Chapter 4

A-Vertex Consecutive Edge Trimagic Total Labeling of Graphs

4.1. Introduction

While in the previous chapter we introduced the new concept edge trimagic total labeling and super edge trimagic total labeling and proved that some standard graphs admits the trimagic labeling. In this chapter we introduce a-vertex consecutive edge trimagic total labeling and has given a-vertex consecutive edge trimagic total labeling for some particular families of graphs.

In section 4.2, we present basic properties of some families of star type graphs that admit a-vertex consecutive edge trimagic total labeling. We prove that the graphs \( (B_{m,n} : 2) \), \( (K_{1,n} : 3) \), the double star \( G(v; nP_3) \) and \( (K_1.p \cup K_1.q \cup K_1.r) \) admits a-vertex consecutive edge trimagic total labeling. In section 4.3, we prove that the corona graphs \( P_3 \odot K_n \), \( C_n \odot K_1 \) and \( C_n \odot K_2 \) admits a-vertex consecutive edge trimagic total labeling. In the last section we prove that the quadrilateral snake \( Q_n \) and the triangular snake \( TS_n \) admits a-vertex consecutive edge trimagic total labeling. Most of the results in this chapter are published in [37].

Definition 4.1.1. A bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\} \) is called a-vertex consecutive edge trimagic total labeling of \( G = G(V, E) \) if \( f \) is an edge trimagic total labeling.
labeling and \( f(V) = \{a+1, a+2, \ldots, a+p\} \), \( 0 \leq a \leq q \). If \( a = 0 \), then the labeling is called a super edge trimagic labeling.

**4.2. Star type graphs**

In this section, we prove that the graphs \( (B_{m, n} : 2) \), \( (K_{1, n} : 3) \), the double star \( G(v; nP_3) \) and the disconnected graph \( (K_{1, p} \cup K_{1, q} \cup K_{1, r}) \) admits a-vertex consecutive edge trimagic total labeling.

**Definition 4.2.1.** [15] \( B_{m, n} \) is a \( (m, n) \) bistar obtained from two disjoint copies of \( K_{1, m} \) and \( K_{1, n} \) by joining the central vertices by an edge. It has \( (m+n+2) \) vertices and \( (m+n+1) \) edges. The graph \( (B_{m, n} : 2) \) obtained from the graph \( B_{m, n} \) by subdividing the middle edge with a new vertex. It has \( (m+n+3) \) vertices and \( (m+n+2) \) edges.

**Definition 4.2.2.** [15] \( (K_{1, n} : 2) \) is a graph obtained from \( B_{n, n} \) by subdividing the middle edge with a new vertex. It has \( (2n+3) \) vertices and \( (2n+2) \) edges. \( (K_{1, n} : 3) \) is a graph obtained from \( (K_{1, n} : 2) \) by joining one more copy of the central vertices of \( (K_{1, n} : 2) \) and subdividing that edge. It has \( (3n+5) \) vertices and \( (3n+4) \) edges.

**Theorem 4.2.3.** The graph \( (B_{m, n} : 2) \) admits a-vertex consecutive edge trimagic total labeling for \( m, n \geq 1 \).

**Proof.** Let \( V = \{u, v, w, u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\} \) be the vertex set and \( E = \{uu_i, vv_j, uw, wv / 1 \leq i \leq m, 1 \leq j \leq n\} \) be the edge set. Then the graph \( (B_{m, n} : 2) \) has \( m+n+3 \) vertices and \( m+n+2 \) edges and \( m, n \geq 1 \).

Define a bijection \( f : V \cup E \rightarrow \{1, 2, \ldots, 2m+2n+5\} \) such that \( f(w) = m+n+3 \), \( f(v) = m+n+4 \), \( f(u) = m+n+5 \), \( f(u_i) = m+2n+5+i, 1 \leq i \leq m \), \( f(v_j) = m+n+5+j, 1 \leq j \leq n \), \( f(v) = m+n+4 \), \( f(u) = m+n+5 \), \( f(u_i) = m+2n+5+i, 1 \leq i \leq m \), \( f(v_j) = m+n+5+j, 1 \leq j \leq n \), \( f(v) = m+n+4 \), \( f(u) = m+n+5 \), \( f(u_i) = m+2n+5+i, 1 \leq i \leq m \), \( f(v_j) = m+n+5+j, 1 \leq j \leq n \),
f(uu_i) = m+1– i, 1 ≤ i ≤ m, f(vv_j) = m+n+1– j, 1 ≤ j ≤ n, f(uw) = m+n+1 and f(wv) = m+n+2.

Now we prove this labeling is a-vertex consecutive edge trimagic total.

For the edges uu_i, 1 ≤ i ≤ m;

f(u)+f(uu_i)+f(u_i) = m+n+5+ m+1– i+m+2n+5+i = 3m+3n+11 = \lambda_1\text{(say)}.

For the edges vv_j, 1 ≤ j ≤ n;

f(v)+f(vv_j)+f(v_j) = m+n+4+m+n+1– j+m+n+5+j = 3m+3n+10 = \lambda_2\text{(say)}.

For the edge uw, f(u)+f(uw)+f(w) = m+n+5+m+n+1+m+n+3 = 3m+3n+9 = \lambda_3\text{(say)}.

