Chapter 3

Wave propagation through a 2D lattice

3.1 Introduction

The study of solitons on discrete lattices dates back to the early days of soliton theory[3] and is of great physical importance. The most important studies are on the effect of anisotropies and nonhomogeneities in the media on wave propagation[73]. Using lasers, it has been shown that the heat flow in solids is closely related to the flow of solitons. Davydov[74], by using some rules of solid state physics, had shown that the idea of soliton propagation is essential in the study of the chemical changes taking place in long protein molecules—which is the basis for the understanding of muscle contraction.

Generally, the relevant nonlinear equations which model these
lattices cannot be solved analytically. Consequently, one looks for possible pulse soliton solutions in the continuum or longwavelength approximation. Only when this approach is not workable, one has to use numerical approaches or simulations. Nevertheless, there exist some lattice models for which the governing equations can be solved exactly[76]. The Fermi-Pasta-Ulam[77] problem together with the explanation of Zabusky and Kruskal can be considered to be the origin of lattice solitons. Zabusky[78,79] first showed that the continuum limit of FPU lattice was the KdV equation. This led to the discovery of lattice solitons. The most remarkable model for the study of lattice solitons is the Toda chain[80]. With nearest neighbour interaction, Toda chain happens to be the only integrable nonlinear model. Its applications in different fields like wave propagation in nerve systems, ladder circuit, chemical reaction in atoms and molecules and ecological systems make it very important and interesting from a physical point of view[81,82]. The general solution to the initial value problem of the Toda lattice has been found[83].

Recently, it has been found that, by considering the weak nonlinear case, it is possible to reduce a large number of one dimensional nonlinear systems to integrable ones. Some physically interesting cases in plasma physics, solid state physics etc. have been reduced to the well known simple
model equations such as Burger’s and KdV equations using weak nonlinear approximation (WNA) [84-85].

The weak nonlinear approximation rests mainly on two assumptions:

1. The amplitude of the wave is small but finite, and
2. The wave is a long wave or a modulation of a monochromatic wave.

As far as these two conditions are satisfied, this method is applicable to inhomogeneous systems including random systems. For such a system, it is desirable to have a consistent method to treat the weak nonlinear phenomena. It is found that the reductive perturbation method (RPM) [86,87] is very useful for carrying out weak nonlinear approximation. It takes into account a competition between nonlinearity and dispersion in a systematic manner. Various cases of nonlinear dynamics in fluids, nonlinear lattices and plasmas are reduced to soliton equations by RPM [88,89]. Then it becomes easy to study the waves analytically and explain the observation of soliton phenomena. If a time-dependent and homogeneous perturbation is added to the nonlinear system, we also obtain soliton systems [85].

Iizuka et al. [90] studied the propagation of nonlinear waves
through an inhomogeneous lattice. They considered a one dimensional system and reduced the equation of motion to the known equations, Korteweg de-Vries (KdV), modified KdV and nonlinear Schrödinger (NLS) equations, for different perturbations using WNA. In this chapter, we extend our studies to a two dimensional lattice and investigate the propagation of nonlinear waves using the continuum approximation. Such models are associated with rather important problems in physics. The continuum approximation to lattice problems is used in many contexts because: (1) continuum approximation is easier for analytical as well as numerical study than its discrete counterpart, and (2) results can be conveniently related to the discrete version in many cases. This approach is regarded as an extension of the RPM and it is extremely useful in describing wave propagation in inhomogeneous media. Here we study the wave propagation through a 2D lattice for three special cases- quadratic nonlinearity, cubic nonlinearity and both of these together. In each case, the equation of motion reduces to different nonlinear equations. For quadratic nonlinearity, we get the well known Kadomtsev-Petviashvili (KP) equation and for cubic nonlinearity, modified KP equation. When both of them are applied together, we arrive at an integro-differential equation.
3.2 Reductive perturbation method (RPM)

In the study of the asymptotic behaviour of nonlinear dispersive waves, Gardner and Morikawa\cite{87} introduced the scale transformations

$$\zeta = \epsilon^a (x - \lambda t)$$

$$\tau = \epsilon^b t$$

This transformation is called the Gardner-Morikawa transformation, and may be derived from the linearized asymptotic behaviour of long waves. They combined this transformation with a perturbation expansion of the dependent variables so as to describe the asymptotic nonlinear behaviour. In that process they arrived at the KdV equation as a single tractable equation describing the asymptotic behaviour of a wave.

