Date: 19-06-2014

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Dear Prof. Jayasekaran,

I am happy to inform you that on the recommendations of the referees, your paper titled “Trimagic Labeling in Digraphs” co-authored with M. Regees (Ref No. 1312016) has been accepted for publication in the Journal of Discrete Mathematical Sciences and Cryptography.

Sincerely,

Bal Kishan Dass

(Chief Editor, JDMSC)

Professor of Mathematics

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EDGE TRIMAGIC LABELING OF SOME GRAPHS

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ABSTRACT: An edge magic total labeling of a \((p, q)\) graph is a bijection \(f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p+q\}\) such that for each edge \(xy \in E(G)\), the value of \(f(x) + f(y) + f(xy)\) is a constant \(k\). If there exists two constants \(k_1\) and \(k_2\), such that \(f(x) + f(y) + f(xy)\) is either \(k_1\) or \(k_2\), it is said to be an edge bimagic total labeling. In this paper we introduce the concept of edge trimagic labeling and prove that, a pyramid graph \(P_4(n)\), \(K_4\) snake graph, wheel snake \(nW_4\) and the fan graph \(F_4\) are edge trimagic total.

AMS SUBJECT CLASSIFICATION: 05C78.

KEYWORDS: Bijection, Magic labeling, Bimagic labeling, Trimagic labeling.

1. INTRODUCTION

We begin with simple, finite and undirected graph \(G = (V(G), E(G))\). A graph labeling is an assignment of integers to elements of a graph, the vertices or edges, or both, subject to some conditions.

The Cartesian product of two graphs \(G_1\) and \(G_2\) denoted by \(G = G_1 \times G_2\) is the graph \(G\) such that \(V(G) = V(G_1) \times V(G_2)\), that is every vertex of \(G_1 \times G_2\) is an ordered pair \((u, v)\), where \(u \in V(G_1)\) and \(v \in V(G_2)\), and two distinct vertices \((u, v)\) and \((x, y)\) are adjacent in \(G_1 \times G_2\) if and only if \(u = x\) and \(vy \in E(G_2)\) or \(v = y\) and \(ux \in E(G_1)\). A wheel \(W_n\) with \(n\) spokes, is a graph that has a centre \(x\) connected to all the \(n\) vertices in the cycle \(C_n\). A fan graph \(F_n\) can be constructed from a wheel by deleting one edge of the \(n\)-cycle[4]. The graphs of the form \(P_n \times C_m\) are called prisms. \(P_m \circ K_n\) is a graph obtained by introducing \(n\) new pendant edges at each vertex of path \(P_m\)[2].

The concept of graph labeling was introduced by Rosa[8] in 1967. The labeling of a graph is a mapping that takes graph elements, the vertices or edges or both, to positive numbers. In 1970 Kotzig and Rosa[7] defined, a magic labeling of graph \(G = (V(G), E(G))\) is a bijection \(f: V \cup E \rightarrow \{1, 2, 3, \ldots, p+q\}\) such that, for each edge \(xy \in E(G)\), \(f(x) + f(y) + f(xy)\) is a magic constant. In 1996, Ringel and LLado called this labeling as edge magic. In 2001, Wallis[10] introduced this as edge magic total labeling. A graph is called edge magic total if it admits an edge magic total labeling. An edge magic total labeling is called a super edge magic total labeling of a graph \(G\) if the vertices are labeled with the smallest positive integers.

In 2004, J. Baskar Babujee[1] introduced the Bimagic labeling of graphs. A graph \(G\) is said to be bimagic, if there exists two distinct magic constants \(k_1\) (or) \(k_2\).
A bijection \( f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\} \) is called an edge bimagic total labeling if \( f(x) + f(y) + f(xy) = k_1 \) (or \( k_2 \)) for each edge \( xy \in E(G) \), where \( k_1 \) and \( k_2 \) are two distinct constants called magic constants[3]. A graph \( G(V, E) \) with edge bimagic total labeling is said to be a super edge bimagic total labeling if it has the additional property that the vertices are labeled with smallest positive integers.

**Theorem 1.1 [3]:** Let \( G \) be \( Py(n) \). Then \( G \) is edge bimagic total for \( n \geq 3 \).

**Theorem 1.2 [3]:** The graph \( nK_4 \) is edge bimagic total for \( n \geq 1 \).

**Theorem 1.3 [3]:** The graph \( nW_4 \) is edge bimagic total for \( n \geq 1 \).

**Theorem 1.4 [4]:** The fan graph \( F_n \) is edge-magic for every positive integer \( n \).

For further references, we use Dynamic survey of graph labelings by J. A. Galian[6]. We follow the notations and terminology of[5]. In this paper, we introduce the concept “**Edge Trimagic Labeling**” and prove, the pyramid graph \( Py(n) \), \( K_4 \) snake graph, wheel snake \( nW_4 \) and a fan graph \( F_n \) are edge trimagic total graphs.

**2. MAIN RESULTS**

**Definition 2.1.** An edge trimagic total labeling of a \((p, q)\) graph \( G \) is a bijective function \( f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, p + q\} \) such that for each edge \( xy \in E(G) \), the value of \( f(x) + f(y) + f(xy) \) is equal to any of the distinct constants \( k_1 \) or \( k_2 \) or \( k_3 \). A graph \( G \) is said to be edge trimagic total if it admits an edge trimagic total labeling. An edge trimagic total labeling is called **super edge trimagic total labeling** if \( G \) has the additional property that the vertices are labeled with smallest positive integers.

**Example 2.2:** Edge trimagic total labeling of \( P_4 \odot K_{4,6} \).

![Diagram](image)

**Figure 1:** \( K_1 = 20 \), \( K_2 = 21 \) and \( K_3 = 25 \)
Example 2.3: Edge trimagic total labeling of $P_2 \odot K_{1,6}$.

Figure 2: $K_1 = 38, K_2 = 44$ and $K_3 = 54$

Example 2.3: Edge trimagic total labeling of $P_3 \odot K_{1,6}$.

Figure 3: $K_1 = 62, K_2 = 63$ and $K_3 = 65$.

Definition 2.5 [3]: The graphs of the form $P_n \times C_3$ are called prisms. These can be viewed as grids on cylinders of height $n - 1$. Let $V(P_n \times C_3) = \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq 3\}$. A pyramid graph is obtained from $P_n \times C_3$ by adding a new vertex $v_0$ adjacent to the three vertices $v_{11}$, $v_{12}$, and $v_{13}$ of $P_n \times C_3$. This graph has $3n + 1$ vertices and $6n$ edges and is denoted by $P_\gamma(n)$. The graph $P_\gamma(5)$ is given in Fig. 4.
Theorem 2.6: The pyramid graph $P_{y}(n)$ is edge trigram total for $n \geq 3$.

Proof: Consider a pyramid graph $P_{y}(n)$. Let $V(P_{y} \times C_{3}) = \{v_{i}/ 1 \leq i \leq n, 1 \leq j \leq 3\}$. The pyramid graph $P_{y}(n)$ has $3n + 1$ vertices and $6n$ edges. Define a bijection $f: V \cup E \rightarrow \{1, 2, ..., 9n + 1\}$ such that $f(v_{i,j}) = 1$, $f(v_{i}) = 3i - 1$, $f(v_{i,j}) = 3i$, $f(v_{i,j}) = 3i + 1$.

![Figure 5: Pyramid Graph $P_{y}(n)$](image)

The edges are defined as, $f(v_{i,j}, v_{i,j+1}) = 9n + 1$, $f(v_{i,j}, v_{i+1,j}) = 9n$, $f(v_{i,j}, v_{i,j+1}) = 9n - 1$, $f(v_{i,j}, v_{i-1,j}) = 9n - 6i + 3$, $f(v_{i,j}, v_{i,j+1}) = 9n - 6i + 2$, $f(v_{i,j}, v_{i,j+1}) = 9n - 6i + 4$, $(i = 1, 2, ..., n)$ and for $j = 1, 2, ..., n - 1$, $f(v_{i,j+1}, v_{i+1,j+1}) = 9n - 6j + 1$, $f(v_{i,j+1}, v_{i+1,j}) = 9n - 6j$, $f(v_{i,j+1}, v_{i+1,j+1}) = 9n - 6j - 1$.

Now, we shall prove this labeling is edge trigram total as follows,

\[ f(v_{o,0}) + f(v_{o,1}) + f(v_{o,2}) = 1 + 2 + 9n + 1 = 9n + 4 = k_{1} \text{(say)}. \]
\[ f(v_{o,0}) + f(v_{o,1}) + f(v_{o,2}) = 1 + 3 + 9n + 4 = k_{2} \text{say}. \]
\[ f(v_{o,0}) + f(v_{o,1}) + f(v_{o,2}) = 1 + 4 + 9n - 1 = 9n + 4 = k_{3} \text{say}. \]
\[ f(v_{o,0}) + f(v_{o,1}) + f(v_{o,2}) = 3i - 1 + 3i + 9n - 6i + 3 = 9n + 2 = k_{4} \text{say}. \]
\[ f(v_{o,0}) + f(v_{o,1}) + f(v_{o,2}) = 3i + 3i + 9n - 6i + 2 = 9n + 3 = k_{5} \text{say}. \]
\[ f(v_{o,0}) + f(v_{o,1}) + f(v_{o,2}) = 3i + 1 + 3i - 1 + 9n - 6i + 4 = 9n + 4 = k_{6} \text{say}. \]
Therefore, for each edge uv, \( f(u) + f(v) + f(uv) \) yields any one of the magic constant \( 9n + 2 \) or \( 9n + 3 \) or \( 9n + 4 \). Hence the pyramid graph \( P_y(n) \) is edge trimagic total for \( n \geq 3 \).

**Theorem 2.7:** The pyramid graph \( P_y(n) \) is super edge trimagic total.

**Proof:** We have proved that the pyramid graph \( P_y(n) \) is edge trimagic total. From the labeling given in the proof of Theorem 2.6, we have \( f(v_{00}) = 1, f(v_{i0}) = 3i - 1, f(v_{0i}) = 3i, f(v_{i1}) = 3i + 1, i = 1, 2, ..., n. \) Put \( i = 1 \), we get \( f(v_{11}) = 2, f(v_{12}) = 3, f(v_{13}) = 4. \) Put \( i = 2 \), we get \( f(v_{21}) = 5, f(v_{22}) = 6, f(v_{23}) = 7 \), and so on. Put \( i = n \), we get \( f(v_{n0}) = 3n - 1, f(v_{n1}) = 3n, f(v_{n2}) = 3n + 1 \). Since \( P_y(n) \) has \( 3n + 1 \) vertices and the \( 3n + 1 \) vertices have the labels \( 1, 2, ..., 3n + 1 \), the pyramid graph \( P_y(n) \) is super edge trimagic total.

**Example 2.8:** The pyramid graph \( P_y(4) \) is super edge trimagic total.

![Figure 6: Super Edge Trimagic Total with \( k_1 = 38, k_2 = 39 \) and \( k_3 = 40 \)](image)

**Definition 2.9.[3] K_4 snake graph:** A triangular snake graph[8] is a connected graph, all of whose blocks are triangles and whose block-cut point is a path. Consider a class of snakes all of whose blocks are the complete graphs on 4 vertices, \( K_4 \), snake graph with \( n \) blocks is denoted by \( nK_4 \), its vertex set is \( V = \{x_i / 1 \leq i \leq n + 1\} \cup \{y_i, w_i / 1 \leq i \leq n\} \) and the edge set is \( E = \{x_i x_{i+1}, x_i y_i, y_i x_{i+1}, x_i w_i, w_i x_{i+1} / 1 \leq i \leq n\} \). The graph \( 3K_4 \) is given in Fig. 7.
Here we prove that the $K_n$ snake graph $nK_n$ is edge trimagic total for $n \geq 4$, and it is super edge trimagic total graph.

**Theorem 2.10:** The $K_n$ snake graph $nK_n$ is edge trimagic total for $n \geq 4$.

**Proof:** Consider the $k^{th}$ block of the graph $nK_n$ and label the vertices and edges as follows.

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 9n + 1\}$ such that $f(x_i) = 3k - 2, f(x_{i+1}) = 3k + 1, f(y_j) = 3k - 1, f(w_l) = 3k, f(x_jx_{j+1}) = 9n - 6k + 5, f(x_jy_j) = 9n - 6k + 7, f(y_jx_{j+1}) = 9n - 6k + 2, f(x_jw_l) = 9n - 6k + 6, f(w_lx_{j+1}) = 9n - 6k + 3$ and $f(y_jw_l) = 9n - 6k + 4$.

Clearly from the $k^{th}$ block of $nK_n$, for $1 \leq k \leq n$.

$f(x_i) + f(y_j) + f(x_jy_j) = 3k - 2 + 3k - 1 + 9n - 6k + 7 = 9n + 4 = k_i$. (say)

$f(x_i) + f(w_l) + f(x_jx_{j+1}) = 3k - 2 + 3k + 9n - 6k + 6 = 9n + 4 = k_i$.

$f(x_i) + f(x_{i+1}) + f(x_jx_{j+1}) = 3k - 2 + 3k + 1 + 9n - 6k + 5 = 9n + 4 = k_i$.

$f(w_l) + f(x_{i+1}) + f(w_lx_{i+1}) = 3k + 3k + 1 + 9n - 6k + 3 = 9n + 4 = k_i$.

$f(y_j) + f(w_l) + f(y_jw_l) = 3k - 1 + 3k + 9n - 6k + 4 = 9n + 3 = k_i$. (say)

$f(x_{i+1}) + f(y_j) + f(x_{i+1}y_j) = 3k + 1 + 3k - 1 + 9n - 6k + 2 = 9n + 2 = k_i$. (say).
Hence for any edge $uv$ in the $k^{th}$ block, the expression $f(u) + f(v) + f(uv)$ yields any one of the constants $(9n + 2)$ or $(9n + 3)$ or $(9n + 4)$. Similar way we can prove that for any edge $uv$ in the graph $nK_4$, the expression $f(u) + f(v) + f(uv)$ yields any one of the constant $(9n + 2)$ or $(9n + 3)$ or $(9n + 4)$.

Therefore, there exist three different magic constants $9n + 2, 9n + 3, 9n + 4$. Hence the $K_4$ snake graph $nK_4$ is edge trimagic total.

**Theorem 2.11.** The $K_4$ snake graph $nK_4$ is super edge trimagic total.

**Proof:** We have proved that $nK_4$ is an edge trimagic total graph. The labeling given in the proof of Theorem 2.10, the vertices of the $k^{th}$ block have labelings $3k - 2, 3k - 1, 3k, 3k + 1$. The first block vertices get labels 1, 2, 3, 4; the second block vertices get labels 4, 5, 6, 7 and so on. Hence the vertices of the $nK_4$ graph has labels 1, 2, ..., $3n + 1$. Therefore, $K_4$ snake graph $nK_4$ is super edge trimagic total.

**Example 2.12:** The $K_4$ snake graph $4K_4$ is super edge trimagic total.

![Figure 9: 4K_4 is Super Edge Trimagic Total](image)

This $K_4$ snake graph $4K_4$ is super edge trimagic total graph with magic constants $k_1 = 40, k_2 = 39$ and $k_3 = 38$.

**Definition 2.13 [3]: Wheel Snake $nW_4$:** A class of snake denoted $nW_4$, all of whose blocks are isomorphic to the wheel $W_4$ and whose block-cut point graph is a path. The vertex set of $nW_4$ is $V = \{x_i | 1 \leq i \leq n + 1\} \cup \{y_i, z_i, w_i | 1 \leq i \leq n\}$ and the edge set is given as $E = \{x_i y_{i+1}, x_i z_{i+1}, x_i w_{i+1}, y_i x_{i+1}, z_i x_{i+1}, w_i x_{i+1} | 1 \leq i \leq n\}$. The graph given in Fig. 10 is $4W_4$.

We prove that the graph $nW_4$ is edge trimagic total for $n \geq 4$ and it is a super edge trimagic total graph.

![Figure 10: The snake 4W_4](image)
Theorem 2.14: The wheel snake graph $nW_4$ is edge trimagic total for $n \geq 14$.

Proof: Consider the $k^{th}$ block of $nW_4$ and label the vertices as follows.

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 12n + 1\}$ such that $f(x_1) = 4k - 3$, $f(y_1) = 4k - 1$, $f(x_{k+1}) = 4k + 1$, $f(z_1) = 4k - 2$, $f(w_1) = 4k$, $f(x_1y_1) = 12n - 8k + 8$, $f(y_1x_{k+1}) = 12n - 8k + 2$, $f(x_1z_1) = 12n - 8k + 9$, $f(x_1w_1) = 12n - 8k + 7$, $f(z_1x_{k+1}) = 12n - 8k + 4$, $f(w_1x_{k+1}) = 12n - 8k + 3$, $f(z_1y_1) = 12n - 8k + 6$ and $f(y_1w_1) = 12n - k + 5$. The labeling on the $k^{th}$ block is given in Fig. 12.