For the edge vw, f(v)+f(vw)+f(w) = m+n+4+m+n+2+m+n+3 = 3m+3n+9 = \lambda_3\text{(say)}.

Hence for each edge uv \in E, the value of f(u)+f(uv)+f(v) yields any of the trimagic constants \lambda_1 = 3m+3n+11, \lambda_2 = 3m+3n+10 and \lambda_3 = 3m+3n+9. This proves that the graph \langle B_{m, n}: 2 \rangle, m, n ≥ 1 admits a-vertex consecutive edge trimagic total labeling for a = m+n+2.

**Example 4.2.4.** A-vertex consecutive edge trimagic total labeling of \langle B_{7, 6}: 2 \rangle with a = 15 and is given in figure 4.1.

![Figure 4.1](image-url)

Figure 4.1. \langle B_{7, 6}: 2 \rangle with \lambda_1 = 50, \lambda_2 = 49 and \lambda_3 = 48.
**Theorem 4.2.5.** The graph \( K_{1,n} : 3 \), \( n \geq 3 \) admits a-vertex consecutive edge trimagic total labeling.

**Proof.** Let \( V = \{u, u_1, u_2, \ldots, u_n, v, v_1, v_2, \ldots, v_n, w, w_1, w_2, \ldots, w_n, x, y\} \) be the vertex set and \( E = \{uu_i, vv_i, ww_i / 1 \leq i \leq n\} \cup \{ux, xv, vy, yw\} \) be the edge set of \( (K_{1,n} : 3) \). Then the graph \( (K_{1,n} : 3) \) has \( 3n+5 \) vertices and \( 3n+4 \) edges.

Define a bijection \( f : V \cup E \rightarrow \{1, 2, \ldots, 6n+9 \} \) such that \( f(u) = 3n+5, f(v) = 3n+6, f(w) = 3n+7, f(x) = 3n+8, f(y) = 3n+9, f(ux) = 3n+4, f(xv) = 3n+3, f(vy) = 3n+2, f(yw) = 3n+1, \) for all \( 1 \leq i \leq n; f(u_i) = 3n+i+9, f(v_i) = 4n+i+9, f(w_i) = 5n+i+9, f(uu_i) = 3n–i+1, f(vv_i) = 2n–i+1 \) and \( f(ww_i) = n–i+1 \).

Now we prove this labeling is a-vertex consecutive edge trimagic total.

For the edges \( uu_i, 1 \leq i \leq n; \)

\[ f(u)+f(uu_i)+f(u_i) = 3n+5+3n–i+1+3n+i+9 = 9n+15 = \lambda_1 \text{(say)}. \]

For the edges \( vv_i, 1 \leq i \leq n; \)

\[ f(v)+f(vv_i)+f(v_i) = 3n+6+2n–i+1+4n+i+9 = 9n+16 = \lambda_2 \text{(say)}. \]

For the edges \( ww_i, 1 \leq i \leq n; \)

\[ f(w)+f(ww_i)+f(w_i) = 3n+7+n–i+1+5n+i+9 = 9n+17 = \lambda_3 \text{(say)}. \]

For the edge \( ux, f(x)+f(ux)+f(u) = 3n+8+3n+4+3n+5 = 9n+17 = \lambda_3. \)

For the edge \( xv, f(x)+f(xv)+f(v) = 3n+8+3n+3+3n+6 = 9n+17 = \lambda_3. \)

For the edge \( vy, f(v)+f(vy)+f(v) = 3n+6+3n+2+3n+9 = 9n+17 = \lambda_3. \)
For the edge $yw, f(y)+f(yw)+f(w) = 3n+9+3n+1+3n+7 = 9n+17 = \lambda_3$.

Hence for each edge $uv \in E$ the value of $f(u)+f(uv)+f(v)$ yields any of the trimagic constants $\lambda_1 = 9n+15$, $\lambda_2 = 9n+16$ and $\lambda_3 = 9n+17$. This proves that the graph $(K_{1,n}; 3)$, $n \geq 3$ admits a-vertex consecutive edge trimagic total labeling for $a = 3n+4$.

**Example 4.2.6.** A-vertex consecutive edge trimagic total labeling of $(K_{1, 6}; 3)$ with $a = 22$ and is given in figure 4. 2.

![Figure 4. 2: $(K_{1, 6}; 3)$ with $\lambda_1 = 69$, $\lambda_2 = 70$ and $\lambda_3 = 71$.](image)

**Theorem 4.2.7.** The double star $G(v; nP_3)$ admits a-vertex consecutive edge trimagic total labeling for even $n$.

**Proof.** Let $V = \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_n\} \cup \{u\}$ be the vertex set and $E = \{uv_i / 1 \leq i \leq n\} \cup \{v_iw_i / 1 \leq i \leq n\}$ be the edge set of the double star $G(v; nP_3)$ for even $n$. The double star $G(v; nP_3)$ has $2n+1$ vertices and $2n$ edges.
Define a bijection $f : V \cup E \to \{1, 2, \ldots, 4n+1\}$ such that $f(u) = 2n+1$, $f(v_i) = 2n+i+1$, $1 \leq i \leq n$; $f(w_i) = 3n+i+1$, $1 \leq i \leq n$;

$$f(uv_i) = \begin{cases} 2n-i+1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ n-i+1, & 1 \leq i \leq n \text{ and } i \text{ is even}, \end{cases}$$

and $f(v_iw_i) = 2n-2i+1$, $1 \leq i \leq n$.