The perturbation method has been developed and formulated in a general way by Taniuti and his collaborators\cite{88,89} and this method is now known as Reductive Perturbation Method (RPM). This method was first established for the reduction of a fairly general nonlinear system to a single tractable nonlinear equation.
3.3 Formulation of the problem

We consider a nonlinear lattice where the masses of the particles are not equal. The force due to the spring between two adjacent particles is assumed to be

\[ F = K(\Delta + \alpha \Delta^2 + \beta \Delta^3 + \ldots) \quad (3.1) \]

where \( \Delta \) is the elongation of the spring and \( K \) is the spring constant. Let \( m_i \) be the mass and \( a_i \) be the displacement of the \( i^{th} \) particle. Then the equation of motion for the \( i^{th} \) particle is

\[ m_i \ddot{a}_i = K[a_{i+1} - a_i + \alpha(a_{i+1} - a_i)^2 + \beta(a_{i+1} - a_i)^3 + \ldots] - K[a_i - a_{i-1} + \alpha(a_i - a_{i-1})^2 + \beta(a_i - a_{i-1})^3 + \ldots] \quad (3.2) \]

We assume that the inhomogeneity is small and does not depend explicitly on time. Let us suppose that

\[ m_i = \bar{m}(1 + \rho) \quad (3.3) \]

\[ \rho = \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \ldots \quad (3.4) \]

where \( \bar{m} \) is the average mass and \( \rho_1, \rho_2, \ldots \) are functions of the lattice site \( i \). Let the lattice spacings be \( h \) in the \( x \) direction and \( k \) in the \( y \) direction. Hence \( a_i = a_i(x, y, t) \)
The following three wave motions will be considered separately:

(a) Slowly varying in \( x, y \) and \( t \) for quadratic nonlinearity, ie, \( \alpha \neq 0, \beta = 0 \)

(b) Slowly varying in \( x, y \) and \( t \) for cubic nonlinearity, ie, \( \alpha = 0, \beta > 0 \)

(c) Slowly varying in \( x, y \) and \( t \) for quadratic nonlinearity along with cubic nonlinearity, ie, \( \alpha > 0, \beta > 0 \)

The continuum case is physically acceptable when the wavelength is very large compared to the spacing of particles in a lattice, ie the wave is so smooth that one can make the Taylor expansion on \( a_{i+1} \). Since we are interested in wave propagation through a 2D lattice, we may expand \( a_{i+1} \) in a Taylor series for two variables:

\[
a_{i+1} = a_i + h a_x + k a_y + \frac{1}{2}[h^2 a_{xx} + 2kh a_x a_y + k^2 a_{yy}] + \quad (3.5)
\]

where \( a_x, a_y, \ldots \) etc are corresponding derivatives of \( a_i \)

**Case (a): Quadratic nonlinearity (\( \alpha \neq 0, \beta = 0 \))**

For \( \beta = 0 \), eqn (3.2) becomes,

\[
m_i \ddot{a}_i = K[a_{i+1} - a_i + \alpha(a_{i+1} - a_i)^2 + \ldots - (a_i - a_{i-1}) - \alpha(a_i - a_{i-1})^2 - \ldots] \quad (3.6)
\]
From eqs (3.3), (3.4), (3.5) and (3.6), we have,

\[
(1 + \rho)\ddot{a}_i = \frac{K}{m}[h^2a_{xx} + k^2a_{yy} + 2hka_{xy} + 2\alpha h^2a_xa_{xx} + 2\alpha k^2a_ya_{yy} + 2\alpha k^2a_ya_{yy} + 2\alpha k^2a_xa_{yy} + \frac{h^4}{12}a_{xxxx} + \ldots] \tag{3.7}
\]

