( \mathcal{S}^3_n \) is the graph obtained from the star graph with \( n \) vertices by adding an edge.

We show for \( m = n \), a signed analogue of the conjecture is true, that is, among all unicyclic signed graphs with \( n \geq 6 \) vertices and \( n \) edges, all signed graphs in \( [S^3_n] \) and \( [S^3_u] \) have the minimal energy.

### 3.2 Switching in signed graphs

The following result can be seen in [85].

**Lemma 3.2.1.** A signed graph is balanced if and only if it switches to an all-positive signing.

The following result shows that there are only two switching classes on signnings of unicyclic graph.
**Theorem 3.2.2.** There exists only two switching classes on the signings of fixed unicyclic graph.

**Proof.** Let $G$ be a unicyclic graph. By Lemma 3.2.1, all balanced unicyclic signed graphs on $G$ comprise one switching class. We show that all unbalanced signed graphs on $G$ are switching equivalent to an unbalanced signed graph with exactly one negative cyclic edge and all other edges positive. Let $S$ be any unbalanced signed graph on $G$. Choose a negative edge $e = (u, v)$. Then $S - e$ is balanced and hence by Lemma 3.2.1, $S - e$ switches to an all positive signed graph. Now return edge $e$ in the all positive signed graph of $S - e$. Again, by Lemma 3.2.1, with this switching $e$ must be a negative edge. Thus $S$ is switching equivalent to an unbalanced signed graph with exactly one negative cyclic edge and all other edges positive.

The following result shows that adjacency matrices of switching equivalent signed graphs are similar by means of a signature matrix.

**Theorem 3.2.3.** Signed graphs $S_1$ and $S_2$ with same underlying graph are switching equivalent if and only if their adjacency matrices satisfy $A(S_2) = D^{-1}A(S_1)D$ for some $(0, \pm 1)$-matrix $D$ whose diagonal has no zeroes.

Theorem 3.2.3 shows that switching equivalent signed graphs are always cospectral. It is not known whether the converse is true or not. However, we have examples of cospectral unbalanced signed graphs whose underlying graphs are non-isomorphic (Signed graphs $S_1$ and $S_2$ in Fig. 3.1). It is shown in Remark 3.3.8 (i) that all unbalanced signed graphs on a fixed unicyclic graph are cospectral. Thus there are only two cospectral classes on the signings of a fixed unicyclic graph, one for balanced and one for unbalanced. Therefore, by Theorem 3.2.2, the following result is true for unicyclic signed graphs.

**Theorem 3.2.4.** Two unicyclic signed graphs with the same underlying graph are cospectral if and only if they are switching equivalent.
3.3 Integral representation for energy of signed graphs and its applications

First we obtain Coulson’s integral formula and then discuss its consequences for signed graphs.

**Theorem 3.3.1.** Let $S$ be a signed graph with $n$ vertices having characteristic polynomial $\phi_S(x)$. Then

$$E(S) = \sum_{j=1}^{n} |x_j| = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{\imath x \phi_S'(\imath x)}{\phi_S(\imath x)} \right) dx,$$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of signed graph $S$, $\imath = \sqrt{-1}$ and $\int_{-\infty}^{\infty} F(x) dx$ denotes principle value of the respective integral.

**Proof.** Let $x_1, x_2, \ldots, x_n$ be the zeroes of polynomial $\phi_S(x)$. Then

$$\phi_S(x) = \prod_{j=1}^{n} (x - x_j) \quad \text{and} \quad \phi_S'(x) = \sum_{j=1}^{n} \prod_{k \neq j} (x - x_k),$$

so that $\frac{\phi_S'(x)}{\phi_S(x)} = \sum_{j=1}^{n} \frac{1}{x - x_j}$.

Using the integrals $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_j^2}{(x_j^2 + x^2)} dx = |x_j|$ and $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_j x}{(x_j^2 + x^2)} dx = 0$, we have

$$|x_j| = |x_j| + 0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_j^2}{(x_j^2 + x^2)} dx + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_j x}{(x_j^2 + x^2)} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_j^2 + \imath x_j x}{(x_j^2 + x^2)} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( 1 - \frac{\imath x}{\imath x - x_j} \right) dx.$$

Therefore, $E(S) = \sum_{j=1}^{n} |x_j| = \frac{1}{\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{n} \left( 1 - \frac{\imath x}{\imath x - x_j} \right) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{\imath x \phi_S'(\imath x)}{\phi_S(\imath x)} \right) dx.$

The following result is a consequence of Coulson’s integral formula.

**Theorem 3.3.2.** If $S$ is a signed graph on $n$ vertices, then

$$E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log |x^n \phi_S(\frac{\imath x}{x})| dx.$$
Proof. By Theorem 3.3.1, we have

\[
E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} (n - \frac{\iota x \phi'_S(\iota x)}{\phi_S(\iota x)}) dx \\
= \frac{1}{\pi} \int_{0}^{\infty} (n - \frac{\iota x \phi'_S(\iota x)}{\phi_S(\iota x)}) dx + \frac{1}{\pi} \int_{0}^{\infty} (n - \frac{\iota x \phi'_S(\iota x)}{\phi_S(\iota x)}) dx
\]

Put \(x = \frac{1}{y}\), so that

\[
E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} (n - \frac{\frac{1}{y} \phi'_S(\frac{1}{y})}{\phi_S(\frac{1}{y})}) \cdot \frac{1}{y^2} dy.
\]

Now, integrating by parts and taking \(u = \frac{1}{y}\) and \(dv = (n - \frac{\frac{1}{y} \phi'_S(\frac{1}{y})}{\phi_S(\frac{1}{y})})\), so that \(du = -\frac{1}{y^2} dy\) and \(v = \log |y^n \phi_S(\frac{1}{y})|\).