Now we prove the property of a-vertex consecutive edge trimagic total labeling.

For the edges $uv_i$, $1 \leq i \leq n$;

For odd $i$, $f(u)+f(uv_i)+f(v_i) = 2n+1+2n–i+1+2n+i+1 = 6n+3 = \lambda_1$(say).

For even $i$, $f(u)+f(uv_i)+f(v_i) = 2n+1+n–i+2+2n+i+1 = 5n+4 = \lambda_2$(say).

For the edges $v_iw_i$, $1 \leq i \leq n$;

$$f(v_i)+f(v_iw_i)+f(w_i) = 2n+i+1+2n–2i+1+3n+i+1 = 7n+3 = \lambda_3$(say).

Which concludes that, there exist three distinct trimagic constants $\lambda_1 = 6n+3$, $\lambda_2 = 5n+4$ and $\lambda_3 = 7n+3$. Hence for each edge $uv \in E$ the value of $f(u)+f(uv)+f(v)$ yields any of the magic constant $\lambda_1$ or $\lambda_2$ or $\lambda_3$.

This proves that the double star $G(v; nP_3)$ admits a-vertex consecutive edge trimagic total labeling with $a = 2n$ for even $n$.

**Example 4.2.8.** A-vertex consecutive edge trimagic total labeling of the double star $G(v; 6P_3)$ with $a = 12$ and is given in figure 4.3.
Figure 4.3. G(v; 6P₃) with λ₁ = 34, λ₂ = 39 and λ₃ = 45.

**Theorem 4.2.9.** The disconnected graph \((K₁, p ∪ K₁, q ∪ K₁, r)\) admits a-vertex consecutive edge trimagic total labeling.

**Proof.** Let \(V = \{u, u₁, u₂, \ldots, u_p, v, v₁, v₂, \ldots, v_q, w, w₁, w₂, \ldots, w_r\}\) be the vertex set and \(E = \{uuᵢ, vvⱼ, wwₖ / 1 ≤ i ≤ p, 1 ≤ j ≤ q, 1 ≤ k ≤ r\}\) be the edge set of the graph \((K₁, p ∪ K₁, q ∪ K₁, r)\). The disconnected graph \((K₁, p ∪ K₁, q ∪ K₁, r)\) has \((p+q+r+3)\) vertices and \((p+q+r)\) edges.

Define a bijection \(f : V ∪ E \rightarrow \{1, 2, \ldots, 2(p+q+r)+3\}\) such that \(f(u) = p+q+r+1, f(v) = p+q+r+2, f(w) = p+q+r+3, f(uᵢ) = p+q+r+3+i, 1 ≤ i ≤ p; f(vⱼ) = 2p+q+r+3+j, 1 ≤ j ≤ q; f(wₖ) = 2p+2q+r+3+k, 1 ≤ k ≤ r; f(uuᵢ) = p+1−i, 1 ≤ i ≤ p; f(vvⱼ) = p+q+1−j, 1 ≤ j ≤ q and f(wwₖ) = p+q+r+1−k, 1 ≤ k ≤ r. \)

Now we prove that this labeling is a-vertex consecutive edge trimagic total.
For the edges $uu_i$, $1 \leq i \leq p$;
\[ f(u) + f(uu_i) + f(u_i) = p + q + r + 1 + p + 1 - i + p + q + r + 3 + i = 3p + 2q + 2r + 5 = \lambda_1 \text{(say)}. \]

For the edges $vv_j$, $1 \leq j \leq q$;
\[ f(v) + f(vv_j) + f(v_j) = p + q + r + 2 + p + q + 1 - j + 2p + q + r + 3 + j = 4p + 3q + 2r + 6 = \lambda_2 \text{(say)}. \]

For the edges $ww_k$, $1 \leq k \leq r$;
\[ f(w) + f(ww_k) + f(w_k) = p + q + r + 3 + p + q + r + 1 - k + 2p + 2q + r + 3 + k = 4p + 4q + 3r + 7 = \lambda_3 \text{(say)}. \]

Hence for each edge $uv \in E$, the value of $f(u) + f(uv) + f(v)$ yields any of the trimagic constants $\lambda_1 = 3p + 2q + 2r + 5$, $\lambda_2 = 4p + 3q + 2r + 6$ and $\lambda_3 = 4p + 4q + 3r + 7$. This proves that the graph $K_{1,p} \cup K_{1,q} \cup K_{1,r}$ admits a-vertex consecutive edge trimagic total labeling with $a = p + q + r$.

**Example 4.2.10.** A-vertex consecutive edge trimagic total labeling of $K_{1,7} \cup K_{1,6} \cup K_{1,8}$ with $a = 21$ and is given in figure 4.4.

Figure 4.4. $K_{1,7} \cup K_{1,6} \cup K_{1,8}$ with $\lambda_1 = 54$, $\lambda_2 = 68$ and $\lambda_3 = 83$. 

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4.3. Corona graphs

In this section, we prove that the corona graphs $P_3 \odot \overline{K}_n$ and $C_n \odot K_1$ admits a-vertex consecutive edge trimagic total labeling.