Now we introduce a change of independent variables \(x\), \(y\) and \(t\) into \(\eta\), \(\zeta\) and \(\tau\):

\[
\eta = \frac{\varepsilon}{h}(x - vt) \tag{3.8}
\]

\[
\zeta = \frac{\varepsilon^2}{k}y \tag{3.9}
\]

\[
\tau = \frac{\varepsilon^3}{24h}t \tag{3.10}
\]

Here \(v\) is the velocity of sound given by \(v = h\sqrt{\frac{K}{m}}\). Again,

\[
a(x, y, t) = -\frac{\varepsilon}{4\alpha} \phi(\eta, \zeta, \tau) \tag{3.11}
\]

Using eqs (3.8), (3.9), (3.10), and (3.11) along with eqn (3.4) in eqn (3.7), we arrive at

\[
(1 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \ldots)(-\frac{\varepsilon^3 v^2}{4\alpha h^2}\phi_{nn} + \frac{\varepsilon^5v}{48\alpha h^2}\phi_{nt} - \frac{\varepsilon^7}{2944\alpha h^2}\phi_{tt}) = \frac{K}{\kappa h}(\frac{-\varepsilon^3}{4\alpha}\phi_{nn} - \frac{\varepsilon^5}{4\alpha}\phi_{\zeta\zeta} - \ldots) \tag{3.12}
\]

Equating coefficients of equal powers of \(\varepsilon\) on either side of the equation;
\[ \varepsilon^3 \quad \frac{K}{\dot{m}} = \frac{v^2}{h^2} \quad (3.13) \]

which gives the velocity \( v \).

\[ \varepsilon^4 \quad \rho_1 \phi_{\eta \eta} = 2 \phi_{\eta \zeta} \quad (3.14) \]

\[ \varepsilon^5 \quad -\rho_2 \phi_{\eta \eta} + \frac{1}{12v} \phi_{\eta \tau} = -\phi_{\tau \zeta} + \frac{1}{2} \phi_{\eta \phi_{\eta \eta} - \frac{1}{12} \phi_{\eta \eta \eta \eta} \quad (3.15) \]

Again applying the change of variables;

\[ X = \eta + 12 \int \rho_2(\tau) d\tau, \quad T = \tau, \quad Y = y \quad (3.16) \]

and

\[ U(X, Y, T') = \phi_{\eta}(\eta, \zeta, \tau) \quad (3.17) \]

Then the equation\( (3.15) \) reduces to

\[ \frac{\partial}{\partial X}(U_T - 6UU_X + U_{XXX}) = -12U_{YY} \quad (3.18) \]

or

\[ U_{TX} - 6U_X^2 - 6UU_{XX} + U_{XXX} + 12U_{YY} = 0 \quad (3.19) \]

Hence, for quadratic nonlinearity, we reduced the equation of motion into the 2 dimensional form of KdV equation (now known as KP equation).
Case (b): Cubic nonlinearity ($\alpha = 0, \beta > 0$)

In this case, the equation of motion becomes

$$m_i a_i'' = K [a_{i+1} - a_i + \beta (a_{i+1} - a_i)^3 + \ldots - (a_i - a_{i-1}) - \beta (a_i - a_{i-1})^3 - \ldots] \quad (3.20)$$

We define $a(x,y,t)$ as:

$$a(x,y,t) = \frac{1}{\sqrt{6\beta}} \phi(\eta, \zeta, \tau) \quad (3.21)$$

Using the same transformations (3.8), (3.9) and (3.10) the equation of motion becomes,

$$(1 + \rho) a_i'' = \frac{K}{m} [h^2 a_{xx} + k^2 a_{yy} + 2 k a_{xy} + 3 \beta h^4 a_i^2 a_{xx} + 3 \beta h^2 k^2 a_i^2 a_{xy} + 3 \beta k^2 h^2 a_i^2 a_{yy} + 3 \beta k^4 a_i^2 a_{xy} + \frac{h^4}{12} a_{xxxx} + 6 \beta h^3 k a_i^2 a_{xy} + 6 \beta k h^3 a_i^2 a_{yy} + \ldots] \quad (3.22)$$