Therefore

\[
E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{1}{y} \phi'_{S}(\frac{1}{y}) \right) \log |y^n \phi_S(\frac{1}{y})| dy \\
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{y^2} \log |y^n \phi_S(\frac{1}{y})| dy.
\]

Using change of variable, the result follows. \(\square\)

**Theorem 3.3.3.** If \(S\) is a signed graph on \(n\) vertices with characteristic polynomial \(\phi_S(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n\), then

\[
E(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log \left( \sum_{j=0}^{\left\lfloor \frac{x}{2} \right\rfloor} (-1)^j a_{2j} x^{2j} \right) + \left( \sum_{j=0}^{\left\lfloor \frac{x}{2} \right\rfloor} (-1)^j a_{2j+1} x^{2j+1} \right) dx.
\]

**Proof.** Let \(\psi(x) = (-\iota x)^n \phi_S(\frac{x}{\iota})\) and note that

\[
\psi(x) = \sum_{j=0}^{n} a_j (-\iota x)^j = \sum_{j=0}^{\left\lfloor \frac{x}{2} \right\rfloor} (-1)^j a_{2j} x^{2j} - \iota \sum_{j=0}^{\left\lfloor \frac{x}{2} \right\rfloor} (-1)^j a_{2j+1} x^{2j+1},
\]

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where $a_0 = 1$ and $a_j = 0$ for $j > n$. By Theorem 3.3.2, $E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log |\psi(x)|$. Substituting the value for $\psi(x)$, the result follows.

![Diagram](image)

Fig. 3.1

We know from [14, Theorem 3.11] that a graph containing at least one edge is bipartite if and only if its spectrum, considered as a set of points on the real axis, is symmetric with respect to the origin. This is not true for signed graphs. There exist non bipartite signed graphs whose spectrum is symmetric about origin. Signed graphs $S_1$ and $S_2$ in Fig. 3.1 are clearly non bipartite, but $\text{spec}(S_1) = \text{spec}(S_2) = \{-\sqrt{5}, -1^{(2)}, 1^{(2)}, \sqrt{5}\}$. We say a signed graph has a pairing property if its spectrum is symmetric about the origin. We denote by $\Delta_n$, the set of all signed graphs on $n$ vertices with pairing property. The next result shows that all the odd coefficients of the characteristic polynomial of a signed graph in $\Delta_n$ are zero and all the even coefficients alternate in sign.

**Lemma 3.3.4.** Let $S \in \Delta_n$, then $\phi_S(x) = x^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j b_j x^{n-2j}$, where $b_j = |a_j|$ and $a_j$ is the $j$-th coefficient of characteristic polynomial for $j = 1, 2, \cdots, n$.

**Proof.** Assume $S \in \Delta_n$. Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ be the positive eigenvalues of $S$, where $p \leq \lfloor \frac{n}{2} \rfloor$. Then

$$\phi_S(x) = x^\delta \prod_{j=1}^{p} (x^2 - \alpha_j^2) = x^\delta \psi(x^2),$$

where $\psi(x^2) = \prod_{j=1}^{p} (x^2 - \alpha_j^2)$ is a polynomial in $x^2$ and $\delta \geq 0$ is a non negative integer. Using the fact that if the zeroes of a polynomial are real and positive then its coefficients alternate in sign, we see that the coefficients of $\psi(x^2)$ and hence $\phi_S(x)$ alternate in sign. Therefore the result follows. \qed
Remark 3.3.5. Let $S$ be a bipartite signed graph. Then $S$ has no odd cycles and consequently no basic figure of odd order. By Theorem 3.1.1, we see that the characteristic polynomial of $S$ is of the form $\phi_S(x) = x^d \psi(x^2)$, where $d = 0$ or $1$ and $\psi(x^2)$ is a polynomial in $x^2$. This shows that $S$ has the pairing property, i.e., $S \in \Delta_n$.

We now define a quasi-order relation for signed graphs in $\Delta_n$ and show it is possible to compare energy of signed graphs in $\Delta_n$.

Given signed graphs $S_1$ and $S_2$ in $\Delta_n$, by Theorem 3.3.4, for $i = 1, 2$, we have

$$\phi_{S_i}(x) = x^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j b_{2j}(S_i)x^{n-2j},$$

where $b_{2j}(S_i)$ are non-negative integers for all $j = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$. If $b_{2j}(S_1) \leq b_{2j}(S_2)$ for all $j = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$, then we define $S_1 \preceq S_2$. If in addition $b_{2j}(S_1) < b_{2j}(S_2)$ for some $j = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$, then we write $S_1 \prec S_2$. Clearly $\preceq$ is a quasi-order relation. The following result which is a consequence of Theorem 3.3.3 shows that the energy increases with respect to this quasi-order relation.