Figure 11: $k^{th}$ Block of $nW_4$.

Figure 12: The $k^{th}$ block of $nW_4$ and its Labeling.
Then for $1 \leq k \leq n$ we get,

$$f(x_i) + f(y_i) + f(x_iy_i) = 4k - 3 + 4k - 1 + 12n - 8k + 8 = 12n + 4 = k_i \text{ (say).}$$

$$f(x_i) + f(z_i) + f(x_i, z_i) = 4k - 3 + 4k - 2 + 12n - 8k + 9 = 12n + 4 = k_i.$$  

$$f(x_i) + f(w_i) + f(x_i, w_i) = 4k - 3 + 4k + 12n - 8k + 7 = 12n + 4 = k_i.$$  

$$f(z_i) + f(x_i, z_i) + f(z_i, x_i, z_i) = 4k - 2 + 4k + 1 + 12n - 8k + 4 = 12n + 3 = k_i \text{ (say).}$$  

$$f(y_i) + f(z_i) + f(y_i, z_i) = 4k - 1 + 4k - 2 + 12n - 8k + 6 = 12n + 3 = k_j.$$  

$$f(y_i) + f(x_i, y_i) + f(y_i, x_i, y_i) = 4k - 1 + 4k + 1 + 12n - 8k + 2 = 12n + 2 = k_j \text{ (say).}$$  

$$f(y_i) + f(w_i) + f(y_i, w_i) = 4k - 1 + 4k + 12n - 8k + 5 = 12n + 4 = k_i.$$  

$$f(w_i) + f(w_i, x_i) + f(w_i, x_i, w_i) = 4k + 4k + 1 + 12n - 8k + 3 = 12n + 4 = k_i.$$  

Hence, for any edge $uv$ in the $k^\text{th}$ block, the expression $f(u) + f(v) + f(uv)$ equals $(12n + 2)$, or $(12n + 3)$ or $(12n + 4)$. Similar way we can prove that for any edge $uv$ in the graph $nW_4$, the expression $f(u) + f(v) + f(uv)$ equals $(12n + 2)$, or $(12n + 3)$ or $(12n + 4)$. Therefore, there exist three different magic constants $12n + 2$, $12n + 3$ and $12n + 4$. Hence the wheel snake graph $nW_4$ is an edge trimagic total graph.

**Theorem 2.15:** The wheel snake graph $nW_4$ is super edge trimagic total.

**Proof:** We have proved that the wheel snake graph $nW_4$ is an edge trimagic total graph. The labeling given in the proof of the Theorem 2.14, the vertices of the $k^\text{th}$ block have labelings $4k - 3, 4k - 2, 4k - 1, 4k$ and $4k + 1$. The first block vertices get labels 1, 2, 3, 4, 5; the second block vertices get labels 6, 7, 8, 9 and so on. Since the vertices of the $nW_4$ graph has labels 1, 2, ..., $4n + 1$, the graph $nW_4$ is super edge trimagic total.

**Example 2.16:** The wheel snake graph $4W_4$ is super edge trimagic total.

![Figure 13: 4W_4 is Edge Trimagic Total](image)

The graph $4W_4$ is super edge trimagic total graph with magic constants $k_1 = 52$, $k_2 = 51$ and $k_3 = 50$. 
Theorem 2.17: The fan graph $F_n$ is edge trimagic total for every positive even integer $n$.

Proof: Let $F_n$ be a fan graph with vertices and edges as follows:

$V(F_n) = \{u\} \cup \{v_i/1 \leq i \leq n\}$

and

$E(F_n) = \{uv_i/1 \leq i \leq n\} \cup \{v_iv_{i+1}/1 \leq i \leq n-1\}.$

Define a bijection $f: V(F_n) \cup E(F_n) \to \{1, 2, \ldots, 3n\}$ such that $f(u) = 1$,

$$f(v_i) = \begin{cases} 
\frac{i+1}{2} + 1, & \text{if } i \text{ is odd} \\
\frac{i+n}{2} + 1, & \text{if } i \text{ is even}
\end{cases}$$

$$f(v_i, v_{i+1}) = 2n - i + 1, \quad i = 1 \text{ to } n - 1.$$

$$f(uv_i) = \begin{cases} 
\frac{3n - i + 1}{2} - 1, & \text{if } i \text{ is odd} \\
\frac{3n - i + 1}{2}, & \text{if } i \text{ is even}
\end{cases}$$

Now we can prove this labeling is edge trimagic total as follows.

For odd $i$, $f(u) + f(v_i) + f(uv_i) = 1 + \frac{i+1}{2} + 1 + 3n - \frac{i+1}{2} - 2 = 3n = k_1$ (say),

For even $i$, $f(u) + f(v_i) + f(uv_i) = 1 + \frac{n+i}{2} + 1 + 3n - \frac{i+1}{2} + 1 = \frac{7n+6}{2} = k_2$ (say).

To find the value of $f(v_i) + f(v_{i+1}) + f(v_i, v_{i+1})$.

For odd $i$, $f(v_i) + f(v_{i+1}) + f(v_i, v_{i+1}) = \frac{i+1}{2} + 1 + \frac{i+1+n}{2} + 1 + 2n - i + 1$

$= \frac{i+1+n+2n-2i}{2} = \frac{3n+8}{2} = k_3$ (say).

For even $i$, $f(v_i) + f(v_{i+1}) + f(v_i, v_{i+1}) = \frac{i+n}{2} + 1 + \frac{i+1}{2} + 1 + 2n - i + 1$

$= \frac{i+n+2+6+4n-2i}{2} = \frac{5n+8}{2} = k_4$.

Hence for each edge $uv_i$, $f(u) + f(v_i) + f(uv_i)$ yields any one of the trimagic constant $K_1 = 3n$ or $K_2 = \frac{7n+6}{2}$ or $K_3 = \frac{3n+8}{2}$ or $K_4 = \frac{5n+8}{2}$. Hence a fan graph $F_n$ is edge trimagic total for every positive even integer $n$. 
Theorem 2.18. A fan graph $F_n$ is super edge trimagic total for even $n$.

**Proof:** We have proved that the fan graph $F_n$ is edge trimagic total for every positive even integer $n$. The labeling given in the proof of theorem 2.17, the vertices get labels $f(v_l) = 1$, $f(v_j) = 2$, $f(v_i) = 3$, ..., $f(v_{n-2}) = (n/2) + 1$ and $f(v_{n-1}) = (n/2) + 2$, $f(v_n) = (n/2) + 3$, ..., $f(v_{n-1}) = n + 1$. Since the fan graph $F_n$ has $n + 1$ vertices and labels with 1, 2, ..., $n + 1$, the fan graph $F_n$ is super edge trimagic total.

**Example 2.19:** A fan graph $F_n$ is super edge trimagic total.

![Diagram of a fan graph](image)

Figure 14: Fan $F_n$ with $k_1 = 18$, $k_2 = 19$ and $k_3 = 24$

3. CONCLUSION

In this paper we have defined the edge trimagic total labeling of a graph and proved some classes of graphs namely pyramid graph $F_p(n)$, $K_5$, snake graph, wheel snake $nW_5$, and fan graph $F_n$ are super edge trimagic total. There may be many interesting trimagic graphs can be constructed in future.

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MORE RESULTS ON EDGE TRIMAGIC LABELING OF GRAPHS

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**ABSTRACT**

An edge magic total labeling of a (p, q) graph is a bijection \(f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}\) such that FOR each edge \(xy \in E(G)\), the value of \(f(x)+f(xy)+f(y)\) is a constant \(k\). If there exists three constants \(k_1\), \(k_2\) and \(k_3\) such that \(f(x)+f(xy)+f(y)\) is either \(k_1\) or \(k_2\) or \(k_3\), it is said to be an edge trimagic total labeling. In this paper we prove that the ladder \(L_n\) (odd \(n\)), triangular ladder \(TL_{nm}\) generalized Petersen graph \(P(n, \frac{n-1}{2})\), the helm graph \(H_n\) and the flower graph \(Fl_n\) are edge trimagic total and super edge trimagic total graphs.

**Keywords:** Function, Bijection, Magic labeling, Trimagic labeling.

**AMS Subject Classification:** 05C78.

**1. INTRODUCTION**

We begin with simple, finite and undirected graph \(G = (V(G), E(G))\). A graph labeling is an assignment of integers to elements of a graph, the vertices or edges, or both subject to certain conditions. The concept of graph labeling was introduced by Rosa in 1967. In 1970, Kotzig and Rosa\(^5\) defined, a magic labeling of graph \(G = (V(G), E(G))\) is a bijection \(f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}\) such that FOR each edge \(xy \in E(G)\), the value of \(f(x)+f(xy)+f(y)\) is a magic constant. In 1996, Ringel and Llado called this labeling as edge magic. In 2001, Wallis \(^6\) introduced this as edge magic total labeling. An edge magic total labeling is called a super edge magic total if the vertices are labeled with smallest positive integers. An edge trimagic total labeling is called a super edge trimagic total if the vertices are labeled with smallest positive integers.

In 2004, J.Baskar Babujee\(^1\) introduced the bimagic labeling of graphs. In 2013, C. Jayasekaran, M. Regees and C. Davidraj\(^3\) introduced the edge trimagic total labeling of graphs. An edge trimagic total labeling of a (p, q) graph \(G\) is a bijection \(f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}\) such that for each edge \(xy \in E(G)\), the value of \(f(x)+f(xy)+f(y)\) is equal to any of the distinct constants \(k_1\) or \(k_2\) or \(k_3\). A graph \(G\) is said to be an edge trimagic total if it admits an edge trimagic total labeling. An edge trimagic total labeling is called super edge trimagic total labeling if \(G\) has the additional property that the vertices are labeled with smallest positive integers. A simple graph in which there is an edge between each pair of vertices is called a complete graph. The complete graph with \(n\) vertices is denoted by \(K_n\). A walk of a graph \(G\) is an alternating sequence of vertices and edges \(v_0, x_1, v_1, \ldots, v_m, x_n, v_n\) beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. It is closed if \(v_0 = v_n\) and is open otherwise. An open walk in which no vertex appears more than once is called a path. A path with \(n\) vertices is denoted by \(P_n\). A ladder \(L_n\) is a graph \(P_n \times P_1\) with \(V(L_n) = \{u_i, v_i / 1 \leq i \leq n\}\) and \(E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_1 v_1 / 1 \leq i \leq n\}\). A triangular ladder \(TL_{nm}\) \(n \geq 2\), is a graph obtained from the ladder \(L_n\) by adding the edges \(u_{i1} v_{i1}\) for \(1 \leq i \leq n-1\). The generalized Petersen graph \(P(n, m)\) is a graph that consists of an outer-cycle \(v_0, y_1, y_2, \ldots, y_{nm}\) a set of \(n\) spokes \(y_i x_i, 0 \leq i \leq n-1\), and \(n\) edges \(x_i x_{i+m}, 0 \leq i \leq n-1\), where all subscripts are taken modulo \(n\). A wheel \(W_n\) with \(n\) spokes is a graph that has a centre \(x\) connected to all the \(n\) vertices in cycle \(C_n\). A helm \(H_n\) is constructed from a wheel \(W_n\) by adding \(n\) vertices of degree one adjacent to each terminal vertex. A flower graph \(Fl_n\) is constructed from a helm \(H_n\) by joining each vertex of degree one to the center.

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For further references, we use Dynamic survey of graph labeling by J. A. Galian[5]. We follow the notations and terminology of [2]. In [4], we introduced the concept edge trimagic and super edge trimagic total labeling and proved, the pyramid graph Py(n), K_4 snake graph, wheel snake nW_4 and a fan graph F_n are edge trimagic total and super edge trimagic total graphs. In this paper, we prove the ladder L_n, triangular ladder TL_n, generalized Petersen graph P(n, n-1), the helm graph H_n and the flower graph Fl_n are edge trimagic total and super edge trimagic total graphs.

2. EDGE TRIMAGIC LABELING FOR SOME FAMILIES OF GRAPHS

In this section, we prove edge trimagic total and super edge trimagic total labeling for the families of graphs like Ladder, Triangular Ladder, generalized Petersen graph, Helm and Flower graphs and give examples for edge trimagic labeling for each of the above graphs.

Theorem: 2.1 The Ladder L_n = P_n x P_2 admits an edge trimagic total labeling for all n ≥ 2.

Proof: Let V = {u_i, v_i | 1 ≤ i ≤ n} be the vertex set and E = {u_i u_{i+1}, v_i v_{i+1} | 1 ≤ i ≤ n-1} be the edge set of the ladder L_n. Then L_n has 2n vertices and 3n-2 edges.

Case: 1 n is odd.
Define a bijection f: V ∪ E → {1, 2, ..., 2n, 2n+1, ..., 5n-2} such that
\[ f(u_i) = \left\{ \begin{array}{ll}
\frac{i+1}{2}, & \text{i is odd} \\
\frac{n+i+1}{2}, & \text{i is even}
\end{array} \right. \]
\[ f(v_i) = \left\{ \begin{array}{ll}
\frac{3n+i}{2}, & \text{i is odd} \\
\frac{2n+i}{2}, & \text{i is even}
\end{array} \right. \]
\[ f(u_i u_{i+1}) = 3n–i, 1 ≤ i ≤ n-1; f(v_i v_{i+1}) = 5n–i–1, 1 ≤ i ≤ n-1 \text{ and } f(u_i v_i) = 4n–i, 1 ≤ i ≤ n. \]

Now we prove this labeling is an edge trimagic total labeling.
Consider the edges u_i u_{i+1}, 1 ≤ i ≤ n-1.
For odd i, f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{i+1}{2} + 3n–i + \frac{n+i+1}{2} = \frac{7n+3}{2} = \lambda_1 (say).
For even i, f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{n+i+1}{2} + 3n–i + \frac{i+1}{2} = \frac{7n+3}{2} = \lambda_2.

Next we consider the edges v_i v_{i+1}, 1 ≤ i ≤ n-1.
For odd i, f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \frac{3n+i}{2} + 5n–i–1 + \frac{2n+i+1}{2} = \frac{15n–1}{2} = \lambda_3 (Say).
For even i, f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = \frac{2n+i}{2} + 5n–i–1 + \frac{3n+i+1}{2} = \frac{15n–1}{2} = \lambda_3.

Finally we consider the edges u_i v_i, 1 ≤ i ≤ n.
For odd i, f(u_i) + f(u_i v_i) + f(v_i) = \frac{i+1}{2} + 4n – i + \frac{3n+i}{2} = \frac{11n+1}{2} = \lambda_3 (say).
For even i, f(u_i) + f(u_i v_i) + f(v_i) = \frac{n+i+1}{2} + 4n–i + \frac{2n+i}{2} = \frac{11n+1}{2} = \lambda_3.

Hence for each edge uv ∈ E, f(u) + f(uv) + f(v) yields any one of the magic constant \[ \lambda_1 = \frac{7n+3}{2}, \lambda_2 = \frac{15n–1}{2} \text{ and } \lambda_3 = \frac{11n+1}{2}. \]

Hence the Ladder L_n is an edge trimagic total when n is odd.

Case: 2 n is even.
Define a bijection f: V ∪ E → {1, 2, ..., 2n, 2n+1, ..., 5n–2} such that
\[ f(u_i) = \left\{ \begin{array}{ll}
\frac{i+1}{2}, & \text{i is odd} \\
\frac{n+i}{2}, & \text{i is even}
\end{array} \right. \]
f(v_i) = \begin{cases} 
\frac{3n+i+1}{2}, & \text{i is odd} \\
\frac{2n+i}{2}, & \text{i is even} 
\end{cases}

f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{i is odd} \\
\frac{n+i+1}{2}, & \text{i is even} 
\end{cases}

f(u_i, u_{i+1}) = 3n - i, 1 \leq i \leq n-1; f(v_{V_i+1}) = 4n - i - 1, 1 \leq i \leq n-1 \text{ and } f(u_{V_i}) = 5n - i - 1, 1 \leq i \leq n.

Now we prove this labeling is an edge trimagic total.