**Theorem 4.3.1.** The graph $P_3 \odot \overline{K}_n$ admits a-vertex consecutive edge trimagic total labeling for even $n$.

**Proof.** Let $V = \{u, v, w, u_i, v_i, w_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{uu_i, vv_i, ww_i, uv, vw / 1 \leq i \leq n\}$ be the edge set of the graph $P_3 \odot \overline{K}_n$. The graph $P_3 \odot \overline{K}_n$ has $3n+3$ vertices and $3n+2$ edges.

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 6n+5\}$ such that $f(u) = 4n–3$, $f(v) = 4n–2$, $f(w) = 4n–1$, $f(u_i) = 4n+i–1$, $1 \leq i \leq n$; $f(v_i) = 5n+i–1$, $1 \leq i \leq n$; $f(w_i) = 6n+i–1$, $1 \leq i \leq n$; $f(uu_i) = 3n–i+1$, $1 \leq i \leq n$; $f(vv_i) = 2n–i+1$, $1 \leq i \leq n$; $f(ww_i) = n–i+1$, $1 \leq i \leq n$; $f(uv) = 3n+2$ and $f(vw) = 3n+1$.

Now we prove this labeling is a-vertex consecutive edge trimagic total.

For the edges $uu_i$, $1 \leq i \leq n$;

$f(u)+f(uu_i)+f(u_i) = 4n–3+3n–i+1+4n+i–1 = 11n–3 = \lambda_1$(say).

For the edges $vv_i$, $1 \leq i \leq n$;

$f(v)+f(vv_i)+f(v_i) = 4n–2+2n–i+1+5n+i–1 = 11n–2 = \lambda_2$(say).

For the edges $ww_i$, $1 \leq i \leq n$;

$f(w)+f(ww_i)+f(w_i) = 4n–1+n–i+1+6n+i–1 = 11n–1 = \lambda_3$(say).
For the edge $uv$, $f(u) + f(uv) + f(v) = 4n - 3 + 3n + 2 + 4n - 2 = 11n - 3 = \lambda_1$.

For the edge $vw$, $f(v) + f(vw) + f(w) = 4n - 2 + 3n + 1 + 4n - 1 = 11n - 2 = \lambda_2$.

Hence for each edge $uv \in E$, the value of $f(u) + f(uv) + f(v)$ yields any of the trimagic constants $\lambda_1 = 11n - 3$, $\lambda_2 = 11n - 2$ and $\lambda_3 = 11n - 1$. Therefore, the graph $P_3 \odot \overline{K}_n$ admits a-vertex consecutive edge trimagic total labeling for $a = 3n + 2$.

**Example 4.3.2.** A-vertex consecutive edge trimagic total labeling of $P_3 \odot \overline{K}_6$ with $a = 20$ is given in figure 4.5.

![Figure 4.5](image)

Figure 4.5. $P_3 \odot \overline{K}_6$ with $\lambda_1 = 63$, $\lambda_2 = 64$ and $\lambda_3 = 65$.

**Theorem 4.3.3.** The Crown graph $C_n \odot K_1$ admits a-vertex consecutive edge trimagic total labeling.

**Proof:** Let $V = \{u_i, v_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{u_iv_i / 1 \leq i \leq n\} \cup \{u_iu_{i+1} / 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ be the edge set of the crown graph $C_n \odot K_1$. The Crown graph $C_n \odot K_1$ has $2n$ vertices and $2n$ edges.

**Case 1.** $n$ is odd.

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 4n\}$ such that
\[ f(u_i) = \begin{cases} 
2n \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{5n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}, 
\end{cases} \]

\[ f(v_i) = \begin{cases} 
3n \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{7n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}, 
\end{cases} \]

\[ f(u_i u_{i+1}) = 2n-i, \ 1 \leq i \leq n-1; \]

\[ f(u_i v_i) = n-i+1, \ 1 \leq i \leq n; \ f(u_n u_1) = 2n. \]

Now we prove this labeling is a vertex consecutive edge trimagic total.

For the edge \( u_n u_1 \),

\[ f(u_n) + f(u_n u_1) + f(u_1) = 2n + 2n + \frac{n+1}{2} + 2n + \frac{i+1}{2} = \frac{13n+3}{2} = \lambda_1 \text{(say)}. \]

**Subcase 1.1.** \( i \) is odd.

For the edges \( u_i u_{i+1}, \ 1 \leq i \leq n-1; \)

\[ f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2n + \frac{i+1}{2} + 2n - i + \frac{5n+i+1}{2} = \frac{13n+3}{2} = \lambda_1. \]

For the edges \( u_i v_i, \ 1 \leq i \leq n; \)

\[ f(u_i) + f(u_i v_i) + f(v_i) = 2n + \frac{i+1}{2} + n - i + 3n + \frac{i+1}{2} = 6n + 2 = \lambda_2 \text{(say)}. \]

**Subcase 1.2.** \( i \) is even.

For the edges \( u_i u_{i+1}, \ 1 \leq i \leq n-1; \)

\[ f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{5n+i+1}{2} + 2n - i + \frac{i+1}{2} + n + 2 = \frac{13n+3}{2} = \lambda_1. \]
For the edges $u_i v_i, 1 \leq i \leq n$;

$$f(u_i) + f(u_i v_i) + f(v_i) = \frac{5n+i+1}{2} + n - i + 1 + \frac{7n+i+1}{2} = 7n+2 = \lambda_3 (\text{say}).$$

Hence for each edge $uv \in E$, the value of $f(u) + f(uv) + f(v)$ yields any of the trimagic constants $\lambda_1 = \frac{13n+3}{2}$, $\lambda_2 = 6n+2$ and $\lambda_3 = 7n+2$.