Theorem 3.3.6. If $S \in \Delta_n$, then

$$E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log[1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} b_{2j}(S)x^{2j}]dx.$$  

In particular, if $S_1, S_2 \in \Delta_n$ and $S_1 \prec S_2$ then $E(S_1) < E(S_2)$.

Put $b_j = |a_j|$, $j = 1, 2, \cdots, n$. Note $b_1 = 0$, $b_2 =$-number of edges of signed graph $S$ and so on. We denote by $m(S, j)$ the number of matchings of $S$ of size $j$. This number is independent of signing. We use the convention that $m(S, 0) = 1$. The following result shows that the even and odd coefficients of the characteristic polynomial of a unicyclic signed graph alternate in sign.

Theorem 3.3.7. Let $S \in S(n, g)$. Then $(-1)^j a_{2j} \geq 0$ for all $j \geq 0$ irrespective of $S$ is balanced or unbalanced and $g$ is odd or even. Moreover, if $g = 2r + 1$, $r \geq 1$, then $(-1)^j a_{2j+1} \geq 0$ (respectively, $\leq 0$) for all $j \geq 0$ if either $r$ is odd and $S$ is
balanced or $r$ is even and $S$ is unbalanced (respectively, if either $r$ is even and $S$ is balanced or $r$ is odd and $S$ is unbalanced).

**Proof.** If $g$ is even, then $S$ is bipartite. By Remark 3.3.5, $a_{2j+1} = 0$ for all $j \geq 0$ and $a_{2j} = (-1)^j b_{2j}$. This gives $(-1)^j a_{2j} = b_{2j} \geq 0$. Also if $g$ is odd, say $g = 2r + 1$, then $S$ is non-bipartite. Now, $a_{2j} = (-1)^j m(S, j)$ which gives $(-1)^j a_{2j} = m(S, j) \geq 0$. The odd coefficients in balanced and unbalanced case are respectively given by

\[
a_{2j+1} = \begin{cases} 
0, & \text{if } 2j + 1 < g, \\
-2(-1)^{j-r} m(S - C_g, j - r), & \text{if } 2j + 1 \geq g.
\end{cases}
\]

and

\[
a_{2j+1} = \begin{cases} 
0, & \text{if } 2j + 1 < g, \\
2(-1)^{j-r} m(S - C_g, j - r), & \text{if } 2j + 1 \geq g.
\end{cases}
\]

From this, the result follows. \qed

**Remark 3.3.8.**

(i) From the above result, it follows that all unbalanced signed graphs on a fixed unicyclic graph are cospectral.

(ii) It is now possible to compare the energy in unicyclic signed graphs of odd girth as well by means of a quasi-order relation defined on $b_j$’s.

Given two unicyclic signed graphs $S_1$ and $S_2$, by Theorem 3.3.7, for $i = 1, 2$, we have

\[
\phi_{S_i(x)} = \sum_{j \geq 0} \left\{( -1)^j b_{2j}(S_i)x^{n-2j} + (-1)^{j+[s]} b_{2j+1}(S_i)x^{n-(2j+1)}\right\},
\]

where $[s] = 1$ if the girth $g_i$ of $S_i$ satisfies $g_i = 2r_i + 1$ with either $r_i$ is even and $S_i$ is balanced or $r_i$ is odd and $S_i$ is unbalanced, otherwise $[s] = 0$. If $b_j(S_1) \leq b_j(S_2)$ for all $j \geq 0$, then we define $S_1 \preceq S_2$. If in addition $b_j(S_1) < b_j(S_2)$ for some $j$, then we write $S_1 \prec S_2$. The following result is a consequence of Theorem 3.3.3.
and it shows that the energy increases with respect to this quasi-order relation.

**Theorem 3.3.9.** Let $S$ be a unicyclic signed graph of order $n$. Then

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log([\sum_{j=0}^{\lfloor n/2 \rfloor} b_{2j}(S)x^{2j})^2 + [\sum_{j=0}^{\lfloor n/2 \rfloor} b_{2j+1}(S)x^{2j+1}]^2] dx.$$ 

In particular, if $S_1$ and $S_2$ are unicyclic signed graphs and $S_1 \prec S_2$, then $E(S_1) < E(S_2)$.

We now show that all signed graphs on a unicyclic graph of odd girth are equienergetic.

**Corollary 3.3.10.** For each positive integer $n \geq 3$, there exists a pair of connected, non cospectral and equienergetic unicyclic signed graphs of order $n$ with one constituent balanced and other constituent unbalanced.

**Proof.** Let $G$ be a unicyclic graph of order $n$ and odd girth $g$. Let $S$ be any balanced signed graph on $G$ and $T$ be any unbalanced signed graph on $G$. Then $S$ and $T$ are non cospectral by Theorem 3.1.2. The coefficients of signed graphs $S$ and $T$ are related as follows

$a_{2j+1}(S) = -a_{2j+1}(T)$ for all $j = 0, 1, 2, \cdots, [\frac{n}{2}]$ and $a_{2j}(S) = a_{2j}(T)$ for all $j = 1, 2, \cdots, [\frac{n}{2}]$. Thus $b_j(S) = b_j(T)$ for all $j = 1, 2, \cdots, n$. By Theorem 3.3.9 $E(S) = E(T)$.

Now we use Theorem 3.3.9 to compare the energies of signed graphs obtained from a unicyclic bipartite graph.