Consider the edges \( u_i u_{i+1}, 1 \leq i \leq n-1. \)

For odd \( i, f(u_i) + f(u_i, u_{i+1}) + f(u_{i+1}) = \frac{i+1}{2} + 3n - i + \frac{n+i+1}{2} = \frac{7n+2}{2} = \lambda_1 (\text{say}). \)

For even \( i, f(u_i) + f(u_i, u_{i+1}) + f(u_{i+1}) = \frac{n+i}{2} + 3n - i + \frac{i+1+1}{2} = \frac{7n+2}{2} = \lambda_1. \)

Consider the edges \( v_i v_{i+1}, 1 \leq i \leq n-1. \)

For odd \( i, f(v_i) + f(v_i, v_{i+1}) + f(v_{i+1}) = \frac{3n+i+1}{2} + 4n - i - 1 + \frac{2n+i+1}{2} = \frac{13n}{2} = \lambda_2 (\text{say}). \)

For even \( i, f(v_i) + f(v_i, v_{i+1}) + f(v_{i+1}) = \frac{2n+i}{2} + 4n - i - 1 + \frac{3n+i+1+1}{2} = \frac{13n}{2} = \lambda_2. \)

Consider the edges \( u_i v_{i+1}, 1 \leq i \leq n. \)

For odd \( i, f(u_i) + f(u_i, v_{i+1}) + f(v_{i+1}) = \frac{i+1}{2} + 5n - i - 1 + \frac{3n+i+1}{2} = \frac{13n}{2} = \lambda_3. \)

For even \( i, f(u_i) + f(u_i, v_{i+1}) + f(v_{i+1}) = \frac{n+i}{2} + 5n - i - 1 + \frac{2n+i}{2} = \frac{13n-2}{2} = \lambda_3. \)

Hence for each edge \( uv \in E, f(u) + f(uv) + f(v) \) yields any one of the magic constant

\[ \lambda_1 = \frac{7n+2}{2}, \lambda_2 = \frac{13n}{2} \text{ and } \lambda_3 = \frac{13n-2}{2}. \]

Hence the Ladder \( L_n \) is an edge trimagic total when \( n \) is even.

Therefore, by case 1 and case 2 the ladder \( L_n \) admits an edge trimagic total labeling.

**Theorem 2.2** The Ladder \( L_n = P_n \times P_2 \) is a super edge trimagic total for all \( n \geq 2. \)

**Proof:** We proved that the Ladder \( L_n = P_n \times P_2 \) is an edge trimagic total graph for all \( n \) with \( 2n \) vertices. The labeling given in Theorem 2.1 is as follows:

When \( n \) is odd,

\[ f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{i is odd} \\
\frac{n+i+1}{2}, & \text{i is even} 
\end{cases} \]

\[ f(v_i) = \begin{cases} 
\frac{3n+i}{2}, & \text{i is odd} \\
\frac{2n+i}{2}, & \text{i is even} 
\end{cases} \]

When \( n \) is even,

\[ f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{i is odd} \\
\frac{n+i}{2}, & \text{i is even} 
\end{cases} \]

\[ f(v_i) = \begin{cases} 
\frac{3n+i+1}{2}, & \text{i is odd} \\
\frac{2n+i}{2}, & \text{i is even} 
\end{cases} \]
Hence the 2n vertices get labels 1, 2, …, 2n. Therefore, the ladder L_n is a super edge trimagic total for all n.

**Example: 2.3** An edge trimagic total labeling of the Ladders L_7 and L_6 are given in figure 1 and figure 2, respectively.

![Figure 1: L_7 with λ_1 = 26, λ_2 = 39 and λ_3 = 52.](image1)

![Figure 2: L_6 with λ_1 = 22, λ_2 = 39 and λ_3 = 38.](image2)

**Theorem: 2.4** The triangular Ladder TL_n admits an edge trimagic total labeling for all n ≥ 2.

**Proof:** Let V = {v_i, u_i | 1 ≤ i ≤ n} be the vertex set and E = \{v_i v_{i+1}, u_i u_{i+1} | 1 ≤ i ≤ n\} \cup \{v_i u_i | 1 ≤ i ≤ n\} \cup \{v_i u_{i+1} | 1 ≤ i ≤ n-1\} be the edge set of the triangular Ladder TL_n. Then TL_n has 2n vertices and 4n–3 edges.

**Case: 1** n is odd.
Define a bijection f: V \cup E \rightarrow \{1, 2, …, 6n–3\} such that f(u_i) = 2i, 1 ≤ i ≤ n; f(v_i) = 2i – 1, 1 ≤ i ≤ n; f(u_i u_{i+1}) = 6n–4i, 1 ≤ i ≤ n–1; f(v_i v_{i+1}) = 6n–4i–2, 1 ≤ i ≤ n–1; f(u_i v_i) = 6n–4i+1, 1 ≤ i ≤ n and f(u_i v_{i+1}) = 6n–4i–1, 1 ≤ i ≤ n–1.

Now, we prove this labeling is an edge trimagic total.

For the edge u_i u_{i+1},
\[ f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2i + 6n–4i + 2(i+1) = 6n + 2 = \lambda_1 \]
(say).

For the edges v_i v_{i+1},
\[ f(v_i) + f(v_i v_{i+1}) + f(v_{i+1}) = 2i–1 + 6n–4i–2 + 2(i+1) – 1 = 6n–2 = \lambda_2 \]
(say).

For the edges u_i v_i,
\[ f(u_i) + f(u_i v_i) + f(v_i) = 2i + 6n–4i+1 + 2i–1 = 6n = \lambda_3 \]
(say).

Also, for the edges, u_i u_{i+1},
\[ f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2i + 6n–4i–1 + 2(i+1) – 1 = 6n = \lambda_3. \]

Hence for each edge uv \in E, f(u) + f(uv) + f(v) yields any one of the magic constant \(\lambda_1 = 6n+2, \lambda_2 = 6n–2\) and \(\lambda_3 = 6n\).

Therefore, the triangular Ladder TL_n admits an edge trimagic total labeling when n is odd.

**Case: 2** n is even.

Define a bijection f: V \cup E \rightarrow \{1, 2, …, 6n–3\} such that
\[ f(u_i) = 2i–1, 1 ≤ i ≤ n; f(v_i) = 2i, 1 ≤ i ≤ n; f(u_i u_{i+1}) = 6n–4i–1, 1 ≤ i ≤ n–1; f(v_i v_{i+1}) = 6n–4i–2, 1 ≤ i ≤ n–1; f(u_i v_i) = 6n–4i+1, 1 ≤ i ≤ n and f(u_i v_{i+1}) = 6n–4i–1, 1 ≤ i ≤ n–1. \]
Now we prove this labeling is an edge trimagic total.

For the edges \( u_i u_{i+1}, 1 \leq i \leq n-1 \),
\[
f(u_i)+f(u_i u_{i+1})+f(u_{i+1}) = 2i-1+6n-4i-1+2(i+1) -1 = 6n-1= \lambda_1 \text{(say)}.
\]

For the edges \( v_i v_{i+1}, 1 \leq i \leq n-1 \),
\[
f(v_i)+f(v_i v_{i+1})+f(v_{i+1}) = 2i+6n-4i-2(i+1) = 6n +2 = \lambda_2 \text{(say)}.
\]

For the edges \( u_i v_i, 1 \leq i \leq n \),
\[
f(u_i)+f(u_i v_i)+f(v_i) = 2i-1+6n-4i+1+2i = 6n = \lambda_3 \text{ (say)}.
\]

Also, for the edges \( u_i v_{i+1}, 1 \leq i \leq n-1 \),
\[
f(u_i)+f(u_i v_{i+1})+f(v_{i+1}) = 2i-1+6n-4i-2+2(i+1) = 6n-1 = \lambda_1.
\]

Hence for each edge \( uv \in E \), \( f(u)+f(uv)+f(v) \) yields any one of the magic constant \( \lambda_1 = 6n-1, \lambda_2 = 6n+2, \) and \( \lambda_3 = 6n \).

Therefore, the triangular Ladder \( TL_n \) admits an edge trimagic total labeling for even \( n \).

Hence by case 1 and case 2, the triangular Ladder \( TL_n \) admits an edge trimagic total labeling.

**Theorem: 2.5** The triangular ladder \( TL_n \) admits a super edge trimagic total labeling.

**Proof:** We have proved that the triangular ladder \( TL_n \) has an edge trimagic total labeling with 2n vertices. The labeling given in the proof of Theorem 2.4, is as follows:

For odd \( n \), \( f(u_i) = 2i, 1 \leq i \leq n \) and \( f(v_i) = 2i-1, 1 \leq i \leq n \).

For even \( n \), \( f(u_i) = 2i-1, 1 \leq i \leq n \) and \( f(v_i) = 2i, 1 \leq i \leq n \).

Hence the 2n vertices get labels 1, 2, ..., 2n. Therefore, the triangular ladder \( TL_n \) admits a super edge trimagic total labeling for all \( n \geq 2 \).

**Example: 2.6** An super edge trimagic total labeling of the triangular ladders \( TL_7 \) and \( TL_6 \) are given in figure 3 and figure 4, respectively.

![Figure 3: TL_7 with magic constants \( \lambda_1 = 40, \lambda_2 = 42 \) and \( \lambda_3 = 44 \).](image1)

![Figure 4: TL_6 with magic constants \( \lambda_1 = 35, \lambda_2 = 36 \) and \( \lambda_3 = 38 \).](image2)
Theorem: 2.7 The generalized Petersen graph $P(n, \frac{n-1}{2})$ admits an edge trimagic total labeling ($n$ is odd).

Proof: Consider a generalized Petersen graph $P(n, \frac{n-1}{2})$ with the vertex set $V = \{x_i, y_i, 0 \leq i \leq n-1\}$ and the edge set $E = \{x_iy_i, 0 \leq i \leq n-1\} \cup \{y_iy_{i+1}, 0 \leq i \leq n-2\} \cup \{x_ix_{i+1}, 0 \leq i \leq n-1\} \cup \{y_0y_{n-1}\}$, where the subscripts taken modulo $n$.

Then $P(n, \frac{n-1}{2})$ has $2n$ vertices and $3n$ edges.

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 5n\}$ such that $f(x_i) = 2n-i, 0 \leq i \leq n-1$; $f(y_i) = n-i, 0 \leq i \leq n-1$; $f(y_iy_{i+1}) = 3n+2i+2, 0 \leq i \leq n-2$; $f(y_0y_{n-1}) = 5n$; $f(x_0y_0) = 3n+1+1$,

$0 \leq i \leq n-1$ and $f(x_ix_{i+1}) = 2n+2i+1, 0 \leq i \leq n-1$.

Now we have to prove that the generalized Petersen graph $P(n, \frac{n-1}{2})$ admits an edge trimagic total labeling.

For the edges $y_iy_{i+1}, 0 \leq i \leq n-2$;

$f(y_i)+f(y_iy_{i+1})+f(y_{i+1}) = n-i+3n+2i+2+2n-(i+1) = 5n+1 = \lambda_1$(say).

For the edge $y_0y_{n-1}$;

$f(y_0)+f(y_0y_{n-1})+f(y_{n-1}) = n-0+5n+n-(n-1) = 6n+1 = \lambda_2$(say).

For the edges $x_iy_{n-i}, 0 \leq i \leq n-1$;

$f(x_i)+f(x_0y_0)+f(y_i) = 2n-i+3n+2i+1+n-i = 6n+1 = \lambda_2$.

For the edges $x_i x_{i+1}, 0 \leq i \leq n-1$ with $i$ taken modulo $n$.

$f(x_i)+f(x_i, x_{i+1})+f(x_{i+1}) = 2n-i+2n+2i+1+2n-(i+1) = 11n+3 = \lambda_3$(say).

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(u)$ yields any one of the magic constants $\lambda_1 = 5n+1, \lambda_2 = 6n+1$ and $\lambda_3 = \frac{11n+3}{2}$.

Therefore, the generalized Petersen graph $P(n, \frac{n-1}{2})$ admits an edge trimagic total labeling.

Theorem: 2.8 The generalized Petersen graph $P(n, \frac{n-1}{2})$ admits a super edge trimagic total labeling.

Proof: We have proved that the generalized Petersen graph $P(n, \frac{n-1}{2})$ admits edge trimagic total labeling. The labeling given in the Theorem 2.7 for the vertices is, $f(x_i) = 2n-i, 0 \leq i \leq n-1$ and $f(y_i) = n-i, 0 \leq i \leq n-1$. Since the vertices get labels $1, 2, \ldots, 2n$ the generalized Petersen graph $P(n, \frac{n-1}{2})$ is a super edge trimagic total.

Example: 2.9 Generalized Petersen graph $P(9, 4)$ is super edge trimagic total.

Figure 5: Petersen graph $P(9, 4)$ with $\lambda_1 = 46, \lambda_2 = 55$ and $\lambda_3 = 51$. 

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Theorem: 2.10 The Helm graph \( H_n \) has an edge trimagic total labeling for every positive even integer \( n \).

Proof: Let \( V = \{u_1, v_1, \ldots, v_{2n+1}\} \) be the vertex set and \( E = \{u_iv_i, v_iw_i / 1 \leq i \leq n\} \cup \{v_{i+1}v_i, 1 \leq i \leq n-1\} \cup \{v_iV_{n+1}\} \) be the edge set of the helm graph \( H_n \). Then \( H_n \) has \( 2n+1 \) vertices and \( 3n \) edges.

Define a bijection \( f: V \cup E \rightarrow \{1, 2, \ldots, 5n+1\} \) such that

\[
\begin{align*}
 f(u) &= 1, \\
 f(v_i) &= \begin{cases} 
 \frac{i+1}{2} + 1, & \text{i is odd} \\
 \frac{i+n}{2} + 1, & \text{i is even}
\end{cases} \\
 f(w_i) &= \begin{cases} 
 n+\frac{i+1}{2} + 1, & \text{i is odd} \\
 n+\frac{i+n}{2} + 1, & \text{i is even}
\end{cases} \\
 f(uv_i) &= \begin{cases} 
 5n-\frac{i+1}{2} + 2, & \text{i is odd} \\
 5n-\frac{n+i}{2} + 2, & \text{i is even}
\end{cases} \\
 f(v_1v_{n+1}) &= 4n+1, f(v_iw_i) = 4n-i+1, 1 \leq i \leq n-1 \text{ and } f(v_iV_{n+1}) = 3n-i+2, 1 \leq i \leq n.
\end{align*}
\]

Now we prove this labeling is an edge trimagic total.

Consider the edges \( uv_i, 1 \leq i \leq n \).

For odd \( i \), \( f(u)+f(uv_i)+f(v_i) = 1 + \frac{i+1}{2} + 1 + 5n-\frac{i+1}{2} + 2 = 5n+4 = \lambda_1 \) (say).

For even \( i \), \( f(u)+f(uv_i)+f(v_i) = 1 + 5n-\frac{n+i}{2} + 2 + \frac{i+n}{2} + 1 = 5n+4 = \lambda_1 \).

Consider the edges \( v_iv_{i+1}, 1 \leq i \leq n-1 \).

For odd \( i \), \( f(v_i)+f(v_iw_i)+f(w_i) = \frac{i+1}{2} + 1 + 4n-i+1 + \frac{i+n}{2} + 1 = 4n+\frac{n}{2} + 4 = \lambda_2 \) (say).

For even \( i \), \( f(v_i)+f(v_iw_i)+f(w_i) = \frac{i+n}{2} + 1 + 4n-i+1 + \frac{i+n}{2} + 1 = 4n+\frac{n}{2} + 4 = \lambda_2 \).

Consider the edges \( v_iw_i, 1 \leq i \leq n \).

For odd \( i \), \( f(v_i)+f(v_iw_i)+f(w_i) = \frac{i+1}{2} + 1 + 3n-i+2 + n+\frac{i+1}{2} + 1 = 4n+5 = \lambda_3 \) (say).

For even \( i \), \( f(v_i)+f(v_iw_i)+f(w_i) = \frac{i+n}{2} + 1 + 3n-i+2 + n+\frac{i+n}{2} + 1 = 5n+4 = \lambda_4 \).

Hence for each edge \( uv \in E \), \( f(u)+f(uv)+f(v) \) yields any one of the constants \( \lambda_1 = 5n+4, \lambda_2 = 4n+\frac{n}{2} + 4 \) and \( \lambda_3 = 4n+5 \).

Therefore, the helm graph \( H_n \) admits an edge trimagic total labeling for every positive even integer \( n \).

Theorem: 2.11 The helm graph \( H_n \) is a super edge trimagic total for even \( n \).

Proof: We have proved that the helm graph \( H_n \) is an edge trimagic total for even \( n \). The labeling given in the proof of Theorem 2.10, the labeling for the vertices are \( f(u) = 1 \),

\[
 f(v_i) = \begin{cases} 
 \frac{i+1}{2} + 1, & \text{i is odd} \\
 \frac{i+n}{2} + 1, & \text{i is even}
\end{cases} \\
 f(w_i) = \begin{cases} 
 n+\frac{i+1}{2} + 1, & \text{i is odd} \\
 n+\frac{i+n}{2} + 1, & \text{i is even}
\end{cases} 
\]
Since the helm graph $H_n$ has $2n+1$ vertices and gets labels 1, 2, ..., $2n+1$, the helm graph $H_n$ is a super edge trimagic total labeling.

