CHAPTER 2
PRELIMINARIES

This chapter highlights basic definitions and results available in the standard literatures [3, 14, 22, 24, 25, 26, 28, 35, 36, 37, 39, 41, 44, 53, 56, 57, 58, 61, 64, 65, 72, 75, 76, 77, 80, 90]. This chapter is divided into six sections. First section contains the basic definition of modules and its substructures. Various results on the substructures are also discussed in this section. The second section deals with sequences of modules. The third section deals with special types of submodules. Fourth section deals with the definition of annihilators, singular module and torsion modules. Various results on these modules are also discussed in this section. The fifth section contains basic definition and results of semi simple module, socle and radical of a module. The sixth section consists of definitions and results on injective modules.

2.1. MODULE AND SUBSTRUCTURES

The concept of module over a ring is a generalization of vector space over a field. In a vector space, the scalars are elements of a field while in a module we shall allow the scalars to be elements of an arbitrary ring.

**Definition 2.1.1:** If $R$ is a ring, a left $R$-module (or a left module over $R$) is an additive abelian group $M$ upon which $R$ acts nicely that is there is a map $R \times M \to M$ with $(r, m) \to r m$ satisfying the following conditions:
(i) \( r (m_1 + m_2) = r m_1 + r m_2 \)

(ii) \((r + s)m = rm + sm\)

(iii) \((rs)m = r(sm), \forall m_1, m_2, m \in M \) and \(r, s \in R\).

Similarly, we can define a right \(R\) -module.

If \(R\) is a ring with unity \(1_R\) (or simply 1) then for a module \(M\), the above map must satisfy the condition (iv).

(iv) \(1_R m = m, \forall m \in M\)

Note that \(M\) is said to be a unitary \(R\) -module if \(R\) is a ring with unity. Also, a unitary \(R\) -module is called a vector space if \(R\) is a division ring.

**Example 2.1.2:** An ideal of a ring \(R\) is an \(R\)-module. In particular, \(R\) itself is an \(R\)-module.

**Example 2.1.3:** Every Abelian group \(G\) is a left (as well as right) module over the ring of integers \(\mathbb{Z}\).

**Definition 2.1.4:** Let \(M\) be a right \(R\) -module. A subgroup \(N\) of \((M, +)\) is called a submodule of \(M\) if \(N\) is closed under multiplication with elements in \(R\), that is, \(nr \in N, \forall n \in N, r \in R\).

An additive subgroup \(N\) of \(M\) is also a right \(R\) -module induced from \(M\) by the operation

\[ N \times R \rightarrow N \]

\[ (n, r) \rightarrow nr, \forall n \in N, r \in R \]
The subset \( \{0\} \) is called the **trivial submodule**, and is denoted by \( (0) \). The module \( M \) is a submodule of itself, an **improper** submodule. If \( M \) has any other submodule, then it is called a **proper** submodule. It can be shown that if \( M \) is a left \( R \)—module, then a subset \( N \subseteq M \) is a submodule if and only if it is nonempty, closed under sums, and closed under multiplication by elements of \( R \).

**Example 2.1.5:** Every right ideal of a ring \( R \) is a right submodule of \( R_R \) and every left ideal of a ring \( R \) is a left submodule of \( _R R \).

**Proposition 2.1.6:** A subset \( N \) of an \( R \)—module \( M \) is a submodule of \( M \) if and only if

1. \( 0 \in N \).
2. \( n_1, n_2 \in N \) implies \( n_1 - n_2 \in N \).
3. \( n \in N \) and \( r \in R \) implies \( nr \in N \).

**Proposition 2.1.7:** Let \( M \) be a left \( R \)—module and let \( N \) be a non-empty subset of \( M \). Then \( RN \) is an \( R \)—submodule of \( M \).

**Theorem 2.1.8:** If \( A \) and \( B \) are two submodules of an \( R \)—module \( M \), then \( A \cap B \) is also a submodule of \( M \).

**Theorem 2.1.9:** Intersection of any family of submodules of a module is a submodule.

**Definition 2.1.10:** Suppose \( M \) be an \( R \)—module. Let \( P \) and \( Q \) be two \( R \)—submodules of \( M \). Then the sum of \( P \) and \( Q \), denoted by \( P + Q \), is defined as

\[
P + Q = \{ x + y \mid x \in P, y \in Q \}
\]
This is an $R$—submodule of $M$ containing $P$ and $Q$.

In general, for any family $\{P_\alpha\}_{\alpha \in I}$ of submodules of $M$;

$$\sum_{\alpha \in I} P_\alpha = \left\{ \sum_{\alpha \in I} x_\alpha | x_\alpha \in P_\alpha, x_\alpha = 0 \text{ except for finitely many } \alpha \text{'s} \right\}$$

This is a submodule of $M$ containing each $P_\alpha$, $\alpha \in I$.

**Definition 2.1.11:** Suppose $M$ and $N$ be two $R$—modules. Then the Cartesian product $P = M \times N$ is again an $R$—submodule.