**Theorem 3.3.11.** Let $G$ be a unicyclic graph of order $n$ and even girth $g$ i.e., bipartite unicyclic graph, and let $S$ be any balanced signed graph on $G$ and $T$ be any unbalanced one. Then

(i) $E(S) < E(T)$ if and only if $g \equiv 0 \pmod{4}$;

(ii) $E(S) > E(T)$ if and only if $g \equiv 2 \pmod{4}$.

**Proof.** Let $G$ be a unicyclic graph of order $n$ and even girth $g \geq 4$ and let $S$ and $T$ respectively be any balanced signed graph and any unbalanced signed graph on
The coefficients of $S$ are given by

$a_{2j}(S) = m(S, j)$ for all $j = 1, 2, \ldots, \frac{g}{2} - 1$; $a_{g+2j}(S) = -2(-1)^j m(S - C_g, j) + (-1)^{\frac{g}{2}+j} m(S, \frac{g}{2} + j)$ for all $j = 0, 1, \ldots, \lfloor \frac{n-g}{2} \rfloor$ and $a_{2j+1}(S) = 0$ for all $j = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

whereas the coefficients of $T$ are given by

$a_{2j}(T) = m(T, j)$ for all $j = 1, 2, \ldots, \frac{g}{2} - 1$; $a_{g+2j}(T) = 2(-1)^j m(T - C_g, j) + (-1)^{\frac{g}{2}+j} m(T, \frac{g}{2} + j)$ for all $j = 0, 1, \ldots, \lfloor \frac{n-g}{2} \rfloor$ and $a_{2j+1}(T) = 0$ for all $j = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Two cases arise here (i) $g \equiv 0 \pmod{4}$ and (ii) $g \equiv 2 \pmod{4}$.

**Case (i) $g \equiv 0 \pmod{4}$.** We have $b_{2j+1}(S) = b_{2j+1}(T) = 0$ for all $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$; $b_{2j}(S) = b_{2j}(T)$ for all $j = 1, 2, \ldots, \frac{g}{2} - 1$; $b_{g+2j}(S) = | -2m(S - C_g, j) + m(S, \frac{g}{2} + j) |$ for all $j = 0, 1, \ldots, \lfloor \frac{n-g}{2} \rfloor$ and $b_{g+2j}(T) = | 2m(T - C_g, j) + m(T, \frac{g}{2} + j) |$ for all $j = 0, 1, \ldots, \lfloor \frac{n-g}{2} \rfloor$.

Clearly, $b_j(S) \leq b_j(T)$ for all $j = 1, 2, \ldots, n$. In particular $b_g(S) < b_g(T)$. Therefore $S < T$ and by Theorem 3.3.9, $E(S) < E(T)$.

The proof of case (ii) follows on similar lines.

3.4 Unicyclic signed graphs with minimal energy

Gill and Acharya [24] obtained the following recurrence formula for the characteristic polynomial of a signed graph.

**Lemma 3.4.1.** Let $S$ be a signed graph and $v$ be its arbitrary vertex. Then

$$
\phi_S(x) = x\phi_{S-v}(x) - \sum_{(w,v) \in \delta} \phi_{(S-v-w)}(x) - 2\sum_{Z \in C^+(v)} \phi_{(S-V(Z))}(x) - \sum_{Z \in C^-(v)} \phi_{(S-V(Z))}(x),
$$

where $C^+(v)$ and $C^-(v)$ denote the set of positive and negative cycles containing vertex $v$.

Now we have the following result.

**Lemma 3.4.2.** Let $S \in S(n, g)$ be unbalanced, and let $(u, v)$ be the pendant edge of $S$ with pendant vertex $v$. Then
\[ b_j(S) = b_j(S - v) + b_{j-2}(S - v - u). \]

**Proof.** Since \( S \) is unicyclic and \( v \) is a pendant vertex, Lemma 3.4.1 takes the form

\[ \phi_S(x) = x\phi_{(S-v)}(x) - \phi_{(S-v-u)}(x) \]

which gives

\[ a_j(S) = a_j(S - v) - a_{j-2}(S - v - u). \]

We now claim that the coefficients \( a_j(S - v) \) and \( a_{j-2}(S - v - u) \) are of opposite signs. In case \( S \) is bipartite, then all is clear by Remark 3.3.5. Assume \( S \) is non bipartite and both the signed graphs \( S - v \) and \( S - v - u \) contain the odd cycle \( C_g \), then claim follows by Theorem 3.3.7. Finally, suppose only \( S - v - u \) is acyclic; for odd \( j \), \( a_j(S - v - u) = 0 \), so \( a_j(S - v) = -a_{j-2}(S - v - u) \) and same holds for even \( j \), since basic figures are only matchings. This proves our claim.

Now \( b_j(S) = |a_j(S)| = |a_j(S - v) - a_{j-2}(S - v - u)| = |a_j(S - v)| + |a_{j-2}(S - v - u)| = b_j(S - v) + b_{j-2}(S - v - u). \)

The following result shows that among all unbalanced unicyclic signed graphs in \( S(n, g) \), \( S_n^g \) has minimal energy.

**Theorem 3.4.3.** Let \( S \in S(n, g) \) be unbalanced and \( S \neq S_n^g \). Then \( S_n^g \prec S \) and \( E(S_n^g) < E(S) \).