**Example: 2.12** The helm graph $H_6$ shown in Figure 6 admits a super edge trimagic total labeling with magic constants 29, 31, and 34.

**Figure 6:** Helm graph $H_6$ with $\lambda_1 = 29$, $\lambda_2 = 31$ and $\lambda_3 = 34$.

**Theorem: 2.13** The flower graph $F_n$ has an edge trimagic total labeling for all $n$.

**Proof:** Let $V = \{v_i, w_i / 1 \leq i \leq n\} \cup \{u\}$ be the vertex set and $E = \{uv_i, v_iw_i, uw_i / 1 \leq i \leq n\} \cup \{v_{i+1}, 1 \leq i \leq n-1\} \cup \{v_nv_1\}$ be the edge set of the flower graph $F_n$. Then the flower graph $F_n$ has $2n+1$ vertices and 4n edges.

Define a bijection $f: V \cup E \rightarrow \{1, 2, ..., 6n+1\}$ such that $f(u) = 1$, $f(v_i) = i+1$, $f(w_i) = n+i+1$, $f(uv_i) = 5n-i+2$, $1 \leq i \leq n$; $f(v_{i+1}) = 4n-2i+1$, $1 \leq i \leq n-1$; $f(v_nv_1) = 4n+1$.

Now, we prove the above labeling is an edge trimagic total.

For the edges $uv_i$, $1 \leq i \leq n$,

$$f(u) + f(uv_i) + f(v_i) = 1 + 5n - i + 2 + i + 1 = 5n+4 = \lambda_1 \text{ (say).}$$

For all the edges $v_iv_{i+1}$, $1 \leq i < n-1$,

$$f(v_i) + f(v_{i+1}) + f(v_i) = i+1 + 4n-2i+1 + i + 1 + 1 = 4n + 4 = \lambda_2 \text{ (say).}$$

For the edges $v_iw_i$, $1 \leq i \leq n$,

$$f(v_i) + f(v_iw_i) + f(w_i) = i+1 + 4n-2i+2 + n + i + 1 = 5n+4 = \lambda_3.$$ 

For the edge $uw_i$, $1 \leq i \leq n$,

$$f(u) + f(uw_i) + f(w_i) = 1 + 6n - i + 2 + n + i + 1 = 7n+4 = \lambda_2 \text{ (say).}$$

And for the edge $v_nv_1$, $f(v_1) + f(v_nv_1) + f(v_n) = 1+1+4n+1+n+1 = 5n+4 = \lambda_1$.

Hence for each edge $uv$, $f(u) + f(uv) + f(v)$ yields any one of the magic constants $\lambda_1 = 5n+4$, $\lambda_2 = 4n+4$ and $\lambda_3 = 7n+4$.

Therefore, the flower graph $F_n$ admits an edge trimagic total labeling for all $n$.

**Theorem: 2.14** The flower graph $F_n$ is a super edge trimagic total for all $n \geq 3$.

**Proof:** We have proved that the flower graph $F_n$ admits an edge trimagic total labeling for $n \geq 3$. The labeling given in the proof of the Theorem 2.13, the labeling for the vertices are, $f(u) = 1$, $f(v_i) = i+1$, $1 \leq i \leq n$; $f(w_i) = n+i+1$, $1 \leq i \leq n$. Since the flower graph $F_n$ has $2n+1$ vertices and get labels 1, 2, ..., $2n+1$, the flower graph $F_n$ is a super edge trimagic total labeling.
Example: 2.15 The flower graph $Fl_6$ given in figure 7 is a super edge trimagic total graph with magic constants 34, 28 and 46.

Figure 7: Flower graph $Fl_6$ with $\lambda_1 = 34$, $\lambda_2 = 28$ and $\lambda_3 = 46$.

CONCLUSION

In this paper, we have proved some classes of graphs namely, the ladder $L_n$, triangular ladder $TL_n$, generalized Petersen graph $P(n, \frac{n-1}{2})$, the helm graph $H_n$ and the flower graph $Fl_n$ are edge trimagic total and super edge trimagic total graphs. There will be many trimagic graphs can be constructed in future.

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SUPER EDGE TRIMAGIC TOTAL LABELING OF GRAPHS

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ABSTRACT

An edge magic total labeling of a (p, q) graph is a bijection f: V(G) ∪ E(G) → {1, 2, …, p+q} such that for each edge uv ∈ E(G), the value of f(u)+f(uv)+f(v) is a constant k. If there exists two constants k₁ and k₂ such that f(u)+f(uv)+f(v) is either k₁ or k₂, it is said to be an edge bimagic total labeling. An edge trimagic total labeling of a (p, q) graph is a bijection f: V(G) ∪ E(G) → {1, 2, …, p+q} such that, for each edge uv ∈ E(G), the value of f(u)+f(uv)+f(v) is either k₁ or k₂ or k₃. In this paper, we prove that the corona graph Cₙ⊙K₂, double ladder Pₙ×Pₙ, quadrilateral snake Qₙ and alternate triangular snake A(TSn) are edge trimagic total and super edge trimagic total.

Keywords: Function, Bijection, Labeling, Magic, Trimagic.

AMS Subject Classification: 05C78.

1. INTRODUCTION

We begin with simple, finite and undirected graph G = (V, E). A graph labeling is an assignment of integers to elements of graph, the vertices or edges or both subject to certain conditions. The concept of graph labeling was introduced by Rosa in 1967. In 1970 Kotzig and Rosa [6] defined, magic labeling of graph G is a bijection f: V∪E→ {1, 2, …, p+q} such that, for each edge uv ∈ E(G), f(u)+f(uv)+f(v) is a magic constant. In 1996, Ringel and Llado called this labeling as edge magic. In 2001, Wallis introduced this as edge magic total labeling. In 2004, J. Baskar Babujee [1, 2] introduced the edge bimagic labeling of graphs.

In 2013, C. Jayasekaran, M. Regees and C. Davidraj [3] introduced the edge trimagic total labeling of graphs. An edge trimagic total labeling of a (p, q) graph G is a bijection f: V∪E→ {1, 2, …, p+q} such that for each edge uv ∈ E(G), the value of f(u)+f(uv)+f(v) is equal to any of the distinct constant k₁ or k₂ or k₃. A graph G is said to be edge trimagic total if it admits an edge trimagic total labeling. An edge trimagic total labeling is called edge super trimagic total labeling if G has the additional property that the vertices are labeled with the smallest positive integers.

An alternate triangular snake A(TSn) is obtained from a path u₁, u₂, …, uₙ by joining u₁ and u₁ to new vertex v₁. That is every alternate edge of a path is replaced by a cycle C₃. A quadrilateral snake Qₙ is obtained from a path u₁, u₂, …, uₙ by joining u₁, u₁, v₁ to new vertices v₁, w₁ respectively and joining v₁, w₁. That is, every edge of a path is replaced by a cycle C₄.

For further references, we use dynamic survey of graph labeling by J. A. Gallian [5]. In [3], we introduced the concept edge trimagic and super edge trimagic total labeling and proved that, some family and classes of graphs are edge trimagic total and super edge trimagic total [3, 4, 7]. In this paper, we prove that the corona graph Cₙ⊙K₂, double ladder Pₙ×Pₙ, quadrilateral snake Qₙ and alternate triangular snake A(TSn) are trimagic total and super edge trimagic total graphs.

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2. SUPER EDGE TRIMAGIC LABELING OF $C_n \square K_2$, $P_n \square P_3$, $Q_n$ and $A(TS_h)$.

In this section we prove that the corona graph $C_n \square K_2$, double ladder $P_n \square P_3$, quadrilateral snake $Q_n$ and alternate triangular snake $A(TS_h)$ are edge trimagic total and super edge trimagic total. And give examples for super edge trimagic total labeling for each of the above graphs.

**Theorem 2.1** The graph $C_n \square K_2$ has an edge trimagic total labeling for positive integer $n$.

**Proof:** Let $V = \{u_i, v_i, w_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{u_i v_i, u_i w_i, v_i w_i | 1 \leq i \leq n\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\}$ be the edge set of the graph $C_n \square K_2$. Then $C_n \square K_2$ has 3n vertices and 4n edges.

Define a bijection $f: V \cup E \to \{1, 2, \ldots, 7n\}$ such that

$f(u_i) = i, 1 \leq i \leq n$; $f(v_i) = n+i, 1 \leq i \leq n$; $f(w_i) = 2n+i, 1 \leq i \leq n$;

$f(u_i u_{i+1}) = 7n-2i, 1 \leq i \leq n-1$; $f(u_i v_i) = 7n-2i+1, 1 \leq i \leq n$; $f(u_i w_i) = 5n-2i+2, 1 \leq i \leq n$;

$f(v_i w_i) = 5n-2i+1, 1 \leq i \leq n$ and $f(u_n u_1) = 7n$.

Now we prove the above labeling is an edge trimagic total.

For the edges $u_i u_{i+1}$, $1 \leq i \leq n-1$;

$f(u_i)+f(u_i u_{i+1})+f(u_{i+1}) = i+7n-2i+i+1 = 7n+1 = \lambda_1$(say).

For the edges $u_i v_i$, $1 \leq i \leq n$;

$f(u_i)+f(u_i v_i)+f(v_i) = i+7n-2i+1+n+i = 8n+1 = \lambda_2$(say).

For the edges $u_i w_i$, $1 \leq i \leq n$;

$f(u_i)+f(u_i w_i)+f(w_i) = i+5n-2i+2+2n+i = 7n+2 = \lambda_3$(say).

For the edges $v_i w_i$, $1 \leq i \leq n$;

$f(v_i)+f(v_i w_i)+f(w_i) = n+i+5n-2i+1+2n+i = 8n+1 = \lambda_2$.

For the edge $u_n u_1$, $f(u_n)+f(u_n u_1)+f(u_1) = n+7n+1 = 8n+1 = \lambda_3$.

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the trimagic constants $\lambda_1 = 7n+1$, $\lambda_2 = 8n+1$ and $\lambda_3 = 7n+2$.

Therefore, the graph $C_n \square K_2$ admits an edge trimagic total labeling for all positive integer $n$.

**Theorem 2.2** The graph $C_n \square K_2$ has a super edge trimagic total labeling.

**Proof:** We proved that the graph $C_n \square K_2$ admits an edge trimagic total labeling. The labeling given in the proof of Theorem 2.1, the vertices get labels 1, 2, ..., 3n. Since the graph $C_n \square K_2$ has 3n vertices and the 3n vertices have labels 1, 2, ..., 3n, the graph $C_n \square K_2$ admits a super edge trimagic total labeling.

**Example 2.3** A super edge trimagic total labeling of the graph $C_n \square K_2$ is given in figure1.

![Figure 1: C_n \square K_2 with \lambda_1 = 43, \lambda_2 = 49 and \lambda_3 = 44.](image-url)
Theorem: 2.4 The double ladder $P_n \times P_3$ admits an edge trimagic total labeling for odd $n$.

Proof: Let $V = \{u_i, v_i, w_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{u_i v_i, v_i w_i / 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1}, w_i w_{i+1} / 1 \leq i \leq n-1\}$ be the edge set of $P_n \times P_3$. Then the double ladder $P_n \times P_3$ has $3n$ vertices and $5n-3$ edges.

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 8n-3\}$ such that

\[
f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{1} \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n+i+1}{2}, & \text{1} \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
2n+\frac{n+i}{2}, & \text{1} \leq i \leq n \text{ and } i \text{ is odd} \\
2n+\frac{i}{2}, & \text{1} \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

\[
f(w_i) = \begin{cases} 
n+\frac{i+1}{2}, & \text{1} \leq i \leq n \text{ and } i \text{ is odd} \\
n+\frac{2i+1}{2}, & \text{1} \leq i \leq n \text{ and } i \text{ is even}
\end{cases}
\]

Let $V = \{u_i, v_i, w_i / 1 \leq i \leq n\}$; $f(v_i) = 5n-i, 1 \leq i \leq n$; $f(u_i u_{i+1}) = 8n-i-2, 1 \leq i \leq n-1$;

Now we prove the above labeling is an edge trimagic total.

Consider the edges $u_i v_i, 1 \leq i \leq n$;

For odd $i$, $f(u_i)+f(u_i v_i)+f(v_i) = \frac{i+1}{2}+6n-i+2n+\frac{n+i}{2} = \frac{17n+1}{2} = \lambda_1$(say).

For even $i$, $f(u_i)+f(u_i v_i)+f(v_i) = \frac{n+i+1}{2}+6n-i+2n+\frac{i}{2} = \frac{17n+1}{2} = \lambda_1$.

Consider the edges $v_i w_i, 1 \leq i \leq n$;

For odd $i$, $f(v_i)+f(v_i w_i)+f(w_i) = 2n+\frac{n+i}{2}+5n-i+n+\frac{i+1}{2} = \frac{17n+1}{2} = \lambda_1$.

For even $i$, $f(v_i)+f(v_i w_i)+f(w_i) = 2n+\frac{i}{2}+5n-i+n+\frac{n+i+1}{2} = \frac{17n+1}{2} = \lambda_1$.

Consider the edges $u_i u_{i+1}, 1 \leq i \leq n-1$;

For odd $i$, $f(u_i)+f(u_i u_{i+1})+f(u_{i+1}) = \frac{i+1}{2}+8n-i-2+\frac{n+i+1}{2} = \frac{17n+1}{2} = \lambda_2$(say).

For even $i$, $f(u_i)+f(u_i u_{i+1})+f(u_{i+1}) = \frac{n+i+1}{2}+8n-i-2+\frac{i+1}{2} = \frac{17n+1}{2} = \lambda_2$.

Consider the edges $v_i v_{i+1}, 1 \leq i \leq n-1$;

For odd $i$, $f(v_i)+f(v_i v_{i+1})+f(v_{i+1}) = 2n+\frac{n+i}{2}+4n-i+2n+\frac{i+1}{2} = \frac{17n+1}{2} = \lambda_1$.

For even $i$, $f(v_i)+f(v_i v_{i+1})+f(v_{i+1}) = 2n+\frac{i}{2}+4n-i+2n+\frac{n+i+1}{2} = \frac{17n+1}{2} = \lambda_1$.

Consider the edges $w_i w_{i+1}, 1 \leq i \leq n-1$;

For odd $i$, $f(w_i)+f(w_i w_{i+1})+f(w_{i+1}) = n+\frac{i+1}{2}+7n-i-1+n+\frac{n+i+1}{2} = \frac{19n+1}{2} = \lambda_3$(say).

For even $i$, $f(w_i)+f(w_i w_{i+1})+f(w_{i+1}) = n+\frac{n+i+1}{2}+7n-i-1+n+\frac{i+1}{2} = \frac{19n+1}{2} = \lambda_3$. 

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Hence for each edge \( uv \in E \), \( f(u)+f(uv)+f(v) \) yields any one of the constant

\[
\lambda_1 = \frac{17n+1}{2}, \quad \lambda_2 = \frac{17n-1}{2} \quad \text{and} \quad \lambda_3 = \frac{19n+1}{2}.
\]

Therefore, the double ladder \( P_n \times P_3 \) admits an edge trimagic total labeling for odd \( n \).

**Theorem: 2.7** The Quadrilateral Snake \( Q_n \) admits an edge trimagic total labeling.

**Proof:** Let \( V = \{u_i \mid 1 \leq i \leq n\} \cup \{v_i, w_i \mid 1 \leq i \leq n-1\} \) be the vertex set and \( E = \{u_iu_{i+1}, u Swan, u_{i-1}w_i \mid 1 \leq i \leq n-1\} \) be the edge set of the Quadrilateral Snake \( Q_n \). Then \( Q_n \) has \( 3n-2 \) vertices and \( 4n-4 \) edges.

Define a bijection \( f: V \cup E \to \{1, 2, \ldots, 7n-6\} \) such that \( f(u_i) = i, 1 \leq i \leq n; f(v_i) = n+i, 1 \leq i \leq n-1; f(w_i) = 2n+i-1, 1 \leq i \leq n-1; f(u_iw_i) = 7n-2i-4, 1 \leq i \leq n-1; f(u_iV_i) = 7n-2i-5, 1 \leq i \leq n-1; f(u_iw_i) = 5n-2i-3, 1 \leq i \leq n-1 \) and \( f(v_iw_i) = 5n-2i-2, 1 \leq i \leq n-1 \).