We observe that $P$ contains $M$ and $N$ as submodules,

Namely

$$M = \{(x,0) \in P | x \in M \} \subseteq P$$

$$N = \{(0,y) \in P | y \in N \} \subseteq P.$$  

The sum of the submodules $M$ and $N$ in $P$ is called the direct sum of the modules $M$ and $N$. This is denoted by $M \oplus N$.

We have

$$M \oplus N = \{(x,0) + (0,y) | x \in M \& y \in N \}$$

$$= \{(x,y) \in P | x \in M \& y \in N \}$$

This sum is direct in the following sense:

(i) Every element of $M \oplus N$ can be uniquely written as a sum of an element in $M$ and an element in $N$, or equivalently,
(ii) \( P = M + N \) with \( M \cap N = \{0\} \).

**Proposition 2.1.12:** Suppose \( M \) and \( N \) be two submodules of a module \( P \) over \( R \). Then \( M \cap N = \{0\} \) if and only if every element \( z \in M + N \) can be uniquely written as \( z = x + y \) with \( x \in M \) and \( y \in N \).

**Definition 2.1.13:** A module \( P \) over \( R \) is called a direct sum of family of submodules \( \{P_\alpha\}_{\alpha \in I} \) if \( P = \sum_{\alpha \in I} P_\alpha \) and every element \( z \in P \) can be written uniquely as \( z = \sum_{\alpha \in I} x_\alpha \), \( x_\alpha \in P_\alpha \), \( x_\alpha = 0 \) except for finitely many \( \alpha \)'s and is denoted by \( P = \bigoplus_{\alpha \in I} P_\alpha \).

**Remark:** It is easy to see that \( \bigoplus_{\alpha \in I} P_\alpha \subseteq \prod_{\alpha \in I} P_\alpha \) and equality holds if and only if \( I \) is finite.

**Definition 2.1.14:** A submodule \( P \) of a module \( M \) is said to be a direct summand or simply a summand of \( M \) if there exists a submodule \( Q \) of \( M \) such that \( M = P \oplus Q \). The submodule \( Q \) is called a supplement of \( P \).

**Proposition 2.1.15:** Let \( M \) be a left \( R \) -- module and let \( N \) be a non-empty subset of \( M \). Then the following statements are equivalent:

(i) \( N \) is a submodule of \( M \).

(ii) \( RN = N \).

(iii) For all \( a, b \in R \) and all \( x, y \in N \), \( ax + by \in N \).

**Lemma 2.1.16:** If \( M \) is a left \( R \) -- module and if \( M_1, M_2, \ldots, M_n \) are submodules of \( M \), then the set \( M_1 + M_2 + \cdots + M_n = \{m_1 + m_2 + \cdots + m_n \mid m_i \in M_i \ (i = 1, 2, 3 \ldots n)\} \) is
also a submodule of \( M \). In fact \( M_1 + M_2 + \cdots + M_n \) is the set of all \( \mathbb{R} \)–linear combination of \( M_1 \cup M_2 \cup \ldots \cup M_n \).

**Lemma 2.1.17:** If \( M \) is a left \( \mathbb{R} \)–module and \( C \) be any arbitrary collection of submodule of \( M \), then the intersection of members of \( C \) is submodule of \( M \).

**Definition 2.1.18:** Let \( M \) be a right \( \mathbb{R} \)–module and \( K \) be a submodule of \( M \). Then the set of cosets \( M/K = \{ x + K \mid x \in M \} \) forms a right \( \mathbb{R} \)–module with respect to the addition and scalar multiplication defined by

\[
(x + K) + (y + K) = (x + y) + K
\]

and \( (x + K) r = x r + K \)

This module \( M/K \) is called the **factor module** of \( M \) by \( K \).

Note that the additive identity and inverses are given by \( K = 0 + \mathbb{R} \) and \( -(x + K) = -x + K \) respectively.

**Proposition 2.1.19:** Suppose \( N \) be a submodule of an \( \mathbb{R} \)–module \( M \). Then the set of submodules of \( M/N \) is naturally bijective with the set of all submodules of \( M \) containing \( N \).

**Definition 2.1.20:** An \( \mathbb{R} \)–module \( M \) is said to be **cyclic** if there is an element \( m_0 \in M \) such that every \( m \in M \) is of the form \( m = r m_0 \) where \( r \in \mathbb{R} \). Also \( m_0 \) is called a generator of \( M \) and we write \( M = (m_0) \).
Definition 2.1.21: Let $M$ be a unital $R$–module and for a fixed element $m \in M$, let $A = \{ r m : r \in R \}$. Then $A$ is a cyclic submodule of $M$ generated by $m$.

Definition 2.1.22: Let $M$ and $N$ be two right $R$–modules. A function $f : M \to N$ is called an $(R$–module) homomorphism if for all $m, m_1, m_2 \in M$ and $r \in R$, 

$$f(m_1 r + m_2) = f(m_1) r + f(m_2).$$

Equivalently, $f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(m r) = f(m) r$.