**Proof.** We prove the result by induction on \( n - g \). If \( n - g = 0 \), the result is vacuously true. Let \( p \geq 1 \) and suppose the result is true for \( n - g < p \). We show it holds for \( n - g = p \). Since \( S \) is unicyclic and \( n > g \), so \( S \) is not a cycle and hence it must have a pendant vertex, say \( v \), and \( v \) is adjacent to a unique vertex say \( u \). By Lemma 3.4.2, we have

\[ b_j(S) = b_j(S - v) + b_{j-2}(S - v - u), \]

\[ b_j(S_n^g) = b_j(S_{n-1}^g) + b_{j-2}(P_{g-1}). \]

By induction hypothesis,

\[ b_j(S - v) \geq b_j(S_{n-1}^g) \] (3.1)
for all $j \geq 0$.

As

$$b_{j-2}(P_{g-1}) = \begin{cases} 0, & \text{if } j \text{ is odd or if } j \text{ is even and } j > g + 1; \\ m(P_{g-1}, \frac{j-2}{2}), & \text{if } j \text{ is even and } j \leq g + 1. \end{cases}$$

Since $S - v - u$ contains the signed path $P_{g-1}$ as its subgraph, therefore if $j$ is odd or $j > l + 1$, then $b_{j-2}(S - u - v) \geq b_{j-2}(P_{g-1})$. If $j$ is even and $j \leq g + 1$, then $b_{j-2}(S - v - u) = m(S - v - u, \frac{(j-2)}{2}) \geq m(P_{g-1}, \frac{(j-2)}{2})$. Therefore, we have

$$b_{j-2}(S - v - u) \geq b_{j-2}(P_{g-1}) \quad (3.2)$$

From (3.1) and (3.2), we see $b_j(S) \geq b_j(S^g_n)$. Also if $S \neq S^g_n$, then $b_2(S - v - u) > g - 2 = b_2(P_{g-1})$. Hence $b_4(S^g_n) < b_4(S)$. The second part follows by Theorem 3.3.9. \qed

The following result shows that $S^4_n$ has minimal energy among all unicyclic signed graphs $S^g_n$, where $n \geq g$, $n \geq 6$ and $g \geq 4$.

**Theorem 3.4.4.** Let $n \geq g$, where $n \geq 6$ and $g \geq 5$. Then $S^4_n \prec S^g_n$ and $E(S^4_n) < E(S^g_n)$.

**Proof.** We use induction on $n - g$ for $n \geq g$, where $n \geq 6$ and $g \geq 5$. By Theorem 3.1.1, we have

$$\phi_{S^4_n}(x) = x^{n-4}\{x^4 - nx^2 + 2(n-2)\}. \quad (3.3)$$

It is enough to show that $b_4(S^4_n) < b_4(S^g_n)$. If $n - g = 0$, then $S^g_n = C_n$. Note that $b_4(C_n) = \frac{n}{2}(n - 3)$ and $b_4(S^4_n) = 2(n - 2)$. Clearly $b_4(S^4_n) < b_4(C_n)$ for all $n \geq 6$.

By Lemma 3.4.2, we have

$$b_4(S^g_n) = b_4(S^g_{n-1}) + b_2(P_{g-1}) = b_4(S^g_{n-1}) + g - 2 = 2(n - 1 - 2) + g - 2 = 2(n - 2) + g - 4 > 2(n - 2), \text{ for } g \geq 5. \quad \Box$$

Now we determine unicyclic unbalanced signed graphs with minimal energy.
**Theorem 3.4.5.** Let $S$ be an unbalanced unicyclic signed graph with $n \geq 6$ vertices and $S \neq S_n^3$. Then $E(S_n^3) < E(S)$.

**Proof.** In view of Theorems 3.4.3 and 3.4.4, it suffices to prove that $E(S_n^3) < E(S_n^4)$ for all $n \geq 6$. By Theorem 3.1.1, we have

$$\phi_{S_n^3}(x) = x^{n-4}\{x^4 - nx^2 + 2x + (n-3)\}. \tag{3.4}$$

From equations (3.3) and (3.4) and Theorem 3.3.9, we have

$$E(S_n^4) - E(S_n^3) = \frac{1}{\pi} \int_0^\infty \frac{1}{x^2} \log \frac{[(1 + nx^2 + 2(\alpha-2)x^4)]^2}{[(1 + nx^2 + (n-3)x^4)^2 + 4x^6]} \, dx.$$ 

Let $f(x) = [1 + nx^2 + 2(n - 2)x^4]^2$ and $g(x) = [(1 + nx^2 + (n - 3)x^4)^2 + 4x^6]$. Then

$$f(x) - g(x) = [1 + nx^2 + 2(n - 2)x^4]^2 - [(1 + nx^2 + (n - 3)x^4)^2 + 4x^6]$$

$$= 2(n-1)x^4 + 2(n-2)(n+1)x^6 + (3n-7)(n-1)x^8 > 0,$$

for all $n \geq 6$. Therefore, $E(S_n^3) < E(S_n^4)$ for all $n \geq 6$. \qed

The following result characterizes unicyclic signed graphs with minimal energy.