Now we prove the above labeling is an edge trimagic total.

For the edges \( u_iu_{i+1}, 1 \leq i \leq n-1 \):

\[
f(u_i)+f(u_iu_{i+1})+f(u_{i+1}) = i+7n-2i-4+i+1 = 7n-3 = \lambda_1 \text{ (say).}
\]

For the edges \( u_iV_i, 1 \leq i \leq n-1 \):

\[
f(u_i)+f(u_iV_i)+f(V_i) = i+7n-2i-5+n+i = 8n-5 = \lambda_2 \text{ (say).}
\]

For the edges \( u_iw_i, 1 \leq i \leq n-1 \):

\[
f(u_i)+f(u_iw_i)+f(w_i) = i+1+5n-2i-3+2n+i-1 = 7n-3 = \lambda_1.
\]

For the edges \( v_iw_i, 1 \leq i \leq n-1 \):

\[
f(v_i)+f(v_iw_i)+f(w_i) = n+i+5n-2i-3+2n+i-1 = 8n-3 = \lambda_3 \text{ (say).}
\]

Hence for each edge \( uv \in E \), \( f(u)+f(uv)+f(v) \) yields any one of the constants \( \lambda_1 = 7n-3, \lambda_2 = 8n-5 \) and \( \lambda_3 = 8n-3 \).

Therefore, the Quadrilateral Snake \( Q_n \) admits an edge trimagic total labeling.

**Theorem: 2.8** The Quadrilateral snake \( Q_n \) admits a super edge trimagic total labeling.

**Proof:** We proved that the Quadrilateral snake \( Q_n \) has an edge trimagic total labeling. The labeling given in the proof of Theorem 2.7, the vertices get labels 1, 2, ..., 3n. Since the Quadrilateral snake \( Q_n \) has 3n vertices and the 3n-2 vertices have labels 1, 2, ..., 3n-2 for all integer \( n \), the Quadrilateral snake \( Q_n \) admits a super edge trimagic total labeling.
**Example: 2.9** A super edge trimagic total labeling of Quadrilateral snake $Q_6$ is given in figure 3.

![Figure 3: Q_6 with $\lambda_1 = 39$, $\lambda_2 = 43$ and $\lambda_3 = 45.$](image)

**Theorem: 2.10** The Alternate triangular snake $A(TS_n)$ admits an edge trimagic total labeling for even $n$.

**Proof:** We consider the following two cases.

**Case: 1** Triangle starts from $u_1$.

Let $V = \{u_i/1 \leq i \leq n\} \cup \{v_j/1 \leq j \leq \frac{n}{2}\}$ be the vertex set and $E = \{v_ju_{2j}, v_ju_{2j}/1 \leq j \leq \frac{n}{2}\} \cup \{u_{i+1}/1 \leq i \leq n–1\}$ be the edge set of the alternate triangular snake $A(TS_n)$. Since the triangle starts from $u_1$, the alternate triangular snake $A(TS_n)$ has $n+\frac{n}{2}$ vertices and $2n–1$ edges. Define a bijection $f: V \rightarrow \{1, 2, \ldots, 3n+\frac{n}{2}–1\}$ such that

$$f(u_i) = \begin{cases} \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ \frac{n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = n+j, \ 1 \leq j \leq \frac{n}{2}; \ f(u_{i+1}) = 3n+\frac{n}{2}–i, \ 1 \leq i \leq n–1; \ f(v_{2j+1}) = 2n+\frac{n}{2}–2j+1, \ 1 \leq j \leq \frac{n}{2}.$$  

Now we prove the above labeling is an edge trimagic total.

Consider the edges $u_{i+1}, \ 1 \leq i \leq n–1$;

For odd $i$, $f(u_i)+f(u_{i+1})+f(v_{2j+1}) = \frac{i+1}{2} + 3n+\frac{n}{2}–i + \frac{n+i+1}{2} = 4n+1 = \lambda_1$(say).

For even $i$, $f(u_i)+f(u_{i+1})+f(v_{2j+1}) = \frac{n+i}{2} + 3n+\frac{n}{2}–2j+1 + n+j = 4n+1 = \lambda_3$.

For the edges $u_{2j}v_{2j}, \ 1 \leq j \leq \frac{n}{2}$,

$$f(u_{2j})+f(u_{2j+1})+f(v_{2j}) = \frac{n+2j}{2} + 2n+\frac{n}{2}–2j+2 + \frac{n+j}{2} = 4n+2 = \lambda_2(say).$$

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the trimagic constants $\lambda_1 = 4n+1$, $\lambda_2 = \frac{7n+2}{2}$ and $\lambda_3 = 4n+2$. Therefore, the alternate triangular snake graph $A(TS_n)$ admits an edge trimagic total labeling when the triangle starts from $u_1$.

**Case: 2** Triangle starts from $u_2$.

Let $V = \{u_i/1 \leq i \leq n\} \cup \{v_j/1 \leq j \leq \frac{n}{2}–1\}$ be the vertex set and $E = \{u_3v_j, u_{2j+1}v_j/1 \leq j \leq \frac{n}{2}–1\} \cup \{u_{i+1}/1 \leq i \leq n–1\}$ be the edge set of the alternate triangular snake $A(TS_n)$. Since the triangle starts from $u_2$, the alternate triangular snake $A(TS_n)$ has $n+\frac{n}{2}–1$ vertices and $2n–3$ edges. Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 3n+\frac{n}{2}–4\}$ such that

$$f(u_i) = \begin{cases} \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ \frac{n+i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_j) = n+j, \ 1 \leq j \leq \frac{n}{2}.$$  

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the trimagic constants $\lambda_1 = 4n+1$, $\lambda_2 = \frac{7n+2}{2}$ and $\lambda_3 = 4n+2$. Therefore, the alternate triangular snake graph $A(TS_n)$ admits an edge trimagic total labeling when the triangle starts from $u_2$. 

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f(v_j) = n+j, 1≤ j ≤ \frac{n}{2} - 1; f(u_{i+1}) = 3n+\frac{n}{2} - i - 3, 1≤ i ≤ n-1;
f(u_{j+1}v_j) = 2n+\frac{n}{2} - 2j - 2, 1≤ j ≤ \frac{n}{2} - 1; f(u_{2j+1}v_{2j}) = 2n+\frac{n}{2} - 2j - 1, 1≤ j ≤ \frac{n}{2} - 1.

Now we prove the above labeling is an edge trimagic total.

Consider the edges u_{i+1}v_{i}, 1≤ i ≤ n-1;

For odd i, f(u_i)+f(u_{i+1})+f(v_i) = \frac{n+1}{2} + 3n+\frac{n}{2} - i - 3 + \frac{n+i+1}{2} = 4n-2 = \lambda_2(say).

For even i, f(u_i)+f(u_{i+1})+f(v_i) = \frac{n+i}{2} + 3n+\frac{n}{2} - i - 3 + \frac{i+1}{2} = 4n-2 = \lambda_1.

For the edges u_{2j+1}v_{2j}, 1≤ j ≤ \frac{n}{2} - 1;

f(u_{2j+1})+f(u_{2j+1}v_{2j})+f(v_{2j}) = \frac{1+2j+1}{2} + 2n+\frac{n}{2} - 2j - 2 + n+j = \frac{7n-2}{2} = \lambda_3(say).

For the edges u_{2j}v_{2j}, 1≤ j ≤ \frac{n}{2} - 1;

f(u_{2j})+f(u_{2j}v_{2j})+f(v_{2j}) = \frac{n+2j}{2} + 2n+\frac{n}{2} - 2j - 1 + n+j = 4n-1 = \lambda_3(say).

Hence for each edge uv∈E, f(u)+f(u)+f(v) yields any one of the trimagic constants \lambda_1 = 4n-2, \lambda_2 = \frac{7n-2}{2} and \lambda_3 = 4n-1. Therefore, the alternate triangular snake graph A(TS_n) admits an edge trimagic total labeling when the triangle starts from u_{2}.

Hence the theorem follows from case 1 and case 2.

**Theorem: 2.11** The Alternate Triangular Snake A(TS_n) admits a super edge trimagic total labeling for even n.

**Proof:** We proved that the Alternate Triangular Snake A(TS_n) has an edge trimagic total labeling. The labeling given in the proof of Theorem 2.10, the vertices get labels 1, 2, ..., n+\frac{n}{2}. Since the Alternate Triangular Snake A(TS_n) has n+\frac{n}{2} vertices and the vertices have labels 1, 2, ..., n+\frac{n}{2} for even integer n, the Alternate Triangular Snake A(TS_n) admits a super edge trimagic total labeling for even n.

**Example: 2.12** A super edge trimagic total labeling of the Alternate Triangular Snake A(TS_{10}) of the triangle starts from u_{1} and triangle starts from u_{2} are given in figure 4 and figure 5 respectively.

![Figure 4: A(TS_{10}) with \lambda_1 = 41, \lambda_2 = 36 and \lambda_3 = 42.](image)

![Figure 5: A(TS_{10}) with \lambda_1 = 38, \lambda_2 = 34 and \lambda_3 = 39.](image)
Theorem: 2.13 The Alternate triangular snake A(TSₙ) admits an edge trimagic total labeling for odd n.

Proof: We consider the following two cases.

Case: 1 Triangle starts from u₁.

Let V = {uᵢ ∣ 1 ≤ i ≤ n} ∪ {v_j ∣ 1 ≤ j ≤ \( \frac{n-1}{2} \)} be the vertex set and E = {uᵢvᵢ, uᵢv_j ∣ 1 ≤ j ≤ \( \frac{n-1}{2} \)} ∪ {uᵢuᵢ⁺₁ ∣ 1 ≤ i ≤ n – 1} be the edge set of the alternate triangular snake A(TSₙ). Since the triangle starts from u₁, the alternate triangular snake A(TSₙ) has \( n + \frac{n-1}{2} \) vertices and 2n – 2 edges.

Define a bijection f: V ⊔ E → {1, 2, ..., 3n + \( \frac{n-1}{2} \) – 2} such that

\[
f(uᵢ) = \begin{cases} 
\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}
\]

\[
f(v_j) = n+j, \quad 1 \leq j \leq \frac{n-1}{2}; \quad f(uᵢvᵢ⁺₁) = 3n + \frac{n-1}{2} – i – 1, 1 \leq i \leq n – 1;
\]

\[
f(uᵢvᵢ+) = 2n + \frac{n-1}{2} – j, 1 \leq j \leq \frac{n-1}{2}; \quad f(uᵢvⱼ) = 2n + \frac{n-1}{2} – 2j – 1, 1 \leq j \leq \frac{n-1}{2}.
\]

Now we prove the above labeling is an edge trimagic total.

Consider the edges uᵢuᵢ⁺₁, 1 ≤ i ≤ n – 1;

For odd i, \( f(uᵢ) + f(uᵢuᵢ⁺₁) + f(uᵢ⁺₁) = \frac{i+1}{2} + 3n + \frac{n-1}{2} – j-1 + \frac{n+i+1}{2} = 4n = \lambda₁ \) (say).

For even i, \( f(uᵢ) + f(uᵢuᵢ⁺₁) + f(uᵢ⁺₁) = \frac{n+i+1}{2} + 3n + \frac{n-1}{2} – i – 1 + \frac{n+1}{2} = 4n = \lambda₁. \)

For the edges uᵢvᵢ⁺₁, 1 ≤ j ≤ \( \frac{n-1}{2} \);

\[
f(uᵢvᵢ⁺₁) + f(uᵢ⁺₁vⱼ) + f(vⱼ) = \frac{2i+1}{2} + 2n + \frac{n-1}{2} – 2j + n+j = \frac{7n-1}{2} = \lambda₂ \) (say).
\]

For the edges uᵢvⱼ, 1 ≤ j ≤ \( \frac{n-1}{2} \);

\[
f(uᵢvⱼ) + f(uᵢ⁺₁vⱼ) + f(vⱼ) = \frac{n+i+1}{2} + 2n + \frac{n-1}{2} – 2j + n+j = 4n + 1 = \lambda₃ \) (say).
\]

Hence for each edge \( uv \in E \), \( f(u) + f(uv) + f(v) \) yields any one of the trimagic constants \( \lambda₁ = 4n, \lambda₂ = \frac{7n-1}{2}, \lambda₃ = 4n+1. \)

Therefore, the alternate triangular snake graph A(TSₙ) admits an edge trimagic total labeling when the triangle starts from u₁.

Case: 2 Triangle starts from u₂.

Let V = {uᵢ ∣ 1 ≤ i ≤ n} ∪ {v_j ∣ 1 ≤ j ≤ \( \frac{n-1}{2} \)} be the vertex set and E = {uᵢvᵢ, uᵢvⱼ ∣ 1 ≤ j ≤ \( \frac{n-1}{2} \)} ∪ {uᵢuᵢ⁻¹ ∣ 1 ≤ i ≤ n – 1} be the edge set of the alternate triangular snake A(TSₙ). Since the triangle starts from u₂, the alternate triangular snake A(TSₙ) has \( n + \frac{n-1}{2} \) vertices and 2n – 2 edges. Define a bijection f: V ⊔ E → {1, 2, ..., 3n + \( \frac{n-1}{2} \) – 2} such that

\[
f(uᵢ) = \begin{cases} 
\frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
\frac{n+i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even} 
\end{cases}
\]

\[
f(v_j) = n+j, \quad 1 \leq j \leq \frac{n-1}{2}; \quad f(uᵢuᵢ⁻¹) = 3n + \frac{n-1}{2} – j-1, 1 \leq j \leq n – 1;
\]

\[
f(uᵢvⱼ) = 2n + \frac{n-1}{2} – 2j – 1, 1 \leq j \leq \frac{n-1}{2}; \quad f(uᵢ⁺₁vⱼ) = 2n + \frac{n-1}{2} – 2j, 1 \leq j \leq \frac{n-1}{2}.
\]
Now we prove the above labeling is an edge trimagic total.

Consider the edges \( u_iu_{i+1}, 1 \leq i \leq n-1 \);

For odd \( i \), \( f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = \frac{i+1}{2} + 3n + \frac{n-1}{2} - i - 1 + \frac{n+i+1}{2} = 4n = \lambda_1 \) (say).

For even \( i \), \( f(u_i) + f(u_iu_{i+1}) + f(u_{i+1}) = \frac{n+i+1}{2} + 3n + \frac{n-1}{2} - i + 1 + \frac{n+i+1}{2} = 4n = \lambda_1 \).

For the edges \( u_{2j+1}v_j, 1 \leq j \leq \frac{n-1}{2} \);

\[ f(u_{2j+1}) + f(u_{2j+1}v_j) + f(v_j) = \frac{2j+1}{2} + 2n + \frac{n-1}{2} - 2j + n + j = \frac{7n+1}{2} = \lambda_2 \) (say).

For the edges \( u_{2j}v_j, 1 \leq j \leq \frac{n-1}{2} \);

\[ f(u_{2j}) + f(u_{2j}v_j) + f(v_j) = \frac{n+2j+1}{2} + 2n + \frac{n-1}{2} - 2j + 1 + n + j = 4n+1 = \lambda_3 \) (say).

Hence for each edge \( uv \in E \), \( f(u) + f(uv) + f(v) \) yields any one of the trimagic constants \( \lambda_1 = 4n, \lambda_2 = \frac{7n+1}{2} \) and \( \lambda_3 = 4n+1 \).

Therefore, the alternate triangular snake graph \( A(TS_n) \) admits an edge trimagic total labeling when the triangle starts from \( u_2 \).

Hence the theorem follows from case 1 and case 2.

**Theorem: 2.14** The Alternate Triangular Snake \( A(TS_n) \) admits a super edge trimagic total labeling for odd \( n \).

**Proof:** We proved that the Alternate Triangular Snake \( A(TS_n) \) has an edge trimagic total labeling. The labeling given in the proof of Theorem 2.13, the vertices get labels 1, 2, \ldots, \( n + \frac{n-1}{2} \) Since the Alternate Triangular Snake \( A(TS_n) \) has \( n + \frac{n-1}{2} \) vertices and the vertices have labels 1, 2, \ldots, \( n + \frac{n-1}{2} \) for odd integer \( n \), the Alternate Triangular Snake \( A(TS_n) \) admits a super edge trimagic total labeling for odd \( n \).

**Example: 2.15** A super edge trimagic total labeling of the Alternate Triangular Snake \( A(TS_n) \) of the triangle starts from \( u_1 \) and triangle starts from \( u_2 \) are given in figure 6 and figure 7, respectively.