The set of all $R$–homomorphisms of $M$ in $N$ is denoted by $\text{Hom}_R(M, N)$. In particular, with this addition and composition of mapping, $\text{Hom}_R(M, M) = \text{End}_R(M)$ forms a ring, called the endomorphism ring of $M$. Also if $f \in \text{End}_R(M)$ then $f$ is called an $R$–endomorphism.

Definition 2.1.23: Let $f : M \to N$ be an $R$–homomorphism. Then

1. $f$ is called $R$–monomorphism (or $R$–monic) if $f$ is injective (one-to-one).
2. $f$ is called $R$–epimorphism (or $R$–epic) if $f$ is surjective (onto).
3. $f$ is called $R$–isomorphism if $f$ is bijective (one-to-one and onto).

Lemma 2.1.24: Suppose $f : M \to N$ and $g : N \to K$ be two module homomorphisms. Then

1. $g \circ f : M \to K$ is a module homomorphism.
2. $\ker f = (0)$ if and only if $f$ is one–one.
3. If $f$ and $g$ are monomorphisms, then so is $g \circ f$.
4. If $f$ and $g$ are epimorphisms, then so is $g \circ f$.
5. If $g \circ f$ is a monomorphism, then so is $f$. 

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(6) If \( g \circ f \) is an epimorphism, then so is \( g \).

**Definition 2.1.25:** Let \( M \) and \( N \) be right \( R \)–modules and let \( f: M \to N \) is an \( R \)–homomorphism. Then the set \( \text{Ker} \ (f) = f^{-1}(0) = \{m \in M : f(m) = 0\} \) is called the kernel of \( f \). Also the set \( \text{Im} \ (f) = f(M) = \{n \in N : n = f(m), \text{for some } m \in M\} \) is called the homomorphic image (or simply image) of \( M \) under \( f \) and is denoted by \( \text{Im} f \).

**Lemma 2.1.26:** Let \( M \) and \( N \) be two right \( R \)–modules and let \( f: M \to N \) be an \( R \)–homomorphism. Then

1. \( \text{Ker} \ (f) \) is a submodule of \( M \).
2. \( \text{Im} \ (f) = f(M) \) is a submodule of \( N \).

Note that if \( C \) is any submodule of \( N \), then \( f^{-1}(C) = \{m \in M : f(m) \in C\} \) is a submodule of \( M \).

**Lemma 2.1.27:** Let \( M \) and \( N \) be right \( R \)–modules and let \( f: M \to N \) is an \( R \)–isomorphism. Then the inverse mapping \( f^{-1}: N \to M \) is an \( R \)–isomorphism.

**Proposition 2.1.28:** If \( f: A \to B \) is a homomorphism of the right \( R \)–module \( A \) to the right \( R \)–module \( B \), then \( f(A) \cong \frac{A}{\text{Ker} \ f} \).

**Proposition 2.1.30:** If \( C \) be a submodule of an \( R \)–module \( M \) then every submodule of \( M/C \) has the form \( B/C \) where \( C \subseteq B \subseteq M \) and \( \frac{M}{B} \cong \frac{M/C}{B/C} \).
**Theorem 2.1.30:** Let $N$, $N_0$, and $M_0$ be submodules of a module $_RM$. The following statements hold:

(i) \[ N_0 / (N_0 \cap M_0) \cong (N_0 + M_0) / M_0 \]

(ii) If $N_0 \subseteq N$, then \( (M / N_0) / (N / N_0) \cong M / N \).

(iii) If $N_0 \subseteq N$, then $N \cap (N_0 + M_0) = N_0 + (N \cap M_0)$.