This proves that the graph $C_n \otimes K_1$ admits $a$-vertex consecutive edge trimagic total labeling for $a = 2n$, for odd $n$.

**Case 2.** $n$ is even.

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 4n\}$ such that

$$f(u_i) = \begin{cases} 
2n+\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{5n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even},
\end{cases}$$

$$f(u_i) = \begin{cases} 
3n+\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{7n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}.
\end{cases}$$

$f(u_i u_{i+1}) = 2n-i, 1 \leq i \leq n-1$;

$f(u_i v_i) = n-i+1, 1 \leq i \leq n-1$;

$f(u_n u_1) = 2n$.

Now we prove this labeling is $a$-vertex consecutive edge trimagic total.

For the edge $u_n u_1$,

$$f(u_n) + f(u_n u_1) + f(u_1) = \frac{5n+n}{2} + 2n + 2n + \frac{1+1}{2} = 7n+1 = \lambda_1 (\text{say}).$$
Subcase 2.1. \(i\) is odd.

For the edges \(u_iu_{i+1}, 1 \leq i \leq n-1;\)

\[
f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = 2n + \frac{i+1}{2} + 2n - i + \frac{5n+i+1}{2} = \frac{13n+2}{2} = \lambda_2 \text{(say)}.\]

For the edges \(u_iv_i, 1 \leq i \leq n;\)

\[
f(u_i) + f(u_iv_i) + f(v_i) = 2n + \frac{i+1}{2} + n - i + 1 + 3n + \frac{i+1}{2} = 6n+2 = \lambda_3 \text{(say)}.\]

Subcase 2.2. \(i\) is even.

For the edges \(u_iu_{i+1}, 1 \leq i \leq n-1;\)

\[
f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = \frac{5n+i}{2} + 2n - i + 2n + \frac{i+1+i+1}{2} = \frac{13n+2}{2} = \lambda_2.\]

For the edges \(u_iv_i, 1 \leq i \leq n;\)

\[
f(u_i) + f(u_iv_i) + f(v_i) = \frac{5n+i}{2} + n - i + 1 + \frac{7n+i}{2} = 7n+1 = \lambda_1.\]

Hence for each edge \(uv \in E,\) the value of \(f(u) + f(uv) + f(v)\) yields any of the trimagic constants \(\lambda_1 = 7n+1, \lambda_2 = \frac{13n+2}{2} \text{ and } \lambda_3 = 6n+2.\)

This proves that the graph \(C_9 \odot K_1\) admits \(a\)-vertex consecutive edge trimagic total labeling for \(a = 2n,\) for even \(n.\)

The theorem follows from case 1 and case 2.

Example 4.3.4. A-vertex consecutive edge trimagic total labeling of the graphs \(C_9 \odot K_1\) and \(C_{10} \odot K_1\) are given in figure 4.6 and figure 4.7, respectively.
Theorem 4.3.5. The graph $C_n \odot \overline{K}_2$ admits a vertex consecutive edge trimagic total labeling.

Proof. Let $V = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_n\}$ be the vertex set and $E = \{u_iv_i / 1 \leq i \leq n\} \cup \{u_iw_i / 1 \leq i \leq n\} \cup \{u_iu_{i+1} / 1 \leq i \leq n-1\} \cup \{u_1u_n\}$ be the edge set of $C_n \odot \overline{K}_2$. Then $C_n \odot \overline{K}_2$ has $3n$ vertices and $3n$ edges.
Case 1. n is odd.

Define a bijection \( f : V \cup E \to \{1, 2, \ldots, 6n\} \) such that

\[
f(u_i) = \begin{cases} 
3n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
3n + \frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
4n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
4n + \frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(w_i) = \begin{cases} 
5n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
5n + \frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[f(u_i, v_i) = 2n - i + 1, 1 \leq i \leq n; \ f(u_i, w_i) = n - i + 1, 1 \leq i \leq n;
\]

\[f(u_i u_{i+1}) = 3n - i, 1 \leq i \leq n - 1 \text{ and } f(u_1 u_n) = 3n.
\]

Now we prove the above labeling is an edge trimagic total.

Consider the edges \( u_i v_i, 1 \leq i \leq n \).

For odd \( i \),
\[
f(u_i) + f(u_i, v_i) + f(v_i) = 3n + \frac{i+1}{2} + 2n - i + 1 + 4n + \frac{i+1}{2} = 9n + 2 = \lambda_1 (\text{say}).
\]

For even \( i \),
\[
f(u_i) + f(u_i, v_i) + f(v_i) = 3n + \frac{n+i+1}{2} + 2n - i + 1 + 4n + \frac{n+i+1}{2} = 10n + 2 = \lambda_2 (\text{say}).
\]

Consider the edges \( u_i w_i, 1 \leq i \leq n \).