**CONCLUSION**

In this paper we proved that the corona graph \( C_n \odot K_2 \), double ladder \( P_xP_3 \), quadrilateral snake \( Q_m \), alternate triangular snake \( A(TS_n) \) are edge trimagic total and super edge trimagic total graphs. There may be many interesting trimagic graphs can be constructed in future.
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A-Vertex Consecutive Edge Trimagic Total Labeling of Graphs

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ABSTRACT
An edge trimagic total labeling of a (p, q) graph G is a bijection f: V(G)∪E(G) → {1, 2, ..., p+q} such that for each edge xy ∈ E(G), the value of f(x)+f(xy)+f(y) is equal to either k₁ or k₂ or k₃. In this paper we prove that the graphs Pₙ, and G(n; m, n) admits a-vertex consecutive edge trimagic total labeling.

Keywords- Function, Bijection, Labeling, Magic, Trimagic; AMS Subject Classification: 05C78.

1. INTRODUCTION
We begin with simple, finite and undirected graph G = (V(G), E(G)). A graph labeling is an assignment of integers to elements of a graph, the vertices or edges or both subject to certain conditions. The concept of graph labeling was introduced by Rosa in 1967. In 1970, Kotzig and Rosa defined, a magic labeling of graph G is a bijection f: V(G) → {1, 2, ..., p+q} such that for each edge xy ∈ E(G), f(x)+f(xy)+f(y) is a magic constant. In 1996, Ringel and Llado called this labeling as edge magic. In 2001, Wallis introduced this as edge magic total labeling [9]. An edge magic total labeling is called a super edge magic total if the vertices are labeled with smallest positive integers. In 2004, J. Baskar Babujee introduced the bimagic labeling of graphs [2]. In 2013, C. Jayasekaran, M. Regees and C. Davidraj introduced the edge trimagic total labeling of graphs [4]. In 2009, J. Baskar Babujee and V. Vishnu Priya introduced a-vertex consecutive edge trimagic total labeling of graphs [1]. In [5, 7, 8] we proved that some classes and families of graphs are edge trimagic total. We recall few definitions to prove some theorems.

Definition 1.1 [4] An edge trimagic total labeling of a (p, q) graph G is a bijection f: V(G)∪E(G) → {1, 2, ..., p+q} such that for each edge xy ∈ E(G), the value of f(x)+f(xy)+f(y) is equal to any of the distinct constants k₁ or k₂ or k₃. A graph G is said to be edge trimagic total, if it admits an edge trimagic total labeling.

Definition 1.2 [1] B_{m,n} is a (m, n) bistar obtained from two disjoint copies of K₁,m and K₁,n by joining the central vertices by an edge. It has (m+n+2) vertices and (m+n+1) edges. The graph (B_{m,n}') obtained from the graph (B_{m,n}) by subdividing the middle edge with a new vertex. It has (m+n+3) vertices and (m+n+2) edges.

Definition 1.3 [1] (K₁,n : 2) is a graph obtained from B_{n,n} by subdividing the middle edge with a new vertex. (K₁,n : 2) is a graph obtained from (K₁,n : 2) by joining one more copy of K₁,n by an edge to one of the central vertices of K₁,n of (K₁,n : 2) and subdividing that edge with a new vertex. The graph (K₁,n : 3) has (3n+5) vertices and (3n+4) edges.

Definition 1.4 [1] The Crown graph Cₙ⊙K₁ is the graph obtained from a cycle Cₙ by attaching a pendent edge at each vertex of the cycle.

Definition 1.5 [1] A graph which is obtained from (K₁,₁ : 2) by attaching (n-2)P₂ at the central vertex of (K₁,₁ : 2) is called a double star and is denoted by G(v; nP₂). It has 2n+1 vertices and 2n edges.

For further references, we use Dynamic survey of graph labeling by J.A. Gallian [3]. In this paper we introduce a-vertex consecutive edge trimagic total labeling, and prove that the graphs Pₙ, (Bₙ,n : 2), (K₁,n : 3), Cₙ⊙K₁, (K₁,p ⊔ K₁,q ⊔ K₁,i) and G(v; nP₂) admits a-vertex consecutive edge trimagic total labeling.

2. MAIN RESULTS
In this section, we introduce a-vertex consecutive edge trimagic total labeling and prove that the graphs Pₙ, (Bₙ,n : 2), (K₁,n : 3), Cₙ⊙K₁, (K₁,p ⊔ K₁,q ⊔ K₁,i) and G(v; nP₂) admits a-vertex consecutive edge trimagic total labeling and give examples for each of the above graphs.

Definition 2.1. A bijection f: V(G)∪E(G) → {1, 2, ..., p+q} is called a-vertex consecutive edge trimagic total labeling of G = (G, E) if f is an edge trimagic total labeling and f(V) = {a+1, a+2, ..., a+p}, 0 ≤ a ≤ q. If a = 0, then the labeling is called a super edge trimagic total.
**Theorem 2.2.** The graph $P_3 \square K_n$ admits a vertex consecutive edge trimagic total labeling for even $n$.

**Proof:** Let $V = \{u, v, w, u_i, v_i, w_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{uu_i, vv_i, wv_i, uv_i / 1 \leq i \leq n\}$ be the edge set of the graph $P_3 \square K_n$. The graph $P_3 \square K_n$ has $3n+3$ vertices and $3n+2$ edges.

Define a bijection $f: V \rightarrow \{1, 2, ..., 6n+5\}$ such that $f(u) = 4n-3$, $f(v) = 4n-2$, $f(w) = 4n-1$, $f(u_i) = 4n+i-1$, $1 \leq i \leq n$; $f(v_i) = 5n+i-1$, $1 \leq i \leq n$; $f(w_i) = 6n+i-1$, $1 \leq i \leq n$; $f(uu_i) = 3n+i+1$, $1 \leq i \leq n$; $f(vv_i) = 2n-i+1$, $1 \leq i \leq n$; $f(wv_i) = n-i+1$, $1 \leq i \leq n$; $f(uv_i) = m+n+3$, $f(vw_i) = m+n+4$, $f(uw_i) = m+n+5$.

Now we prove this labeling is a vertex consecutive edge trimagic total.

For the edges $uu_i$, $1 \leq i \leq n$;
\[
f(u)+f(uu_i)+f(u_i) = 4n-3+3n-i+1+4n+i-1 = 11n-3 = \lambda_i (say).
\]

For the edges $vv_i$, $1 \leq i \leq n$;
\[
f(v)+f(vv_i)+f(v_i) = 4n-2+2n-i+1+5n+i-1 = 11n-2 = \lambda_i (say).
\]

For the edges $ww_i$, $1 \leq i \leq n$;
\[
f(w)+f(ww_i)+f(w_i) = 4n-1+n-i+1+6n+i-1 = 11n-1 = \lambda_i (say).
\]

For the edge $uv$, $f(u)+f(uv)+f(v) = 4n-3+3n+2+4n-2 = 11n-3 = \lambda_i$.

For the edge $vw$, $f(v)+f(vw)+f(w) = 4n-2+3n+1+4n-1 = 11n-2 = \lambda_i$.

Hence for each edge $uv \in E$, the value of $f(u)+f(uv)+f(v)$ yields any of the trimagic constants $\lambda_i = 11n-3$, $\lambda_2 = 11n-2$ and $\lambda_3 = 11n-1$. Therefore, the graph $P_3 \square K_n$ admits a vertex consecutive edge trimagic total labeling for $a = 3n+2$.

**Example 2.3.** A vertex consecutive edge trimagic total labeling of $P_3 \square K_6$ with $a = 20$ is given in figure 1.

![Fig.1. $P_3 \square K_6$ with $\lambda_1 = 63$, $\lambda_2 = 64$ and $\lambda_3 = 65$.](image)

**Theorem 2.4.** The graph $(B_{m,n})^2$ admits a vertex consecutive edge trimagic total labeling for $(m, n \geq 1)$.

**Proof:** Let $V = \{u, v, w, u_i, v_i, w_i / 1 \leq i \leq m, 1 \leq j \leq n\}$ be the vertex set and $E = \{uu_i, vv_i, wv_i, uv_i / 1 \leq i \leq m, 1 \leq j \leq n\}$ be the edge set. Then the graph $(B_{m,n})^2$ has $m+n+3$ vertices and $m+n+2$ edges and $m, n \geq 1$.

Define a bijection $f: V \cup E \rightarrow \{1, 2, ..., 2m+2n+5\}$ such that $f(w) = m+n+3$, $f(v) = m+n+4$, $f(u) = m+n+5$, $f(u_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(v_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(uu_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(vv_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(uw_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(wv_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(uv_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(vw_i) = m+i+2n+5$, $1 \leq i \leq m$, $f(uw) = m+n+1$ and $f(wv) = m+n+2$.

Now we prove this labeling is a vertex consecutive edge trimagic total.

For the edges $uu_i$, $1 \leq i \leq m$;
\[
f(u)+f(uu_i)+f(u_i) = m+n+5+m+1-i+m+2n+5 = 3m+3n+11 = \lambda_i (say).
\]

For the edges $vv_i$, $1 \leq j \leq n$;
\[
f(v)+f(vv_i)+f(v_i) = m+n+4+m+n+1-j+m+n+5+j = 3m+3n+10 = \lambda_j (say).
\]

For the edge $uw$,
\[
f(u)+f(uw)+f(w) = m+n+5+m+1-1+m+2n+5 = 3m+3n+11 = \lambda_i (say).
\]
\[ f(u)+f(uw)+f(w) = m+n+5+m+n+1+m+n+3 = 3m+3n+9 = \lambda_3 \text{(say)}. \]

For the edge vw,
\[ f(v)+f(vw)+f(w) = m+n+4+m+n+2+m+n+3 = m+3n+9 = \lambda_3 \text{(say)}. \]

Hence for each edge uv \in E, the value of \( f(u)+f(uv)+f(v) \) yields any of the trimagic constants \( \lambda_1 = 3m+3n+11, \lambda_2 = 3m+3n+10 \) and \( \lambda_3 = 3m+3n+9 \). This proves that the graph \( (B_{m,n}: 2) \) admits a-vertex consecutive edge trimagic total labeling for \( (m, n \geq 1) \) for \( a = m+n+2 \).

**Example 2.5.** The graph \( (B_{7,6}: 2) \) admits a-vertex consecutive edge trimagic total labeling with \( a = 15 \).

**Theorem 2.6.** The graph \( (K_{1,n}: 3) \), \( n \geq 3 \) admits a-vertex consecutive edge trimagic total labeling.

**Proof:** Let \( V = \{u, v, w, x, y, u_i, v_i, w_i \mid 1 \leq i \leq n \} \) be the vertex set and \( E = \{u_i, v_i, w_i \mid 1 \leq i \leq n\} \cup \{ux, xv, vy, yw\} \) be the edge set of \( (K_{1,n}: 3) \). Then the graph \( (K_{1,n}: 3) \) has \( 3n+5 \) vertices and \( 3n+4 \) edges.

Define a bijection \( f: V \cup E \rightarrow \{1, 2, \ldots, 6n+9\} \) such that \( f(u) = 3n+5, f(v) = 3n+6, f(w) = 3n+7, f(x) = 3n+8, f(y) = 3n+9, f(ux) = 3n+4, f(xv) = 3n+3, f(vy) = 3n+2, f(yw) = 3n+1 \), for all \( 1 \leq i \leq n; f(u_i) = 3n+i+9, f(v_i) = 4n+i+9, f(w_i) = 5n+i+9, f(uu_i) = 3n-i+1, f(vv_i) = 4n-i+1 \) and \( f(ww_i) = n-i+1 \).

Now we prove this labeling is a-vertex consecutive edge trimagic total.

For the edges uu_i, \( 1 \leq i \leq n; \)
\[ f(u)+f(uu_i)+f(u_i) = 3n+5+3n-i+1+3n+i+9 = 9n+15 = \lambda_1 \text{(say)}. \]

For the edges vv_i, \( 1 \leq i \leq n; \)
\[ f(v)+f(vv_i)+f(v_i) = 3n+6+2n-i+1+4n+i+9 = 9n+16 = \lambda_2 \text{(say)}. \]

For the edges ww_i, \( 1 \leq i \leq n; \)
\[ f(w)+f(ww_i)+f(w_i) = 3n+7+n-i+1+5n+i+9 = 9n+17 = \lambda_3 \text{(say)}. \]

For the edge ux, \( f(x)+f(ux)+f(u) = 3n+8+3n+4+3n+5 = 9n+17 = \lambda_3 \).

For the edge xv, \( f(x)+f(xv)+f(v) = 3n+8+3n+3+3n+6 = 9n+17 = \lambda_3 \).

For the edge vy, \( f(v)+f(vy)+f(v) = 3n+6+3n+2+3n+9 = 9n+17 = \lambda_3 \).

For the edge yw, \( f(y)+f(yw)+f(w) = 3n+9+3n+1+3n+7 = 9n+17 = \lambda_3 \).

Hence for each edge uv \in E, the value of \( f(u)+f(uv)+f(v) \) yields any of the trimagic constants \( \lambda_1 = 9n+15, \lambda_2 = 9n+16 \) and \( \lambda_3 = 9n+17 \). This proves that the graph \( (K_{1,n}: 3) \), \( n \geq 3 \) admits a-vertex consecutive edge trimagic total labeling for \( a = 3n+4 \).

**Example 2.7.** The graph \( (K_{1,6}: 3) \) admits a-vertex consecutive edge trimagic total labeling with \( a = 22 \).
Fig. 3. \((K_1 \circ 3)\) with \(\lambda_1 = 69, \lambda_2 = 70\) and \(\lambda_3 = 71\).

**Theorem 2.8.** The Crown graph \(C_n \circ K_1\) admits a-vertex consecutive edge trimagic total labeling.

**Proof:** Let \(V = \{v_i, 1 \leq i \leq n\}\) be the vertex set and \(E = \{u_i, 1 \leq i \leq n\} \cup \{u_iu_{i+1}, 1 \leq i \leq n-1\} \cup \{u_{n}u_1\}\) be the edge set of the crown graph \(C_n \circ K_1\). The Crown graph \(C_n \circ K_1\) has \(2n\) vertices and \(2n\) edges.

**Case 1.** \(n\) is odd.

Define a bijection \(f: V \cup E \rightarrow \{1, 2, \ldots, 4n\}\) such that
\[
\begin{align*}
\text{for } v_i & : \quad \frac{3n+i+1}{2}, \quad 1 \leq i \leq n \text{ and } i \text{ is odd}, \\
\text{for } v_i & : \quad \frac{3n+i+1}{2}, \quad 1 \leq i \leq n \text{ and } i \text{ is even}, \\
\text{for } u_iu_{i+1} & : \quad 2n-i, \quad 1 \leq i \leq n-1; \\
\text{for } u_iu_1 & : \quad \frac{5n+i+1}{2}, \quad 1 \leq i \leq n \text{ and } i \text{ is odd}, \\
\text{for } u_iu_1 & : \quad \frac{5n+i+1}{2}, \quad 1 \leq i \leq n \text{ and } i \text{ is even}, \\
\end{align*}
\]

Then \(f(u_iu_{i+1}) = 2n-i, 1 \leq i \leq n-1; f(u_iu_1) = \frac{5n+i+1}{2}, 1 \leq i \leq n\) and \(f(u_{n}u_1) = 2n\).

Now we prove this labeling is a-vertex consecutive edge trimagic total.

For the edge \(u_iu_{i+1}\),
\[
\begin{align*}
\text{for } u_iu_{i+1} & : \quad 2n-i+2n = 4n-i = \frac{13n+3}{2} = \lambda_1. \\
\end{align*}
\]

**Subcase 1.1.** \(i\) is odd.

For the edges \(u_iu_{i+1}\), \(1 \leq i \leq n-1\);
\[
\begin{align*}
\text{for } u_iu_{i+1} & : \quad 2n-i+2n = 4n-i = \frac{13n+3}{2} = \lambda_1. \\
\end{align*}
\]

For the edges \(u_iu_1\), \(1 \leq i \leq n\);
\[
\begin{align*}
\text{for } u_iu_1 & : \quad 2n+i+1 = n+i+1+3n = 6n+2 = \lambda_2. \\
\end{align*}
\]

**Subcase 1.2.** \(i\) is even.

For the edges \(u_iu_{i+1}\), \(1 \leq i \leq n-1\);
\[
\begin{align*}
\text{for } u_iu_{i+1} & : \quad 2n+i+1 = n+i+1+3n = 6n+2 = \lambda_2. \\
\end{align*}
\]

For the edges \(u_iu_1\), \(1 \leq i \leq n\);
\[
\begin{align*}
\text{for } u_iu_1 & : \quad \frac{5n+i+1}{2} - i + 2n = \frac{13n+3}{2} = \lambda_1. \\
\end{align*}
\]

Hence for each edge \(uv \in E\), the value of \(f(u)+f(uv)+f(v)\) yields any of the trimagic constants \(\lambda_1 = \frac{13n+3}{2}\), \(\lambda_2 = 6n+2\) and \(\lambda_3 = 7n+2\). This proves that the graph \(C_n \circ K_1\) admits a-vertex consecutive edge trimagic total labeling for \(a = 2n\), when \(n\) is odd.