**Theorem 3.4.6.** Among all unicyclic signed graphs with $n \geq 6$ vertices, all signed graphs in $[S_n^3]$ and $[S_n^4]$ have minimal energy. Moreover, for $n = 3, 4$ and 5 all signed graphs in $[S]$ have minimal energy, where $S$ is one of the signed graphs $C_3$ or $C_3^*$ or $C_4$ or $S_5$.

**Proof.** A manual calculation shows that for $m = 3, 4$ and 5 all signed graphs in $[S]$ have minimal energy, where $S$ is one of the signed graphs $C_3$ or $C_3^*$ or $C_4$ or $S_5$. As in Theorem 3.3.10, $E(S_n^3) = E(S_n^3)$. By Theorems 3.1.4 and 3.4.5 and noting that all graphs in a switching class are equienergetic, the result follows. \qed
3.5 Equienergetic signed graphs

Two signed graphs are said to be isomorphic if their underlying graphs are isomorphic such that the signs are preserved. Any two isomorphic signed graphs are obviously cospectral. There exist unbalanced non isomorphic cospectral signed graphs, e.g., signed graphs $S_1$ and $S_2$ in Fig. 3.1. Two signed graphs $S_1$ and $S_2$ of same order are said to be equienergetic if $E(S_1) = E(S_2)$. Cospectral signed graphs are obviously equienergetic, therefore in view of Theorem 3.1.2, the problem of equienergetic signed graphs reduces to problem of construction of non cospectral pairs of equienergetic signed graphs such that for every pair not both signed graphs are balanced. In this regard, we have shown for each positive integer $n \geq 3$, there exists a pair of connected, non cospectral and equienergetic unicyclic signed graphs on $n$ vertices with one constituent balanced and the other unbalanced.

We note that the spectral radius of $S$, $\rho(S) = \max_{1 \leq k \leq n} |x_k|$ is an eigenvalue of $S$ for every $S \in \Delta_n$. The following Lemma gives the spectrum of Cartesian and Kronecker product of two signed graphs in terms of that of the corresponding signed graphs [22].

**Lemma 3.5.1.** Let $S_1$ and $S_2$ be two signed graphs with respective eigenvalues $\xi_1, \xi_2, \ldots, \xi_{n_1}$ and $\zeta_1, \zeta_2, \ldots, \zeta_{n_2}$. Then
(i) the eigenvalues of $S_1 \times S_2$ are $\xi_i + \zeta_j$, for all $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$;
(ii) the eigenvalues of $S_1 \otimes S_2$ are $\xi_i \zeta_j$, for all $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.

We have the following result.

**Lemma 3.5.2.** (i) $E(S_1 \otimes S_2) = E(S_1)E(S_2)$
(ii) For each $n \geq 3$, $(K_n, -K_n)$ is a pair of non cospectral and equienergetic signed graphs with one constituent balanced and the other unbalanced.
(iii) For all positive integers $m, n \geq 2$, the signed graphs $S = -K_m \times -K_n$ and $T = -K_m \otimes -K_n$ are non cospectral equienergetic signed graphs with $S$ unbalanced and $T$ balanced.

**Proof.** Let $x_1, x_2, \ldots, x_{n_1}$ be eigenvalues of $S_1$ and $y_1, y_2, \ldots, y_{n_2}$ be eigenvalues of $S_2$. By Lemma 3.5.1, eigenvalues of $S_1 \otimes S_2$ are $x_i y_j$, where $i = 1, 2, \ldots, n_1$ and
\(j = 1, 2, \ldots, n_2\). Therefore, \(E(S_1 \otimes S_2) = \sum_{i,j} |x_i y_j| = \sum_{i=1}^{n_1} |x_i| \sum_{j=1}^{n_2} |y_j| = E(S_1)E(S_2)\).

This proves part (i).

(ii) We know that for each positive integer \(n \geq 3\), \(\text{spec}(K_n) = \{-1^{(n-1)}, n-1\}\) so that \(\text{spec}(-K_n) = \{1-n, 1^{(n-1)}\}\). Therefore, \(E(K_n) = E(-K_n) = 2(n-1)\). Note that \(K_n\) is balanced whereas \(-K_n\) is unbalanced.

(iii) We have, \(\text{Spec}(S) = \{(2-m-n, (2-m)^{(n-1)}, (2-n)^{(m-1)}, 2^{(m-1)(n-1)}\} \neq \{(1-m)(1-n), (1-m)^{(n-1)}, (1-n)^{(m-1)}, 1^{(m-1)(n-1)}\} = \text{spec}(T)\). Therefore \(S\) and \(T\) are non cospectral. Also, \(E(S) = |2-m-n|+(n-1)|2-m|+(m-1)|2-n|+(m-1)(n-1)|2| = 4(m-1)(n-1). By part (i), \(E(T) = E(-K_m \otimes -K_n) = E(-K_m)E(-K_n) = 4(m-1)(n-1).\) Therefore \(S\) and \(T\) are equienergetic. \(S\) is unbalanced and \(T\) is balanced follows from Theorem 3.1.2.

The following result characterizes a signed graph \(S\) in \(\Delta_n\) for which the Cartesian product and Kronecker product of \(S\) with \(K_2\) are unbalanced, non cospectral and equienergetic.