For odd \( i \),
\[
f(u_i) + f(u_i, w_i) + f(w_i) = 3n + \frac{i+1}{2} + n - i + 1 + 5n + \frac{i+1}{2} = 9n + 2 = \lambda_1.
\]

For even \( i \),
\[
f(u_i) + f(u_i, w_i) + f(w_i) = 3n + \frac{n+i+1}{2} + n - i + 1 + 5n + \frac{n+i+1}{2} = 10n + 2 = \lambda_2.
\]

Consider the edges \( u_i u_{i+1}, 1 \leq i \leq n - 1 \).
For odd i, \( f(u_i) + f(u_{i+1}) + f(u_{i+1}) = 3n + \frac{n+1}{2} + 3n - i + 3n + \frac{n+i+1}{2} = \frac{19n+3}{2} = \lambda_3 \text{(say)}. \)

For even i, \( f(u_i) + f(u_{i+1}) + f(u_{i+1}) = 3n + \frac{n+i+1}{2} + 3n - i + 3n + \frac{i+1}{2} = \frac{19n+3}{2} = \lambda_3. \)

For the edge \( u_1u_n, f(u_1) + f(u_1u_n) + f(u_n) = 3n + \frac{1+i+1}{2} + 3n + 3n + \frac{n+1}{2} = \frac{19n+3}{2} = \lambda_3. \)

Hence for each edge \( uv \in E, f(u) + f(uv) + f(v) \) yields any one of the constants \( \lambda_1 = 9n+2, \lambda_2 = 10n+2 \) and \( \lambda_3 = \frac{19n+3}{2}. \)

Therefore, the graph \( C_n \circ \tilde{K}_2 \) admits a-vertex consecutive edge trimagic total labeling for odd \( n \) with \( a = 3n. \)

**Case 2.** \( n \) is even.

Define a bijection \( f : V \cup E \rightarrow \{1, 2, ..., 6n\} \) such that

\[
\begin{align*}
    f(u_i) &= \begin{cases} 
        3n + \frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
        3n + \frac{n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
    \end{cases} \\
    f(v_i) &= \begin{cases} 
        4n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
        4n + \frac{n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
    \end{cases} \\
    f(w_i) &= \begin{cases} 
        5n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
        5n + \frac{n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
    \end{cases}
\end{align*}
\]

\( f(u_1v_1) = 2n - i + 1, 1 \leq i \leq n; \ f(u_iw_i) = n - i + 1, 1 \leq i \leq n; \)

\( f(u_1u_{i+1}) = 3n - i, 1 \leq i \leq n - 1 \) and \( f(u_1u_n) = 3n. \)

Now we prove the above labeling is an edge trimagic total.
Consider the edges $u_iv_i$, $1 \leq i \leq n$.

For odd $i$, $f(u_i)+f(u_iv_i)+f(v_i) = 3n+\frac{i+1}{2}+2n-i+1+4n+\frac{i+1}{2} = 9n+2 = \lambda_1$(say).

For even $i$, $f(u_i)+f(u_iv_i)+f(v_i) = 3n+\frac{n+i}{2}+2n-i+1+4n+\frac{n+i}{2} = 10n+1 = \lambda_2$(say).

Consider the edges $u_iw_i$, $1 \leq i \leq n$.

For odd $i$, $f(u_i)+f(u_iw_i)+f(w_i) = 3n+\frac{i+1}{2}+n-i+1+5n+\frac{i+1}{2} = 9n+2 = \lambda_1$.

For even $i$, $f(u_i)+f(u_iw_i)+f(w_i) = 3n+\frac{n+i}{2}+n-i+1+5n+\frac{n+i}{2} = 10n+1 = \lambda_2$.

Consider the edges $u_iu_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_i)+f(u_iu_{i+1})+f(u_{i+1}) = 3n+\frac{i+1}{2}+3n-i+3n+\frac{n+i+1}{2} = \frac{19n+2}{2} = \lambda_3$(say).

For even $i$, $f(u_i)+f(u_iu_{i+1})+f(u_{i+1}) = 3n+\frac{n+i}{2}+3n-i+3n+\frac{i+1+1}{2} = \frac{19n+2}{2} = \lambda_3$.

For the edge $u_1u_n$, $f(u_1)+f(u_1u_n)+f(u_n) = 3n+\frac{i+1}{2}+3n+3n+\frac{n+n}{2} = \frac{19n+2}{2} = \lambda_3$.

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the constants

$\lambda_1 = 9n+2$, $\lambda_2 = 10n+1$ and $\lambda_3 = \frac{19n+2}{2}$.

Therefore, the graph $C_n \odot \overline{K}_2$ admits a-vertex consecutive edge trimagic total labeling for even $n$ with $a = 3n$.

The theorem follows from Case 1 and Case 2.

**Example 4.3.6.** A-vertex consecutive edge trimagic total labeling of $C_5 \odot \overline{K}_2$ and $C_8 \odot \overline{K}_2$ are given in figure 4.8 and figure 4.9, respectively.
4.4. Snake graphs

In this section, we prove that the quadrilateral snake $Q_n$ and the triangular snake $T_{S_n}$ graphs admits a-vertex consecutive edge trimagic total labeling.