**Case 2.** \(n\) is even.

Define a bijection \(f: V \cup E \rightarrow \{1, 2, \ldots, 4n\}\) such that
\[ f(u_i) = \begin{cases} 
2n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
5n + \frac{i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even},
\end{cases} \]

\[ f(u_i) = \begin{cases} 
3n + \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\
7n + \frac{i}{2}, & 1 \leq i \leq n \text{ and } i \text{ is even},
\end{cases} \]

\[ f(u_i) = 2n - i, \ 1 \leq i \leq n - 1; \]
\[ f(u_i v_i) = n - i + 1, \ 1 \leq i \leq n - 1 \text{ and } f(u_i u_i) = 2n. \]

Now we prove this labeling is a-vertex consecutive edge trimagic total.

For the edge \( u_i u_{i+1} \), \( f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2n + 2n + \frac{i+1}{2} = 7n + 1 = \lambda_1 \text{(say)}. \)

**Subcase 2.1.** \( i \) is odd.

For the edges \( u_i u_{i+1}, 1 \leq i \leq n - 1; \)
\[ f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = 2n + 2n - i + \frac{5n + i + 1}{2} = \frac{13n + 2}{2} = \lambda_2 \text{(say)}. \]

For the edges \( u_i v_i, 1 \leq i \leq n; \)
\[ f(u_i) + f(u_i v_i) + f(v_i) = 2n + \frac{i+1}{2} + n - i + 3n + \frac{i+1}{2} = 6n + 2 = \lambda_3 \text{(say)}. \]

**Subcase 2.2.** \( i \) is even.

For the edges \( u_i u_{i+1}, 1 \leq i \leq n - 1; \)
\[ f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = \frac{5n + i}{2} + 2n - i + \frac{13n + 2}{2} = \lambda_2. \]

For the edges \( u_i v_i, 1 \leq i \leq n; \)
\[ f(u_i) + f(u_i v_i) + f(v_i) = \frac{5n + i}{2} + n - i + 1 + \frac{7n - i}{2} = 7n + 1 = \lambda_1. \]

Hence for each edge \( uv\in E \), the value of \( f(u) + f(uv) + f(v) \) yields any of the trimagic constants \( \lambda_1 = 7n + 1, \lambda_2 = \frac{13n + 2}{2} \), and \( \lambda_3 = 6n + 2 \). This proves that the graph \( C_n \square K_1 \) admits a-vertex consecutive edge trimagic total labeling for \( a = 2n \), when \( n \) is even.

The theorem follows from case 1 and case 2.

**Example 2.9.** A-vertex consecutive edge trimagic total labeling of the graphs \( C_9 \square K_1 \) and \( C_{10} \square K_1 \) are given in figure 4 and figure 5, respectively.

![Figure 4: \( C_9 \square K_1 \) with \( \lambda_1 = 56, \lambda_2 = 60 \) and \( \lambda_3 = 65 \).](image)

![Figure 5: \( C_{10} \square K_1 \) with \( \lambda_1 = 66, \lambda_2 = 62 \) and \( \lambda_3 = 71 \).](image)
Theorem 2.10. The disconnected graph \((K_{i,p} \cup K_{i,q} \cup K_{i,r})\) admits a-vertex consecutive edge trimagic total labeling.

Proof: Let \(V = \{u_1, u_2, \ldots, u_{p}, v_1, v_2, \ldots, v_{q}, w_1, w_2, \ldots, w_{r}\}\) be the vertex set and \(E = \{uu_1, vv_1, wv_1/1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r\}\) be the edge set of the graph \((K_{i,p} \cup K_{i,q} \cup K_{i,r})\). The disconnected graph \((K_{i,p} \cup K_{i,q} \cup K_{i,r})\) has \((p+q+r)\) vertices and \((p+q+r)\) edges.

Define a bijection \(f: V \rightarrow \{1, 2, \ldots, 2(p+q+r+3)\}\) such that \(f(u) = p+q+r+1, f(v) = p+q+r+2, f(w) = p+q+r+3, f(uu_1) = p+q+r+3+i, 1 \leq i \leq p; f(vv_1) = 2p+q+r+3+j, 1 \leq j \leq q; f(wv_1) = 2p+2q+r+3+k, 1 \leq k \leq r; f(uu_1) = p+1-i, 1 \leq i \leq p; f(vv_1) = p+q+1-j, 1 \leq j \leq q; and f(wv_1) = p+q+r+1-k, 1 \leq k \leq r.

Now we prove that this labeling is a-vertex consecutive edge trimagic total. For the edges \(uu_1, 1 \leq i \leq p; f(u)+f(uu_1)+f(u) = p+q+r+1+p+1-i+p+q+r+3+i = 3p+2q+2r+5 = \lambda_i\) (say).

For the edges \(vv_1, 1 \leq j \leq q; f(v)+f(vv_1)+f(v) = p+q+r+2+p+q+r+1+j+2p+2q+r+3+j = 4p+3q+2r+6 = \lambda_2\) (say).

For the edges \(ww_1, 1 \leq k \leq r; f(w)+f(ww_1)+f(w) = p+q+r+3+p+q+r+1-k+2p+2q+r+3+k = 4p+4q+3r+7 = \lambda_3\) (say).

Hence for each edge \(uv \in E\), the value of \(f(u)+f(uv)+f(v)\) yields any of the trimagic constants \(\lambda_i = 3p+2q+2r+5, \lambda_2 = 4p+3q+2r+6\) and \(\lambda_3 = 4p+4q+3r+7\). This proves that the graph \((K_{i,p} \cup K_{i,q} \cup K_{i,r})\) admits a-vertex consecutive edge trimagic total labeling for \(a = p+q+r\).

Example 2.11. The graph \((K_{1,7} \cup K_{1,6} \cup K_{1,8})\) admits a vertex consecutive edge trimagic total labeling with \(a = 21\).
\( = 7n + 3 = \lambda_3 \text{(say)}. \)

Hence for each edge \( uv \in E \), the value of \( f(u) + f(uv) + f(v) \) yields any of the trimagic constants \( \lambda_1 = 6n + 3 \), \( \lambda_2 = 5n + 4 \) and \( \lambda_3 = 7n + 3 \). This proves that the double star \( G(v; nP_3) \) admits a-vertex consecutive edge trimagic total labeling with \( a = 2n \) for even \( n \).

**Example 2.13.** A-vertex consecutive edge trimagic total labeling of the double star \( G(v; 6P_3) \) with \( a = 12 \) is given in figure 7.

![Figure 7](image_url)

**Fig. 7.** \( G(v; 6P_3) \) with \( \lambda_1 = 34 \), \( \lambda_2 = 39 \) and \( \lambda_3 = 45 \).

**REFERENCES**


Edge Trimagic Total Labeling for Disconnected Graphs

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Abstract - An edge trimagic total labeling of a graph G(V, E) with p vertices and q edges is a bijection f from the set of vertices and edges to 1, 2, …, p+q such that for every edge uv in E, f(u)+f(uv)+f(v) is either λ₁, λ₂ or λ₃. An edge trimagic total graph is called a super edge trimagic total if f(V) = {1, 2, …, p}. An edge trimagic total graph is called a superior edge trimagic total if f(E) = {1, 2, …, q}. In this paper we prove the disconnected graphs nP₃, (K₁, p)∪(K₁, q∪K₁, r), nP₂∪K₁, n+1, t copies of the sun graph Sₙ, nC₄ and nC₆ admits edge trimagic total labeling.

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Keywords: Function, Graph, Labeling, Magic labeling, Trimagic labeling.

I. INTRODUCTION

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. All graphs considered here are finite, simple and undirected. The useful survey on graph labeling by J.A. Gallian(2012) can be found in [4].

A walk of a graph G is an alternating sequence of vertices and edges beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A walk which begins and ends at the same vertex is called a closed walk. If the terminal vertices are distinct, the walk is known as open walk. An open walk in which no vertex appears more than once is called a path. A graph is said to be connected if there is at least one path between every pair of vertices otherwise, it is disconnected. The union of two graphs G₁ = (V₁, E₁) and G₂ = (V₂, E₂) is another graph G₃ = G₁∪G₂ whose vertex set V₃ = V₁∪V₂ and the edge set E₃ = E₁∪E₂. A simple graph in which there exists an edge between every pair of vertices is called a complete graph; the complete graph with n vertices is denoted by Kₙ. A bigraph G is a graph whose vertex set V can be partitioned into two subsets V₁ and V₂ such that every edge of G joins a point of V₁ to a point of V₂. If G contains every edge joining V₁ and V₂, then G is a complete bigraph. If V₁ and V₂ have m and n vertices, we write G = Kₘ,ₙ.

A star is a complete bigraph K₁, n. A sun is a cycle Cₙ with an edge terminating in a vertex of degree one attached to each vertex, it is denoted by Sₙ. The t copies of the sun graph Sₙ is denoted by tSₙ[8]

In 1970, Kotzing and Rosa introduced edge magic total labeling [7]. In 2004, J. Basker Bubujee introduced edge bimagic labeling of graphs [1]. In 2013, C. Jayasekaran, M. Regees and C. Davidraj introduced edge trimagic total labeling of graphs[5]. We proved that some families of graphs are edge trimagic total in [5, 6, 9, 10, 11]. In this paper, we prove that the disconnected graphs K₁, p∪K₁, q∪K₁, r (p ≤ q ≤ r), nP₂∪K₁, n+1, t copies of the sun graph tSₙ, nC₄ and nC₆ are super edge trimagic total and nP₃ is superior edge trimagic total.

II. EDGE TRIMAGIC LABELING FOR DISCONNECTED GRAPHS

In this section we define the superior edge trimagic total labeling and prove that the disconnected graphs nP₃, K₁, p∪K₁, q∪K₁, r (p ≤ q ≤ r), nP₂∪K₁, n+1, t copies of the sun graph tSₙ, nC₄ and nC₆ are edge trimagic total.

1) Definition: An edge trimagic total labeling of a (p, q) graph G is a bijection f: V∪E→{1, 2, …, p+q} such that for each edge
2) Theorem: The graph \( nP_3 \) admits an edge trimagic total labeling for even \( n \).

Proof: Let \( V = \{ u_i, v_i, w_i / 1 \leq i \leq n \} \) be the vertex set and \( E = \{ u_i v_i, v_i w_i / 1 \leq i \leq n \} \) be the edge set of the disconnected graph \( nP_3 \). The graph \( nP_3 \) has 3n vertices and 2n edges.

Define a bijection \( f: V \cup E \rightarrow \{ 1, 2, \ldots, 5n \} \) such that

\[
 f(u_i) = 3n + \frac{i+1}{2}, \quad f(w_i) = 3n - \frac{n}{2} + \frac{i}{2} \quad \text{for} \quad 1 \leq i \leq n, \; i \equiv 1 \pmod{2};
\]

\[
 f(u_i) = \frac{n+1+i}{2}, \quad f(v_i) = \frac{n+1+i}{2} \quad \text{for} \quad 1 \leq i \leq n, \; i \equiv 0 \pmod{2};
\]

\[
 f(u_i v_i) = \frac{n+1+i}{2}, \quad f(v_i w_i) = \frac{n+1+i}{2} \quad \text{for} \quad 1 \leq i \leq n, \; i \equiv 1 \pmod{2};
\]

\[
 f(u_i v_i) = n + \frac{n}{2} + \frac{i}{2}, \quad f(v_i w_i) = n + \frac{i}{2} \quad \text{for} \quad 1 \leq i \leq n, \; i \equiv 0 \pmod{2}.
\]

Now we have to prove that the graph \( nP_3 \) has three distinct trimagic constants \( \lambda_1, \lambda_2 \) and \( \lambda_3 \).

Consider the edges \( u_i v_i \):

For \( 1 \leq i \leq n \) and \( i \equiv 1 \pmod{2} \),

\[
 f(u_i) + f(u_i v_i) + f(v_i) = 3n + \frac{i+1}{2} + \frac{n+1+i}{2} + 5n - i + 1
\]

\[
 = 9n + 2 = \lambda_1 \text{(say)}.
\]

Consider the edges \( v_i w_i \):

For \( 1 \leq i \leq n \) and \( i \equiv 0 \pmod{2} \),

\[
 f(v_i) + f(v_i w_i) + f(w_i) = 5n - i + 1 + n + \frac{i+1}{2} + 3n - \frac{n}{2} + \frac{i}{2}
\]

\[
 = 8n + 2 = \lambda_2 \text{(say)}.
\]

Hence for each edge \( uv \in E \), \( f(u) + f(uv) + f(v) \) yields any one of the trimagic constants \( \lambda_1 = 9n + 2 \), \( \lambda_2 = \frac{17n+2}{2} \) and \( \lambda_3 = 8n+2 \).

Hence the graph \( nP_3 \) admits an edge trimagic total labeling for even \( n \).

3) Theorem: The graph \( nP_3 \) admits a superior edge trimagic total labeling for even \( n \).

Proof: We have proved that the graph \( nP_3 \) has an edge trimagic total labeling when \( n \) is even. The labeling given in the proof of the Theorem 2, the edges get labels \( f(u_i v_i) = \frac{n+1+i}{2}, \; f(v_i w_i) = \frac{n+1+i}{2} \) for \( 1 \leq i \leq n \) and \( i \equiv 1 \pmod{2} \); \( f(u_i v_i) = n + \frac{n}{2} + \frac{i}{2}, \; f(v_i w_i) = n + \frac{i}{2} \) for \( 1 \leq i \leq n \) and \( i \equiv 0 \pmod{2} \). Clearly the graph \( nP_3 \) has 2n edges and the 2n edges get labels 1, 2, …, 2n. Hence the graph \( nP_3 \) has a superior edge trimagic total when \( n \) is even.

4) Example: A superior edge trimagic total labeling of the disconnected graph \( 4P_3 \) is given in fig.1.
5) Theorem: The graph $nP_3$ admits an edge trimagic total labeling for odd $n$.

Proof: Let $V = \{u_i, v_i, w_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{u_iv_i, v_iw_i / 1 \leq i \leq n\}$ be the edge set of the disconnected graph $nP_3$. The graph $nP_3$ has $3n$ vertices and $2n$ edges.

Define a bijection $f: V \cup E \rightarrow \{1, 2, ..., 5n\}$ such that

- $f(u_i) = 3n + \frac{n-1}{2} + \frac{i+1}{2}$ for $1 \leq i \leq n$ and $i \equiv 1 \pmod{2}$;
- $f(u_i) = 2n + \frac{n-1}{2} + \frac{i+1}{2}$ for $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$;
- $f(v_i) = 5n - i + 1$;
- $f(u_iv_i) = n + \frac{n-1}{2} + \frac{i+1}{2}$ for $1 \leq i \leq n$ and $i \equiv 1 \pmod{2}$;
- $f(u_iv_i) = n + \frac{n-1}{2} + \frac{i+1}{2} + 1$ for $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$;
- $f(v_iw_i) = n + \frac{i+1}{2} + 1$ for $1 \leq i \leq n$ and $i \equiv 1 \pmod{2}$;
- $f(v_iw_i) = n + \frac{i+1}{2} + 1 + 1$ for $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$.

Now we have to prove the graph $nP_3$ has three different trimagic constants $\lambda_1, \lambda_2, \lambda_3$.

Consider the edges $u_iv_i$;

- For $1 \leq i \leq n$ and $i \equiv 1 \pmod{2}$,
  
  
  $f(u_i)+f(u_iv_i)+f(v_i) = 3n + \frac{n-1}{2} + \frac{i+1}{2} + \frac{n-1}{2} + \frac{i+1}{2} + 5n - i + 1 = 9n + 2 = \lambda_1$ (say).

- For $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$,
  
  $f(u_i)+f(u_iv_i)+f(v_i) = 2n + \frac{n-1}{2} + \frac{i+1}{2} + \frac{n-1}{2} + \frac{i+1}{2} + 5n - i + 1 = \frac{17n + 2}{2} = \lambda_2$ (say).

Consider the edges $v_iw_i$;

- For $1 \leq i \leq n$ and $i \equiv 1 \pmod{2}$,
  
  $f(v_i)+f(v_iw_i)+f(w_i) = 5n - i + 1 + \frac{i+1}{2} + \frac{n-1}{2} + \frac{i+1}{2} - 1 = 8n + 1 = \lambda_3$ (say).