**Theorem 3.5.3.** Let \(S\) be an unbalanced signed graph in \(\Delta_n\) with at least one edge having eigenvalues \(x_1, x_2, \ldots, x_n\). Then \(S \times K_2\) and \(S \otimes K_2\) are unbalanced, noncospectral and equienergetic if and only if \(|x_j| \geq 1,\) for all \(j = 1, 2, \ldots, n.\)

**Proof.** By Theorem 3.1.2, it is clear that \(S \in \Delta_n\) is unbalanced if and only if both \(S \times K_2\) and \(S \otimes K_2\) are unbalanced. We first suppose that \(|x_j| \geq 1\) for all \(j = 1, 2, \ldots, n.\) Let \(x_1 \geq x_2 \geq \cdots \geq x_n.\) Assume \(x_1, x_2, \ldots, x_k\) are positive and \(x_{k+1}, x_{k+2}, \ldots, x_n\) are negative.

Also

\[
E(S \times K_2) = \sum_{j=1}^{k} (|x_j + 1| + |x_j - 1|) + \sum_{j=k+1}^{n} (|x_j + 1| + |x_j - 1|).
\]

As \(|x_j| \geq 1\) for all \(j = 1, 2, \ldots, n,\) we have
\[ E(S \times K_2) = \sum_{j=1}^{k} (|x_j| + 1 + |x_j| - 1) + \sum_{j=k+1}^{n} (|x_j| - 1 + |x_j| + 1) \]
\[ = 2 \sum_{j=1}^{k} |x_j| + 2 \sum_{j=k+1}^{n} |x_j| = 2 \sum_{j=1}^{n} |x_j| = 2E(S) \]
\[ = E(S)E(K_2) = E(S \otimes K_2). \]

Note that \( x_1 + 1 \in \text{spec}(S \times K_2) \) but \( x_1 + 1 \notin \text{spec}(S \otimes K_2) \), therefore \( S \times K_2 \) and \( S \otimes K_2 \) are non cospectral.

Conversely, suppose \( |x_s| < 1 \) for some \( s \). Because of pairing property, we can assume \( x_s \geq 0 \). Choose a real number \( \alpha_s \) such that \( x_s + \alpha_s = 1 \). Therefore, \( |x_s + 1| + |x_s - 1| = 1 + x_s + \alpha_s = 2 > 2|x_s| \). Suppose \( |x_j| \geq 1 \) for \( j = 1, 2, \ldots, k \) and \( |x_j| < 1 \) for \( j = k + 1, k + 2, \ldots, n \). Then as before
\[ \sum_{j=1}^{k} (|x_j + 1| + |x_j - 1|) = 2 \sum_{j=1}^{k} |x_j| \quad \text{and} \quad \sum_{j=k+1}^{n} (|x_j + 1| + |x_j - 1|) > 2 \sum_{j=k+1}^{n} |x_j|. \]

Therefore
\[ E(S \times K_2) = \sum_{j=1}^{k} (|x_j+1|+|x_j-1|) + \sum_{j=k+1}^{n} (|x_j+1|+|x_j-1|) > 2 \sum_{j=1}^{n} |x_j| = E(S \otimes K_2), \]
a contradiction.

**Example 3.5.4.** Consider signed graphs \( S_1 \) and \( S_2 \) in Fig. 3.1. Clearly, eigenvalues of \( S_1 \) and \( S_2 \) have absolute value at least 1, therefore by Theorem 3.5.3, \( S_i \times K_2 \) and \( S_i \otimes K_2 \) are unbalanced, non cospectral and equienergetic for \( i = 1, 2 \).

We know from [22] the eigenvalues of a positive and negative cycles with \( n \) vertices are given by the following result

**Lemma 3.5.5.** The eigenvalues of \( C_n \) and \( C_n \) are respectively given by \( x_k = 2 \cos \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1 \) and \( x_k = 2 \cos \frac{(2k+1)\pi}{n}, \quad k = 0, 1, \ldots, n-1 \).

From Lemma 3.5.5 one can derive the following energy formulae. For proof see Theorem 4.3.1
\[ E(C_n) = \begin{cases} 
4 \cot \frac{\pi}{n}, & \text{if } n = 4k, \\
4 \csc \frac{\pi}{n}, & \text{if } n = 4k + 2, \\
2 \csc \frac{\pi}{2n}, & \text{if } n = 2k + 1.
\end{cases} \]

and

\[ E(C_n) = \begin{cases} 
4 \csc \frac{\pi}{n}, & \text{if } n = 4k, \\
4 \cot \frac{\pi}{n}, & \text{if } n = 4k + 2, \\
2 \csc \frac{\pi}{2n}, & \text{if } n = 2k + 1.
\end{cases} \]

From energy formulae we see that for each odd \( n \geq 3 \), \( E(C_n) = E(C_n) \), where \( C_n \) is balanced and \( C_n \) is unbalanced as already proved in Theorem 3.3.10 but here we have exact formulae for energy.

![Diagram of signed graphs](image)

It is easy to see that there does not exist a pair of non cospectral and equienergetic signed graphs on 3 vertices with both constituents unbalanced. The following result proves the existence of a pair of connected, non cospectral and equienergetic signed graphs on \( n \geq 4 \) vertices with both the constituents unbalanced.

**Theorem 3.5.6.** For each positive integer \( n \geq 4 \), there exists a pair of connected, non cospectral equienergetic signed graphs of order \( n \) with both constituents unbalanced.