**Theorem 4.4.1** The Quadrilateral snake $Q_n$ admits a-vertex consecutive edge trimagic total labeling with $a = 4n - 4$.

**Proof.** Let $V = \{u_i, v_i, w_i / 1 \leq i \leq n-1\} \cup \{u_n\}$ be the vertex set and $E = \{u_i v_i, v_i w_i, u_i u_{i+1}, u_{i+1} w_i / 1 \leq i \leq n-1\}$ be the edge set of the Quadrilateral Snake $Q_n$. Then $Q_n$ has $4n-4$ edges

Figure 4.8: $C_5 \otimes K_2$ with $\lambda_1 = 47$, $\lambda_2 = 52$ and $\lambda_3 = 49$.

Figure 4.9: $C_8 \otimes K_2$ with $\lambda_1 = 74$, $\lambda_2 = 81$ and $\lambda_3 = 77$. 
and $3n-2$ vertices.

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 7n-6\}$ such that $f(u_i) = 4n+i-4$, $1 \leq i \leq n$; $f(v_i) = 5n+i-4$, $1 \leq i \leq n-1$; $f(w_i) = 6n+i-5$, $1 \leq i \leq n-1$; $f(u_iu_{i+1}) = 4n-2i-2$, $1 \leq i \leq n-1$; $f(u_iv_i) = 4n-2i-3$, $1 \leq i \leq n-1$; $f(u_{i+1}w_i) = 2n-2i-1$, $1 \leq i \leq n-1$ and $f(v_iw_i) = 2n-2i$, $1 \leq i \leq n-1$.

Now we prove the above labeling is an edge trimagic total.

For the edges $u_iu_{i+1}$, $1 \leq i \leq n-1$;

$$f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = 4n+i-4+4n-2i-2+4n+i+1-4 = 12n-9 = \lambda_1 (\text{say}).$$

For the edges $u_iv_i$, $1 \leq i \leq n-1$;

$$f(u_i) + f(u_iv_i) + f(v_i) = 4n+i-4+4n-2i-3+5n+i-4 = 13n-11 = \lambda_2 (\text{say}).$$

For the edges $u_{i+1}w_i$, $1 \leq i \leq n-1$;

$$f(u_{i+1}) + f(u_{i+1}w_i) + f(w_i) = 4n+i+1-4+2n-2i-1+6n+i-5 = 12n-9 = \lambda_1.$$

For the edges $v_iw_i$, $1 \leq i \leq n-1$;

$$f(v_i) + f(v_iw_i) + f(w_i) = 5n+i-4+2n-2i+6n+i-5 = 13n-9 = \lambda_3 (\text{say}).$$

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the constants $\lambda_1 = 12n-9$, $\lambda_2 = 13n-11$ and $\lambda_3 = 13n-9$. Therefore, the Quadrilateral snake $Q_n$ admits a-vertex consecutive edge trimagic total labeling with $a = 4n-4$.

**Example 4.4.2.** A-vertex consecutive edge trimagic total labeling of the Quadrilateral snake $Q_6$ is given in figure 4.10.
Figure 4.10. $Q_6$ with $\lambda_1 = 63$, $\lambda_2 = 67$ and $\lambda_3 = 69$ with $a = 20$.

**Theorem 4.4.3.** The triangular snake graph $TS_n$ admits a-vertex consecutive edge trimagic total labeling for $n \geq 3$.

**Proof.** Let $V = \{u_i, v_i / 1 \leq i \leq n-1\} \cup \{u_n\}$ be the vertex set and $E = \{u_i v_i / 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} / 1 \leq i \leq n-1\}$ be the edge set of the triangular snake graph $TS_n$. Then the triangular snake graph $TS_n$ has $2n-1$ vertices and $3n-3$ edges.

**Case 1.** $n$ is odd.

Define a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, 5n-4\}$ such that

$$f(u_i) = \begin{cases} 
3n+\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
3n+\frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}$$

$$f(v_i) = \begin{cases} 
4n+\frac{i+1}{2}, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\
4n+\frac{n+i-1}{2}, & 1 \leq i \leq n-1 \text{ and } i \text{ is even}
\end{cases}$$

$$f(u_i v_i) = \begin{cases} 
2n-i-1, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\
n-i, & 1 \leq i \leq n-1 \text{ and } i \text{ is even}
\end{cases}$$

$$f(u_{i+1} v_i) = \begin{cases} 
2n-i-2, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\
n-i+1, & 1 \leq i \leq n-1 \text{ and } i \text{ is even}
\end{cases}$$

and $f(u_i u_{i+1}) = 3n-i-2, 1 \leq i \leq n-1$. 

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Now we prove the above labeling is an edge trimagic total.

Consider the edges $u_i v_i$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_i) + f(u_i v_i) + f(v_i) = 3n + \frac{i+1}{2} - 3 + 2n - i - 1 + 4n + \frac{i+1}{2} - 3 = 9n - 6 = \lambda_1$ (say).

For even $i$, $f(u_i) + f(u_i v_i) + f(v_i) = 3n + \frac{n+i+1}{2} - 3 + n - i + 4n + \frac{n+i-1}{2} - 3 = 9n - 6 = \lambda_1$.