**Theorem 2.1.31:** Suppose $f: M \to N$ is an epimorphism of $R$-modules with $P = \text{Ker} \ f$. Then there exists a unique isomorphism $g: \frac{M}{P} \to N$ such that $f = g \circ \mu$ where $\mu$ is the natural map given by $\mu: M \to M/P$, $x \to x + P$, that is, the following diagram is commutative.

```
 M ----> N
 |    |    |
 V \  V  \ 
 M/P ----> N
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**Theorem 2.1.32:** Suppose $P, N$ be submodules of a module $M$. Then there exist natural isomorphisms

(i) \[ \frac{P+N}{P} \cong \frac{N}{N \cap P} \]

(ii) \[ \frac{P+N}{N} \cong \frac{P}{P \cap N} \]
Lemma 2.1.33: If $B' \subset B \subset M$ and $C' \subset C \subset M$ then

$$\frac{B' + (B \cap C)}{B' + (B \cap C')} \cong \frac{C' + (B \cap C)}{C' + (B' \cap C)}$$

2.2. SEQUENCES OF MODULES

Definition 2.2.1: A pair of $R -$homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be exact at $B$ if $\text{Im} f = \text{Ker} g$.

A finite sequence of $R -$homomorphisms

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \ldots \xrightarrow{f_{n-1}} A_n$$

is said to be exact if $\text{Im} f_i = \text{Ker} f_{i+1}$ for $i = 1, 2, \ldots n - 1$.

An infinite sequence of $R -$homomorphisms,

$$\ldots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \longrightarrow \ldots$$

is called an exact sequence if $\text{Im} f_i = \text{Ker} f_{i+1}$ for all $i \in \mathbb{Z}$.

An exact sequence of the form

$$0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$$

is called a short exact sequence.

Proposition 2.2.2: A short exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is exact if and only if $f$ is monic, $g$ is epic and $\text{Im} f = \text{Ker} g$. 
Example 2.2.3: If $A$ and $B$ are any modules then the sequence

$$ 0 \rightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \rightarrow 0 $$

is an exact sequence, where $i$ and $\pi$ are the canonical injection and projection respectively.

If $C$ is a submodule of a module $D$ then the sequence

$$ 0 \rightarrow C \xrightarrow{i} D \xrightarrow{p} D/C \rightarrow 0 $$

is an exact sequence, where $i$ is the inclusion map and $p$ is the canonical epimorphism.

Proposition 2.2.4: Let the following sequence of $R$–homomorphisms

$$ 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 $$

be an exact sequence. Then the following conditions are equivalent.

(i) There is an $R$–homomorphism $h: C \rightarrow B$ such that $g \circ h = 1_C$.

(ii) There is an $R$–homomorphism $k: B \rightarrow A$ such that $k \circ f = 1_A$.

(iii) $B \cong A \oplus C$

Definition 2.2.5: A short exact sequence of the form

$$ 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 $$

Is said to be split or split exact sequence if there exist homomorphisms $\overline{f} : B \rightarrow A$ and $\overline{g} : C \rightarrow B$ such that $\overline{f} \circ f = 1_A$ and $g \circ \overline{g} = 1_C$. 
Proposition 2.2.6: Let $M, N$ be left $R$–modules.

(a) Let $f: M \to N$ and $g: N \to M$ be $R$–homomorphisms such that $f \circ g = 1_N$. Then $M = \text{ker}(f) \oplus \text{Im}(g)$.

(b) A one-to-one $R$–homomorphism $g: N \to M$ splits if and only if $\text{Im}(g)$ is a direct summand of $M$.

(c) A onto $R$–homomorphism $f: M \to N$ splits if and only if $\text{ker}(f)$ is a direct summand of $M$.

Proposition 2.2.7: Let $L$, $M$ and $N$ be left $R$–modules. Let $g: L \to M$ be a one-to-one $R$–homomorphism, and let $f: M \to N$ be an onto $R$–homomorphism such that $\text{Im}(g) = \text{ker}(f)$. Then $g$ is split if and only if $f$ is split, and in this case $M \cong L \oplus N$.

Corollary 2.2.8: The following conditions are equivalent for the module $_RM$:

(1) Every submodule of $M$ is a direct summand.

(2) Every one-to-one $R$–homomorphism into $M$ splits.

(3) Every onto $R$–homomorphism of $M$ splits.

2.3. SPECIAL TYPES OF SUBMODULES

Definition 2.3.1: A submodule $C$ of a module $M$ is said to be large (essential) if it has non-zero intersection with every non-zero submodule of $M$. Thus if $B$ is any non-zero submodule of $M$, then $B \cap C \neq 0$. We also say that $M$ is an essential extension of $C$.

We note that $M$ has always at least one essential extension, since $M \leq_e M$. 

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Also $0 \leq_e M$ if and only if $M = 0$

A monomorphism $f: L \to M$ is said to be **essential** if $\text{Im } f$ is an essential submodule of $M$.

Note that a submodule $K \subset M$ is essential if and only if the inclusion map $K \to M$ is an essential monomorphism.

**Example 2.3.2:** In $\mathbb{Z}$, every non-zero submodule is essential.

**Example 2.3.3:** Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a ring under addition and multiplication modulo 8. So $R$ is a module over itself. Here $I = \{0, 2, 4, 6\}, J = \{0, 3, 6\}$ and $K = \{0, 4\}$ are submodules of $R$, and $I \cap J = \{0, 6\}$, and $I \cap K = \{0, 4\}$. Thus $I$ is a large submodule of $R$.

The following example shows the existence of a submodule which is not large.