**Proof.** Case 1. When \( n \) is odd. Assume \( n \geq 5 \) is an odd integer. Consider the signed graphs \( S_{n,1} \) and \( S_{n,2} \) with vertex and edge sets given by

\[ V(S_{n,1}) = V(S_{n,2}) = \{ v_1, v_2, \ldots, v_n \}, \]
\[ \mathcal{E}(S_{n,1}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1}), \ldots, (v_n, v_1), [v_1, v_k]\} \]

and

\[ \mathcal{E}(S_{n,2}) = \{[v_1, v_2], (v_2, v_3), \ldots, (v_k, v_{k+1}), \ldots, (v_n, v_1), (v_1, v_k)\}, \]

where \((u, v)\) means edge from vertex \(u\) to \(v\) is positive and \([u, v]\) means edge from \(u\) to \(v\) is negative and we choose vertex \(v_k\) such that the positive integer \(k\) is even. The signed graphs so constructed are shown in Fig. 3.2.

As both the signed graphs have only one even cycle \(C_k\) and their underlying graphs are same, it follows by Theorem 3.1.1 that \(a_{2j}(S_{n,1}) = a_{2j}(S_{n,2})\), for all \(j = 1, 2, \ldots, \frac{n-1}{2}\).

Also, the odd coefficients of \(S_{n,1}\) are given by

\[ a_{2j-1}(S_{n,1}) = 0 \quad \text{for all} \quad j = 1, 2, \ldots, \frac{n-k+1}{2} \]

and

\[ a_{n-k+2+2j}(S_{n,1}) = \left\{ \begin{array}{ll}
2(-1)^{j+1}m(S_{n,1} - C_{n-k+2}, j) & \text{if} \ j = 0, 1, 2, \ldots, \frac{k-4}{2}, \\
2((-1)^{\frac{k+2}{2}}m(S_{n,1} - C_{n-k+2}, \frac{k-2}{2}) - 1) & \text{if} \ j = \frac{k-2}{2},
\end{array} \right. \]

whereas the odd coefficients of \(S_{n,2}\) are given by

\[ a_{2j-1}(S_{n,2}) = 0 \quad \text{for all} \quad j = 1, 2, \ldots, \frac{n-k+1}{2} \]

and

\[ a_{n-k+2+2j}(S_{n,2}) = \left\{ \begin{array}{ll}
2(-1)^{j+1}m(S_{n,2} - C_{n-k+2}, j) & \text{if} \ j = 0, 1, 2, \ldots, \frac{k-4}{2}, \\
2((-1)^{\frac{k+2}{2}}m(S_{n,2} - C_{n-k+2}, \frac{k-2}{2}) + 1) & \text{if} \ j = \frac{k-2}{2}. \end{array} \right. \]

It is clear that \(a_{2j}(S_{n,1}) = a_{2j}(S_{n,2})\), for all \(j = 1, 2, \ldots, \frac{n-1}{2}\) and \(a_{2j-1}(S_{n,1}) = -a_{2j-1}(S_{n,2})\) for all \(j = 1, 2, \ldots, \frac{n+1}{2}\). Thus \(S_{n,1}\) and \(S_{n,2}\) are non cospectral. From the relation between coefficients of these two signed graphs, it follows that \(\phi_{S_{n,1}}(-x) = -\phi_{S_{n,2}}(x)\) which gives \(spec(S_{n,1}) = -spec(S_{n,2})\). Thus \(E(S_{n,1}) = E(S_{n,2})\).

**Case 2.** When \(n\) is even. Assume \(n \geq 6\) is even. Consider the signed graphs \(S_{n,3}\) and \(S_{n,4}\) with vertex and edge sets given by

\[ V(S_{n,3}) = V(S_{n,4}) = \{v_1, v_2, \ldots, v_n\}, \]

\[ \mathcal{E}(S_{n,3}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_{k+1}), \ldots, (v_{n-1}, v_1), [v_1, v_k], (v_k, v_n)\}, \]

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and

\[ \mathcal{E}(S_{n,4}) = \{ [v_1, v_2], (v_2, v_3), \ldots, (v_{k}, v_{k+1}), \ldots, (v_{n-1}, v_1), (v_1, v_k)(v_k, v_n) \}, \]

where \( k \) is even. The signed graphs so constructed are shown in Fig. 3.3. As in Case 1, it is easy to check that \( S_{n,3} \) and \( S_{n,4} \) are two non cospectral equienergetic signed graphs. Clearly, all the signed graphs are unbalanced.

For \( n = 4 \), consider the signed graphs \( S_1 \) and \( S_2 \) as shown in Fig. 3.4. By Theorem 3.1.1, the characteristic polynomials of \( S_1 \) and \( S_2 \) are \( \phi_{S_1}(x) = x^4 - 5x^2 + 4 \) and \( \phi_{S_2}(x) = x^4 - 6x^2 + 8x - 3 \) so that \( \text{spec}(S_1) = \{-2, -1, 1, 2\} \) and \( \text{spec}(S_2) = \{-3, 1^{(3)}\} \). That is, \( S_1 \) and \( S_2 \) are non cospectral. Also \( E(S_1) = E(S_2) = 6 \) and \( S_1 \) and \( S_2 \) are unbalanced. \( \square \)

3.6 Conclusion

We conclude this Chapter with the following open problem.

**Problem 3.6.1.** Characterize signed graphs having pairing property.

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