Consider the edges $u_{i+1} v_i$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_{i+1}) + f(u_{i+1} v_i) + f(v_i) = 3n + \frac{n+i+1}{2} - 3 + 2n - i - 2 + 4n + \frac{i+1}{2} - 3$

$$= \frac{19n-13}{2} = \lambda_2$$ (say).

For even $i$, $f(u_{i+1}) + f(u_{i+1} v_i) + f(v_i) = 3n + \frac{i+1}{2} - 3 + n - i + 1 + 4n + \frac{n+i-1}{2} - 3$

$$= \frac{17n-9}{2} = \lambda_3$$ (say).

Consider the edges $u_i u_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, $f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 3n + \frac{i+1}{2} - 3 + 3n - i - 2 + 3n + \frac{n+i+1}{2} - 3$

$$= \frac{19n-13}{2} = \lambda_2.$$}

For even $i$, $f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 3n + \frac{n+i+1}{2} - 3 + 3n - i - 2 + 3n + \frac{i+1}{2} - 3 = \frac{19n-13}{2} = \lambda_2$.

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the constants

$\lambda_1 = 9n - 6$, $\lambda_2 = \frac{19n-13}{2}$ and $\lambda_3 = \frac{17n-9}{2}$. Therefore, the triangular snake graph $TS_n$ admits an edge trimagic total labeling for odd $n$.  

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**Case 2.** \( n \) is even.

Define a bijection \( f : V \cup E \to \{1, 2, \ldots, 5n-4\} \) such that

\[
f(u_i) = \begin{cases} 
3n + \frac{i+1}{2} - 3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
3n + \frac{n+i}{2} - 3, & 1 \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
4n + \frac{i+1}{2} - 3, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\
4n + \frac{n+i}{2} - 3, & 1 \leq i \leq n-1 \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(u_i v_i) = \begin{cases} 
2n - i - 1, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\
n - i, & 1 \leq i \leq n-1 \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(u_{i+1} v_i) = \begin{cases} 
2n - i - 2, & 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \\
n - i - 1, & 1 \leq i \leq n-1 \text{ and } i \text{ is even}
\end{cases}
\]

and \( f(u_i u_{i+1}) = 3n - i - 2, 1 \leq i \leq n-1 \).

Now we prove the above labeling is an edge trimagic total.

Consider the edges \( u_i v_i, 1 \leq i \leq n-1 \).

For odd \( i \),
\[
f(u_i) + f(u_i v_i) + f(v_i) = 3n + \frac{i+1}{2} - 3 + 2n - i - 1 + 4n + \frac{i+1}{2} - 3 = 9n - 6 = \lambda_1 \text{ (say)}.\]

For even \( i \),
\[
f(u_i) + f(u_i v_i) + f(v_i) = 3n + \frac{n+i}{2} - 3 + n - i + 4n + \frac{n+i}{2} - 3 = 9n - 6 = \lambda_1.\]

Consider the edges \( u_{i+1} v_i, 1 \leq i \leq n-1 \).

For odd \( i \),
\[
f(u_{i+1}) + f(u_{i+1} v_i) + f(v_i) = 3n + \frac{n+i+1}{2} - 3 + 2n - i - 2 + 4n + \frac{i+1}{2} - 3
\]
\[
= \frac{19n-14}{2} = \lambda_2 \text{ (say)}.\]
For even $i$, 
\[
 f(u_{i+1}) + f(u_{i+1}v_i) + f(v_i) = 3n + \frac{i+1+1}{2} - 3 + n - 1 + 4n + \frac{n+i}{2} - 3
 = \frac{17n-12}{2} = \lambda_3 \text{(say)}.
\]

Consider the edges $u_iu_{i+1}$, $1 \leq i \leq n-1$.

For odd $i$, 
\[
 f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = 3n + \frac{i+1}{2} - 3 + 3n - i - 2 + 3n + \frac{n+i+1}{2} - 3 = \frac{19n-14}{2} = \lambda_2.
\]

For even $i$, 
\[
 f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = 3n + \frac{n+i}{2} - 3 + 3n - i - 2 + 3n + \frac{i+1+1}{2} - 3 = \frac{19n-14}{2} = \lambda_2.
\]

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the constants $
\lambda_1 = 9n-6$, $\lambda_2 = \frac{19n-14}{2}$ and $\lambda_3 = \frac{17n-12}{2}$. Therefore, the triangular snake graph $TS_n$ admits an edge trimagic total labeling for even $n$.

The theorem follows from case 1 and case 2.

**Example 4.4.4.** A-vertex consecutive edge trimagic total labeling of triangular snake graphs $TS_9$ and $TS_8$ are given in figure 4.11 and figure 4.12 respectively.

![Figure 4.11. TS_9 with \( \lambda_1 = 75, \lambda_2 = 79 \) and \( \lambda_3 = 72 \).](image1)

![Figure 4.12. TS_8 with \( \lambda_1 = 66, \lambda_2 = 69 \) and \( \lambda_3 = 62 \).](image2)

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We have dealt with some interesting results in a-vertex consecutive edge trimagic total labeling of some graphs throughout this chapter. Another interesting topic is edge trimagic total labeling of disconnected graphs. In the next chapter we will explore some results on edge trimagic total labeling of disconnected graphs.