**Example 2.3.4:** Let $R = \{0, 1, 2, 3, 4\}$ be a ring under addition and multiplication modulo 5. So $R$ is a module over itself. Here $I = \{0, 2, 4\}$ and $J = \{0, 3\}$ are submodules of $R$ and $I \cap J = \{0\}$. Thus $I$ is not a large submodule of $R$.

**Lemma 2.3.5:** An $R$–module $M$ is an essential extension of an $R$–module $N$ if and only if for any $0 \neq x \in M$ there exists $a \in N$ such that $0 \neq xa \in N$.

**Proposition 2.3.6:** The following statements hold:

(a) If $A \leq B \leq M$ then $A \leq_e M$ if and only if $A \leq_e B \leq_e M$.

(b) If $A \leq_e B \leq M$ and $A' \leq_e B' \leq M$ then $A \cap A' \leq_e B \cap B'$.

(c) If $f: B \to M$ is a homomorphism and $A \leq_e M$ then $f^{-1}(A) \leq_e B$. 

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(d) If \( \{ A_\alpha \} \) is an independent family of submodules of \( M \) and if \( A_\alpha \leq_e B_\alpha \leq M \) for each \( \alpha \) then \( \{ B_\alpha \} \) is an independent family and \( \bigoplus A_\alpha \leq_e \bigoplus B_\alpha \).

**Definition 2.3.7:** Let \( A \) be a submodule of \( M \). A **relative complement** for \( A \) in \( M \) is any submodule \( B \) of \( M \) which is maximal with respect to the property \( A \cap B = 0 \). Such submodule \( B \) always exists, by virtue of Zorn’s lemma; in fact any submodule \( B_0 \) of \( M \) satisfying \( A \cap B_0 = 0 \) can be enlarged to a relative complement for \( A \). If \( A \) is actually a direct summand of \( M \), say \( M = A \oplus B \) then the complementary summand \( B \) is a relative complement for \( A \).

The importance of relative complement is that they can be used to construct essential submodules.

**Proposition 2.3.8:** Let \( A \subseteq M \). If \( B \) is any relative complement for \( A \) in \( M \), then \( A \oplus B \leq_e M \).

**Definition 2.3.9:** A submodule \( B \) of a module \( M \) is said to be **closed submodule** of \( M \) if \( B \) has no proper essential extensions inside \( M \), that is, the only solution of the relation \( B \leq_e N \leq M \) is \( N = B \).

For example, 0 and \( M \) are always closed submodules of \( M \). Also every direct summand of \( M \) is a closed submodule of \( M \).

**Proposition 2.3.10:** If \( B \subseteq M \) then the following conditions are equivalent:

(a) \( B \) is a closed submodules of \( M \).

(b) \( B \) is a relative complement for some \( A \subseteq M \).
(c) If \( A \) is a relative complement for \( B \) in \( M \), then \( B \) is a relative complement for some \( A \) in \( M \).

(d) If \( B \leq K \leq e M \) then \( K/B \leq e M/B \).

**Proposition 2.3.11:** Let \( A \leq B \leq M \). If \( A \) is closed in \( B \) and \( B \) is closed in \( M \), then \( A \) is closed in \( M \).

**Definition 2.3.12:** If \( N \) is a submodule of an \( R \) – module \( M \) then the closure of \( N \) in \( M \) is a closed submodule \( K \) of \( M \) such that \( N \leq e K \) and is denoted by \( cl_M(N) \).

**Definition 2.3.13:** An \( R \) – module \( M \) is called a simple (or irreducible) module if

(i) \( M \neq (0) \)

(ii) The only submodules of \( M \) are \( (0) \) and \( M \).

**Example 2.3.14:** If \( p \) is a prime number then \( \mathbb{Z}/p\mathbb{Z} \) is a simple \( \mathbb{Z} \) – module.

**Lemma 2.3.15 (Schur’s lemma):** Suppose \( N \) and \( M \) are two simple \( R \) – modules. Then any \( R \) – homomorphism \( f: M \to N \) is either 0 or an isomorphism.

In particular, \( D = \text{End}_R(M) \) is a division ring.

**Definition 2.3.16:** A submodule \( K \) of \( M \) is called a **maximal submodule** of \( M \) if \( K \neq M \) and it is not properly contained in any proper submodule of \( M \), that is, \( K \) is maximal in \( M \) if \( K \neq M \) and for every \( A \subset M, K \subset A \) implies \( K = A \).
**Definition 2.3.17:** A submodule $N$ of $M$ is called **minimal submodule** of $M$ if $N \neq 0$ and it has no proper submodules of $M$, that is, $N$ is minimal in $M$ if $N \neq 0$ and for every nonzero submodules $A$ of $M$, $A \subseteq N$ implies $A = N$.

**Proposition 2.3.18:** Let $M$ and $N$ be right $R$-module. If $f : M \to N$ is an epimorphism with $\text{Ker}(f) = K$ then there is a unique isomorphism $\sigma : M/K \to N$ such that $\sigma(m + K) = f(m)$ for all $m \in M$.

**Proposition 2.3.19:** Let $K$ be a submodule of $M$. A factor module $M/K$ is simple if and only if $K$ is a maximal submodule of $M$.

### 2.4. Annihilators, Singular Modules and Torsion Modules

**Definition 2.4.1:** Let $M$ be a right $R$-module. For each $X \subseteq M$ the **right annihilator** of $X$ in $R$ is defined by

$$r_R(X) = \{ r \in R : xr = 0, \quad \forall x \in X \}$$