Substituting (3.4) along with (3.21), and equating powers of $\epsilon$ on either side, we get, for $\epsilon^4$

$$a_2 \phi_{xx} - \frac{\phi_{\eta \xi}}{12} = \phi_{\zeta \zeta} + \frac{1}{12} \phi_{\eta \eta} + \frac{1}{12} \phi_{\xi \xi} \phi_{\eta \eta} \quad (3.23)$$

Again introducing the change of variables as in the previous case, we arrive at,

$$\frac{\partial}{\partial X} (-U_T - U_{XXX} - 6U^2U_X) = 12U_{YY} \quad (3.24)$$
This equation is called modified KP equation.

Case (c): Quadratic nonlinearity along with cubic nonlinearity

\( (\alpha > 0, \beta > 0) \)

In this case, the equation of motion becomes

\[
m_i\dddot{a}_i = K[a_{i+1} - a_i + \alpha(a_{i+1} - a_i)^2 + \beta(a_{i+1} - a_i)^3] - (a_i - a_{i-1}) - \alpha(a_i - a_{i-1})^2 - \beta(a_i - a_{i-1})^3
\]

(3.26)

We define \( a(x, y, t) \) as:

\[
a(x, y, t) = A\phi(\eta, \zeta, \tau)
\]

(3.27)

where \( A \) is a constant. Using the same transformations (3.8), (3.9) and (3.10) the equation of motion becomes,

\[
(1 + \rho)a_i'' = \frac{K}{m} [ h^2 a''_x + k^2 a''_y + 2kh a'_x a'_y + 3\beta h^4 a_2^2 a''_x \\
+ 2\alpha k^2 a'_x a''_y + 2\alpha k^2 a'_y a''_x + 2\alpha k^3 a'_y a'_y + \frac{h^4}{12} a'''_{xxx} \\
+ 3\beta k^2 a_2^2 a''_x + 3\beta k^2 h^2 a_2^2 a''_y \\
+ 3\beta k^4 a_2^2 a''_y + \frac{h^4}{12} a_{xxxx} + 6\beta h^3 k a_2^2 a'_{xy} + 6\beta k^3 a_2^2 a'_{xy} + ...] \quad (3.28)
\]
Substituting (3.4) along with (3.27) in (3.28), and equating powers of ε on either side, we get, for ε⁴

$$\rho_2 \phi_{\eta \eta} - \frac{\phi_{\eta \eta}}{12\nu} = \phi_{\zeta \zeta} + \frac{1}{12} \phi_{\eta \eta \eta \eta} + \phi_{\eta \eta}^2 \phi_{\eta \eta} + 2\phi_{\zeta} \phi_{\eta \eta}$$  \hspace{1cm} (3.29)

Again introducing the change of variables as in the previous case, we arrive at,

$$\frac{\partial}{\partial X} (U_X - 12U_{XX} X - 12U^2 U_X) - 12U_{YY} = 24U_{YY} U_X + 24 \frac{\partial}{\partial Y} \int U dX \hspace{1cm} (3.30)$$

This equation represents an integro-differential equation. This can be identified as a modified form of KP equation with the terms on the right hand side representing perturbations. This means that the system becomes more perturbed as we apply the quadratic and cubic nonlinearities together.

### 3.4 Conclusion

In this chapter, we have performed the problem of nonlinear wave propagation through a two dimensional lattice with nonuniform mass distribution. We have considered weak nonlinear approximation for (a) quadratic nonlinearity, (b) cubic nonlinearity and (c) quadratic nonlinearity along with cubic nonlinearity. Using RPM we reduced the equations of motion into
three nonlinear equations for the three different cases. We derived Kadomtsev Petviashvili(KP) equation, modified KP and an integro-differential equation respectively for these three cases. The question of integrability of these equations is examined in the next chapter.