Similarly, the left annihilator of $X$ in $R$ is defined by

$$l_R(X) = \{ r \in R : rx = 0, \quad \forall x \in X \}$$

For a singleton $\{x\}$, we usually abbreviated to $r_R(x)$ (resp.$l_R(x)$)

**Proposition 2.4.2:** Let $M$ be a right $R$-module, let $S$ and $T$ be two subsets of $M$ and $A$ and $B$ be two subset of $R$. Then

1. $r_R(S)$ is a right ideal of $R$.
2. $S \subseteq T$ implies $r_R(T) \subseteq r_R(S)$.
3. $A \subseteq B$ implies $l_M(B) \subseteq l_M(A)$.
4. $S \subseteq l_M(r_R(S))$ and $A \subseteq r_M l_M(A)$.
**Proposition 2.4.3:** Let $M$ be a right $R$–module over an arbitrary ring $R$, the set $Z(M) = \{ x \in M \mid r_R(x) \text{ is essential in } R \}$ is a submodule of $M$.

**Definition 2.4.4:** The submodule $Z(M) = \{ x \in M \mid r_R(x) \text{ is essential in } R \}$ is called singular submodule of $M$. The module $M$ is called a **singular** module if $Z(M) = M$. It is called a **nonsingular** module if $Z(M) = 0$.

**Lemma 2.4.5:** $Z(M)$ is a singular submodule of $M$. Then

(i) $Z(M).soc(R_R) = 0$, where $soc(R_R)$ denotes the socle of $R_R$.

(ii) If $f: M \to N$ is any $R$–homomorphism, then $f(Z(M)) \subseteq Z(N)$.

(iii) If $M \subseteq N$, then $Z(M) = M \cap Z(N)$.

**Example 2.4.6:** Any simple ring is nonsingular.

**Proposition 2.4.7:** A module $C$ is nonsingular if and only if $\text{Hom}_R(A, C) = 0$ for all singular modules $A$.

**Proposition 2.4.8:** A module $C$ is singular if and only if there exists a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ such that $\phi$ is an essential monomorphism.

**Definition 2.4.9:** Let $R$ be a commutative integral domain and let $M$ be an $R$–module.

The set $T(M) = \{ m \in M : mr = 0, \text{ for some nonzeror } r \in R \}$ is a submodules of $M$, called **torsion submodule** of $M$. The module $M$ is called a torsion module if $T(M) = M$.

While $M$ is torsion free module if $T(M) = 0$. 

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**Proposition 2.4.10:** For any module $M$ over a commutative integral domain, the quotient $M/T(M)$ is torsion free.

### 2.5. Socle, Semisimple and Radical of Modules

**Definition 2.5.1:** Let $M$ be an $R$-module. The **socle** of $M$, denoted by $soc(M)$, is the sum of all simple submodules of $M$. If there are no such submodules, then $soc(M) = 0$.

**Proposition 2.5.2:** For any module $M$, $soc(M)$ is a direct sum of simple modules. In particular, every semisimple module is a direct sum of simple modules.

**Proposition 2.5.3:** Let $A \leq M$. Then $A$ is an intersection of all essential submodules of $M$ if and only if $soc(M) \leq A$. In particular, $soc(M)$ is the intersection of all the essential submodules of $M$.

**Proposition 2.5.4:** Let $M$ be a right $R$-module. Then $Soc(M) \leq_{e} M$ if and only if every non-zero submodule of $M$ contains a minimal submodule.

**Definition 2.5.5:** Let $M$ be an $R$-module. The **radical** of $M$, denoted by $Rad(M)$, is defined as the intersection of all maximal submodules of $M$.

Note that if $M$ has no maximal submodules then $Rad(M) = M$.

**Proposition 2.5.6:** Let $M$ be a right $R$-module. A right $R$-module $M$ is finitely generated if and only if $Rad(M) \ll M$ and $M/Rad(M)$ is finitely generated.
**Definition 2.5.7:** Let $M$ be a right $R$–module. The module $M$ is said to be **semisimple** (or completely reducible) if $soc(M) = M$, that is, if $M$ is the sum of all its simple submodules. A ring $R$ is called **semisimple** if the Jacobson radical $J(R)$ is zero.

For example, the ring of integers is semisimple.

**Proposition 2.5.8:** A module $M$ is semisimple if and only if every submodule of $M$ is a summand.

**Corollary 2.5.9:** Submodule and quotient module of a semisimple module is semisimple.

**Proposition 2.5.10:** An $R$–module $M$ is semisimple if it satisfies any of the following equivalent conditions:

(a) $M$ is a sum of simple submodules.

(b) $M$ is a direct sum of simple submodules.

(c) Every submodule of $M$ has a complement.

### 2.6. INJECTIVE MODULE

In this section injective modules and their sub-structures along with their properties are discussed.

**Definition 2.6.1:** An $R$–module $E$ is called an **injective module** if for any monomorphism $\alpha: A \to B$ of $R$–modules and for any $R$–homomorphism $\beta: A \to E$, there exists a homomorphism $\mu: B \to E$ such that $\beta = \alpha \circ \mu$, that is, such that the following diagram commutes.
Example 2.6.2: $Q$ is an injective $\mathbb{Z}$ module. Consider the diagram of $\mathbb{Z}$ modules where $n \mathbb{Z}$ is an integer and $i: n\mathbb{Z} \to \mathbb{Z}$ is the natural inclusion. We seek a map $g: \mathbb{Z} \to Q$, making the following commutative diagram.

Set $q := f(n)$ and define $g(x) = \frac{xq}{n}, \quad \forall \ x \in \mathbb{Z}$.

Then $g$ is a well-defined.

Furthermore $\forall \ n \ a \in n \mathbb{Z}$,

$$f(a \ n) = a \ f(n) = a \ q$$

$$= a \ n \ \left( \frac{q}{n} \right)$$

$$= a \ n \ g(1)$$

$$= g(a \ n)$$

$$= g \circ i(a \ n)$$

Thus $Q$ is an injective $\mathbb{Z} -$module.
Example 2.6.3: \( \mathbb{Z} \) is not an injective \( \mathbb{Z} \)-module. Consider the homomorphism \( f: 2\mathbb{Z} \to \mathbb{Z} \) given by the rule \( f(2n) = n \quad \forall \ n \in \mathbb{Z} \).

If we draw the diagram with a map \( g: \mathbb{Z} \to \mathbb{Z} \), then

\[
1 = f(2) = g(i(2)) = 2 \cdot g(1)
\]

This can’t hold.

Therefore \( \mathbb{Z} \) is not injective as a module over itself.

Remark: All modules over a semi simple ring are injective.

Proposition 2.6.4: A left \( R \)-module \( Q \) is said to be injective if it satisfies one of the following equivalent conditions:

(i) If \( X \) and \( Y \) are left \( R \)-modules and \( f: X \to Y \) is a monomorphism and \( g: X \to Q \) is an arbitrary module homomorphism, then there exists \( h: Y \to Q \) such that \( h \circ f = g \), that is, the diagram commutes.

(ii) Any short exact sequence

\[
0 \to Q \to M \to K \to 0
\]

of left \( R \)-modules splits.

(iii) If \( L \) is a submodule of some other left \( R \)-module \( M \) such that \( M \) is the external direct sum of \( L \) and \( K \), that is, \( L + K = M \) and \( L \cap K = \{0\} \).
(iv) The contravariant functor $\text{Hom}(-, Q)$ from the category of left $R$-modules to
the category of abelian groups is exact.

(v) Any homomorphism $g: I \to Q$ defined on left ideal $I$ of $R$ can be extended to all
of $R$.

(vi) **(Baer’s Criterion)** for a left ideal $I$ of $R$ and for each map $f: I \to Q$
homomorphism) there corresponds $q \in Q$ such that $f(x) = xq$, $\forall x \in I$.

**Example 2.6.5:** Let $R$ be a domain. We claim that its quotient field $Q(R)$, is injective
over $R$. By using Baer’s criterion, let $\varphi: I \to Q(R)$ be an $R$-linear map where $I$ is an
ideal of $R$.

If $I = 0$ extend by the zero map. Otherwise let $0 \neq i \in I$ and define the following map

$$\gamma: R \to Q(R) \text{ by } r \mapsto r \frac{\varphi(i)}{i}$$

This map is obviously $R$-linear and if $j \in I$,

$$\gamma(j) = j \frac{\varphi(i)}{i} = i \frac{\varphi(j)}{i} = \varphi(j)$$

**Proposition 2.6.6:** Every module is isomorphic to a submodule of an injective module.

**Proposition 2.6.7:** An $R$-module $M$ is injective if and only if it is a direct summand of
every module of which it is a submodule.

**Example 2.6.8:** The zero module $(0)$ is injective. For any $R$-module $M$ we have $M =
(0) + M$, which shows that $(0)$ is the direct summand of any module that contains it. Thus
$(0)$ is injective.

**Proposition 2.6.9:** Any $R$-module can be imbedded in an injective $R$-module.
**Proposition 2.6.10:** Let $M \neq 0$ and $N$ be two $R$-modules and $\alpha: M \to N$ be a monomorphism. Then the following are equivalent:

1. Every nonzero submodule of $N$ has a nonzero intersection with $\alpha(M)$.
2. Every nonzero element of $N$ has a nonzero multiple in $\alpha(M)$.
3. If $\beta \circ \alpha$ is injective for a homomorphism $\beta: N \to Q$, then $Q$ is injective.

**Proposition 2.6.11:** A module $Q$ is injective if and only if it has no proper essential extensions.

**Proposition 2.6.12:** An $R$-module $M$ is injective if and only if any monomorphism $E \to M$ splits.

**Proposition 2.6.13:** For any integral domain $R$ with field of fraction $K$, the $R$-module $K$ is injective.

**Proposition 2.6.14:** Let $\{E_k \mid k \in K\}$ be a family of modules, then $\prod_{k \in K} E_k$ is an injective if and only if each $E_k$ is injective.

**Proposition 2.6.15:** A submodule $N$ of an injective module $M$ is injective if and only if $N$ is closed in $M$.

**Proposition 2.6.16:** A submodule of an injective $R$-module is injective if and only if the ring $R$ is artinian semisimple.

**Definition 2.6.17:** A ring $R$ is said to be left hereditary if every left ideal is projective; it is said to be right hereditary if every right ideal is projective.

**Example 2.6.18:** A semisimple ring is both left and right hereditary.
Proposition 2.6.19: A ring $R$ is right hereditary if and only if quotients of right injective $R$—modules are injective.

Definition 2.6.20: An $R$—module $M$ is the injective hull or injective envelope of an $R$—module $Q$ if it is both an essential extension of $Q$ and an injective module. An injective hull or injective envelope of an $R$—module $M$ is generally denoted by $E(M)$.

Example 2.6.21: We have seen that $Q$ is an injective as a $\mathbb{Z}$—module. Moreover for any $\frac{p}{q} \in Q$ ($q \neq 0$), there exists $q \in \mathbb{Z}$ such that $q \cdot \frac{p}{q} = p \in \mathbb{Z}$. Thus $Q$ is an essential extension of $\mathbb{Z}$ and hence $Q$ is the injective hull of $\mathbb{Z}$.

Proposition 2.6.22: Any $R$—module $M$ has an injective hull which is unique upto an isomorphism extending the identity of $M$.

Proposition 2.6.23: Let $N$ be an extension of $M$. Then the following statements are equivalent:

(i) $N$ is a maximal essential extension of $M$.

(ii) $N$ is an essential extension of $M$ and is injective.

(iii) $N$ is a minimal injective extension of $M$.

Note: If the above statements are equivalent for a normal extension $N$ of an $R$—module $M$ then $N$ is the injective hull of $M$.

Definition 2.6.24: An $R$—module $M$ is said to be a quasi-injective if for any submodule $L \subseteq M$, any $f \in \text{Hom}_R(L, M)$ can be extended to an endomorphism of $M$. 

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Alternately, an $R$-module $M$ is said to be a **quasi-injective** if for any given module $L$, any homomorphism $f: L \rightarrow M$ and monomorphism $g: L \rightarrow M$, there exists an $h \in \text{End}_R(M)$ such that $f = h \circ g$, that is, this diagram commutes.

Note: Any injective module is always quasi injective.

But the converse is not true.

For example, any simple module $M_R$ is obviously quasi injective, since the only submodules of $M$ are $(0)$ and $M$. But a simple module need not always be injective.

**Definition 2.6.25:** An $R$-module $M$ is said to be a **pseudo-injective** if for any given module $L$ and monomorphisms $f: L \rightarrow M$ and $g: L \rightarrow M$, there exists an $h \in \text{End}_R(M)$ such that $f = h \circ g$, that is, the following diagram commutes.

**Proposition 2.6.26:** The following implication holds:

\[
\text{Injective} \Rightarrow \text{Quasi-injective} \Rightarrow \text{Pseudo-injective}.
\]
**Example 2.6.2 [37]:** Every quasi-injective module is pseudo injective. But the converse is not true.

For example, let $R$ be an algebra over $\mathbb{Z}/(2)$ having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$n_3$</td>
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<tr>
<td>$e_2$</td>
<td>0</td>
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<td>$n_4$</td>
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<td>$n_4$</td>
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</tbody>
</table>

We see that the (right) $R$-module $M = e_2 R$ is pseudo injective but not quasi injective.

**Example 2.6.28 [86]:** Let $F = \mathbb{Z}/2$ and $A = F[X]$. Then $A/(x)$ is a $(A/(x) - A/(x^2))$ bimodule in the natural way and $R = \left\{ \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \mid u, v \in A/(x), w \in A/(x^2) \right\}$ is a ring with usual binary operations. Let $M$ be the right ideal $\left\{ \begin{pmatrix} 0 & 0 \\ v & w \end{pmatrix} \mid v \in A/(x), w \in A/(x^2) \right\}$. Then $M_R$ is pseudo injective but not quasi injective.
Proposition 2.6.29: A direct summand of a quasi injective module is always quasi injective. In general, a direct sum of two quasi injective modules need not be quasi injective.

Definition 2.6.30: A module is said to be indecomposable if it cannot be written as the direct sum of two non-zero modules, that is, if \( M = A \oplus B \) then \( M \) is said to indecomposable if and only if either \( A = (0) \) or \( B = (0) \).

Theorem 2.6.31: For any injective right \( R \)-module \( M \) over a ring \( R \). The following conditions are equivalent.

(1) \( M \) is indecomposable.

(2) \( M \) is non-zero and is the injective hull of every nonzero submodule.

(3) \( M \) is uniform module.

(4) \( M \) is the injective hull of a uniform module.

(5) \( M \) is the injective hull of a uniform cyclic module.

(6) \( M \) has a local endomorphism ring.

Definition 2.6.32: A regular element or a nonzero divisor in a ring \( R \) is an element \( x \) such that \( rx \neq 0 \) and \( xr \neq 0 \), for all non-zero \( r \in R \).

Definition 2.6.33: A right \( R \)-module \( M \) is divisible if \( Mx = M \) for all regular element \( x \in R \).

Proposition 2.6.34: \( \mathbb{Z}_n \) is divisible as a module over itself.

Proposition 2.6.35: Any injective right \( R \)-module is divisible.
**Proposition 2.6.36:** A $\mathbb{Z}$-module $Q$ is injective if and only if it is divisible.

**Definition 2.6.37:** A ring $R$ is von Neumann regular if $a \in aRa$ for each $a \in R$.

**Proposition 2.6.38:** The following statements are equivalent for a ring $R$.

1. $R$ is von Neumann regular.
2. Every principal right ideal is a direct summand.
3. Every finitely generated right ideal is a direct summand.