- For $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$,
  
  $f(v_i)+f(v_iw_i)+f(w_i) = 5n - i + 1 + n + \frac{i+1}{2} + 2n + \frac{n-1}{2} + \frac{i+1}{2} = \frac{17n + 2}{2} = \lambda_3$ (say).

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the trimagic constants $\lambda_1, \lambda_2, \lambda_3$.

Therefore, the graph $nP_3$ admits an edge trimagic total labeling when $n$ is odd.

6) Theorem: The graph $nP_3$ admits superior edge trimagic total labeling when $n$ is odd.

Proof: We have proved that the graph $nP_3$ has edge trimagic total labeling when $n$ is odd. The labeling given in the proof of the Theorem 5, the edges get labels for $i \equiv 1 \pmod{2}$, $f(u_iv_i) = \frac{n+1}{2} + \frac{i+1}{2}$ and $f(v_iw_i) = \frac{i+1}{2}$ also for $i \equiv 0 \pmod{2}$; $f(u_iv_i) = n + \frac{n-1}{2} + \frac{i+1}{2} + 1$, $f(v_iw_i) = n + \frac{i+1}{2} + 1, 1 \leq j \leq n$. Clearly the graph $nP_3$ has $2n$ edges and the $2n$ edges get labels $1, 2, ..., 2n$. Hence the graph $nP_3$ is a superior edge trimagic total.

7) Corollary: The disconnected graph $nP_3$ admits an edge trimagic total labeling for all $n$.

8) Example: The disconnected graph $5P_3$ given in fig. 2 is a superior edge trimagic total graph.

![Graph 5P_3 with trimagic constants](image-url)
9) **Theorem:** The graph \( K_{i,p} \cup K_{i,q} \cup K_{i,r} \), \((p \leq q \leq r)\) admits an edge trimagic total labeling.

**Proof:** Let \( V = \{ u_i, v_j, w_k \mid 1 \leq i \leq p+1; 1 \leq j \leq q+1; 1 \leq k \leq r+1 \} \) be the vertex set and \( E = \{ u_i u_j, v_j v_k, w_k w_i \mid 2 \leq i \leq p+1; 2 \leq j \leq q+1; 2 \leq k \leq r+1 \} \) be the edge set of the graph \( (K_{i,p} \cup K_{i,q} \cup K_{i,r}) \).

Define a bijection \( f: V \rightarrow \mathbb{N} \) such that \( f(u_i) = 1; f(v_j) = 2; f(w_k) = 3 \);

\( f(u_i) = p+q+r+5, 2 \leq i \leq p+1; f(v_j) = q+r+5, 2 \leq j \leq q+1; f(w_k) = r+5, 2 \leq k \leq r+1. \)

Hence for each edge \( uv \in E \), \( f(u)+f(uv)+f(v) \) yields any one of the magic constants \( \lambda_1 = 2(p+q+r) + 8, \lambda_2 = 2(p+q+r) + 9 \) and \( \lambda_3 = 2(p+q+r) + 10. \)

Therefore, the disconnected graph \( K_{i,p} \cup K_{i,q} \cup K_{i,r} \), \((p \leq q \leq r)\) admits an edge trimagic total labeling.

10) **Theorem:** The graph \( nP_2 \cup K_{i,p} \cup K_{i,q} \cup K_{i,r} \), \((p \leq q \leq r)\) has a super edge trimagic total labeling.

**Proof:** We have proved that the graph \( K_{i,p} \cup K_{i,q} \cup K_{i,r} \), \((p \leq q \leq r)\) has an edge trimagic total labeling. The labeling given in the proof of the Theorem 9, the vertices get labels \( f(u_i) = 1; f(v_j) = 2; f(w_k) = 3 \);

\( f(u_i) = p+q+r+5, 2 \leq i \leq p+1; f(v_j) = q+r+5, 2 \leq j \leq q+1; f(w_k) = r+5, 2 \leq k \leq r+1. \)

Hence the graph \( (K_{i,p} \cup K_{i,q} \cup K_{i,r}) \cup (nP_2) \) is a super edge trimagic total.

11) **Example:** The graph \( (K_{i,6} \cup K_{i,8} \cup K_{i,10}) \) given in fig. 3 is a super edge trimagic total.

![Fig. 3 (K_{i,6} \cup K_{i,8} \cup K_{i,10}) with \lambda_1 = 56, \lambda_2 = 57 and \lambda_3 = 58.](http://www.ijmttjournal.org)

12) **Theorem:** The graph \( nP_2 \cup K_{i,n+1} \) has an edge trimagic total labeling when \( n \) is odd.

**Proof:** Let \( V = \{ u_i, v_j, w_k \mid 1 \leq i \leq n \cup \{ w, w_j \mid 1 \leq j \leq n + 1 \} \) be the vertex set and \( E = \{ u_i v_j, 1 \leq i \leq n \cup \{ w w_j \mid 1 \leq j \leq n + 1 \} \) be the edge set of the disconnected graph \( nP_2 \cup K_{i,n+1} \). The graph \( nP_2 \cup K_{i,n+1} \) has \( 3n+2 \) vertices and \( 2n+1 \) edges.

Define a bijection \( f: V \cup E \rightarrow \{ 1, 2, \ldots, 5n+3 \} \) such that

\( f(v_i) = n+ \frac{i+1}{2}, f(u_i) = 2n+1+ \frac{1}{2} \) for \( 1 \leq i \leq n, i \equiv 1(\text{mod} \, 2), f(v_i) = n+1+ \frac{1}{2}, f(u_i) = 2n+1+1+ \frac{1}{2} \) for \( 1 \leq i \leq n, i \equiv 0(\text{mod} \, 2). \)
\[ f(v_i) = 4n + 3 - i, \quad 1 \leq i \leq n, \]
\[ f(w_j) = 2(n+1) - \left( \frac{j-1}{2} \right), \quad f(ww_i) = 4n + \frac{n+1}{2} + 2 \quad \text{for} \quad 1 \leq j \leq n+1 \text{ and} \]
\[ j \equiv 1 \pmod{2}, \]
\[ f(w_j) = \frac{n+1}{2} - \frac{1}{2} + 1, \quad f(ww_i) = 4n + \frac{n+1}{2} + \frac{i}{2} + 2 \quad \text{for} \quad 1 \leq j \leq n+1 \]

and \( j \equiv 0 \pmod{2} \) and \( f(w) = n+1 \).

Now we prove that the graph \( nP_2 \cup K_{1,n+1} \) have three trimagic constants \( \lambda_1, \lambda_2 \) and \( \lambda_3 \).

Consider the edges \( v_iu_i \):

For \( 1 \leq i \leq n \) and \( i \equiv 1 \pmod{2} \):
\[ f(v_i) + f(v_iu_i) + f(u_i) = n + \frac{i+1}{2} + 4n + 3 - i + 2n + \frac{i+1}{2} + 2 \]
\[ = 7n + 7 = \lambda_1 \text{ (say)}. \]

For \( 1 \leq i \leq n \) and \( i \equiv 0 \pmod{2} \):
\[ f(v_i) + f(v_iu_i) + f(u_i) = \frac{n+1}{2} + \frac{1}{2} + 4n + 3 - i + 2n + \frac{n+1}{2} + \frac{i}{2} + 2 \]
\[ = 7n + 6 = \lambda_2 \text{ (say)}. \]

Consider the edges \( ww_j \):

For \( 1 \leq j \leq n+1 \) and \( j \equiv 1 \pmod{2} \):
\[ f(w_j) + f(ww_j) + f(w_j) = n + 1 + 4n + \left( \frac{j-1}{2} + 1 \right) + 2 + 2(n+1) - \left( \frac{j-1}{2} \right) \]
\[ = 7n + 6 = \lambda_2 \text{ (say)}. \]

For \( 1 \leq j \leq n+1 \) and \( j \equiv 0 \pmod{2} \):
\[ f(w_j) + f(ww_j) + f(w_j) = n + 1 + 4n + \frac{n+1}{2} + \frac{i}{2} + 2 + \frac{n+1}{2} - \frac{j}{2} + 1 \]
\[ = 6n + 5 = \lambda_3 \text{ (say)}. \]

Hence for each edge \( uv \in E \), \( f(u) + f(uv) + f(v) \) yields any one of the trimagic constants \( \lambda_1 = 7n + 7, \lambda_2 = 7n + 6 \) and \( \lambda_3 = 6n + 5 \).

Thus the graph \( nP_2 \cup K_{1,n+1} \) has an edge trimagic total labeling when \( n \) is odd.

13) Theorem: The graph \( nP_2 \cup K_{1,n+1} \) admits a super edge trimagic total labeling for odd \( n \).

Proof: We have proved that the graph \( nP_2 \cup K_{1,n+1} \) has edge trimagic total labeling when \( n \) is odd. The labeling given in the proof of the Theorem 12, the vertices get labels \( f(v_i) = n + \frac{i+1}{2} + 1, \quad f(u_i) = 2n + \frac{i+1}{2} + 2 \) for \( 1 \leq i \leq n \) and \( i \equiv 1 \pmod{2} \) and \( f(v_i) = \frac{n+1}{2} + \frac{1}{2} \), \( f(u_i) = 2n + \frac{n+1}{2} + \frac{1}{2} + 2 \) for \( 1 \leq i \leq n \) and \( i \equiv 0 \pmod{2} \). Also \( f(w) = n + 1, f(w_j) = 2(n+1) - \left( \frac{j-1}{2} \right) \) for \( 1 \leq j \leq n+1 \) and \( j \equiv 1 \pmod{2} \) and \( f(w_j) = \frac{n+1}{2} - \frac{j}{2} + 1 \) for \( 1 \leq j \leq n+1, j \equiv 0 \pmod{2} \). Clearly the graph \( nP_2 \cup K_{1,n+1} \) has \( 3n+2 \) vertices and get labels \( 1, 2, \ldots, 3n+2 \).

Hence the graph \( nP_2 \cup K_{1,n+1} \) is a super edge trimagic total.

14) Example: The disconnected graph \( 5P_2 \cup K_{1,6} \) given in fig.4 is a super edge trimagic total.

![Graph 5P2∪K1,6](http://www.ijmttjournal.org)

Fig. 4 Graph 5P2∪K1,6 with \( \lambda_1 = 42, \lambda_2 = 41 \) and \( \lambda_3 = 35 \).
15) Theorem: The graph $nP_2 \cup K_{1, n+1}$ has an edge trimagic total labeling when $n$ is even.

Proof: Let $V = \{u_i, v_i / 1 \leq i \leq n\} \cup \{w, w_j / 1 \leq j \leq n + 1\}$ be the vertex set and $E = \{v_iu_i / 1 \leq i \leq n\} \cup \{ww_j / 1 \leq j \leq n + 1\}$ be the edge set of the disconnected graph $nP_2 \cup K_{1, n+1}$. The graph $nP_2 \cup K_{1, n+1}$ has $3n+2$ vertices and $2n+1$ edges.

Define a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, 5n + 3\}$ such that

- $f(v_i) = n + \frac{n+1}{2} + 1$, $f(u_i) = 2n - \frac{n-1}{2} + 1$, $f(v_i) = n + \frac{n+1}{2} + 1$, $f(u_i) = 2n + \frac{n-1}{2} + 2$, $1 \leq i \leq n$, $i \equiv 1 \pmod{2}$,
- $f(v_i) = 0$, $f(u_i) = 2n + \frac{n-1}{2} + 2$, $1 \leq i \leq n$, $i \equiv 0 \pmod{2}$,
- $f(v_iu_i) = 4n + 3 - i$, $1 \leq i \leq n$;
- $f(w) = 2(n+1) + \frac{n}{2}$, $f(ww_j) = 4n + \frac{n+1}{2} + 3$, $j = 1, 2, \ldots, n+1$, $j \equiv 1 \pmod{2}$,
- $f(w) = -1$, $f(ww_j) = 4n + \frac{n+1}{2} + 3$, $j = 1, 2, \ldots, n+1$, $j \equiv 0 \pmod{2}$.

Now we have to prove the graph $nP_2 \cup K_{1, n+1}$ has three distinct trimagic constants $\lambda_1, \lambda_2$ and $\lambda_3$.

Consider the edges $v_iu_i$:
For $1 \leq i \leq n$ and $i \equiv 1 \pmod{2}$;

$$f(v_i) + f(v_iu_i) + f(u_i) = n + 1 + 4n + 3 - i + 2n + \frac{n+1}{2} + 1 = \frac{12n+12}{2} = \lambda_1 \text{(say)}.$$  

For $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$;

$$f(v_i) + f(v_iu_i) + f(u_i) = 0 + 1 + 4n + 3 - i + 2n + \frac{n+1}{2} + 2 = 7n+6 = \lambda_2 \text{(say)}.$$ 

Consider the edges $ww_j$:
For $1 \leq j \leq n + 1$ and $j \equiv 1 \pmod{2}$;

$$f(w) + f(ww_j) + f(w_j) = 2(n+1) + 4n + \frac{n+1}{2} + 3 + n + \frac{n+1}{2} + 2 = 7n+6 = \lambda_2 \text{(say)}.$$ 

Hence for each edge $uv \in E$, $f(u) + f(uv) + f(v)$ yields any one of the trimagic constants $\lambda_1, \lambda_2, \lambda_3$.

Thus the graph $nP_2 \cup K_{1, n+1}$ admits an edge trimagic total labeling when $n$ is even.

16) Theorem: The graph $nP_2 \cup K_{1, n+1}$ has a super edge trimagic total labeling for even $n$.

Proof: We have proved that the graph $nP_2 \cup K_{1, n+1}$ has an edge trimagic total labeling when $n$ is even. The labeling given in the proof of the theorem 15, the vertices get labels $f(v_i) = n + \frac{n+1}{2} + 1$, $f(u_i) = 2n - \frac{n-1}{2} + 1$, $f(v_i) = n + \frac{n+1}{2} + 1$, $f(u_i) = 2n + \frac{n-1}{2} + 2$, $1 \leq i \leq n$, $i \equiv 1 \pmod{2}$,

$$f(v_iu_i) = 4n + 3 - i$$ for $1 \leq i \leq n$ and $i \equiv 0 \pmod{2}$,

$$f(w) = 2(n+1) + \frac{n}{2}$$ for $j = 1 \to n+1$ and $j \equiv 1 \pmod{2}$,

$$f(w_j) = 4n + \frac{n+1}{2} + 3$$ for $j = 1 \to n+1$, $j \equiv 0 \pmod{2}$ and $f(w) = \frac{n}{2} + 1$.

Clearly the graph $nP_2 \cup K_{1, n+1}$ has $3n+2$ vertices and get labels $1, 2, \ldots, 3n+2$. Hence the graph $nP_2 \cup K_{1, n+1}$ admits a super edge trimagic total labeling when $n$ is even.

17) Corollary: The disconnected graph $nP_2 \cup K_{1, n+1}$ admits a super edge trimagic labeling for all $n$.

18) Example: The graph $6P_2 \cup K_{1, 7}$ given in fig. 5 is a super edge trimagic total.
19) **Theorem:** For $n \geq 3$ and $t \geq 1$, the $t$ copies the sun graph $tS_n$ admits an edge trimagic total labeling.

**Proof:** Let $tS_n$ be the $t$ copies of the sun graph $S_n$ with the vertex set $V = \{v_i^j, u_i^j : 1 \leq i \leq n; 1 \leq j \leq t\}$ and the edge set $E = \{v_i^j v_{i+1}^j, v_i^j u_j^1 / 1 \leq i \leq n-1; 1 \leq j \leq t\} \cup \{v_i^j v_{i+1}^j / 1 \leq j \leq t\}$.

Define a bijection $f : V(tS_n) \cup E(tS_n) \rightarrow \{1, 2, \ldots, 4nt\}$ such that

$\begin{align*}
f(v_i^j v_{i+1}^j) &= n(j-1)+i, \\
f(v_i^j u_j^1) &= nt+n(j-1)+i, \\
f(v_i^j v_{i+1}^j) &= 4nt-2n(j-1)-2(i-1)-1, \\
1 \leq i \leq n-1, 1 \leq j \leq t.
\end{align*}$

Now we have to prove that the graph $tS_n$ has three distinct trimagic constants $\lambda_1, \lambda_2$ and $\lambda_3$.

For the edges $v_i^j v_{i+1}^j$, $1 \leq j \leq t$;

$\begin{align*}
f(v_i^j v_{i+1}^j) + f(v_i^j u_j^1) + f(v_i^j v_{i+1}^j) &= n(j-1)+1+4nt-2n(j-1)-2(n-1)+n(j-1)+n \\
&= 4nt-n+3 = \lambda_1 \text{(say)}.
\end{align*}$

For the edges $v_i^j v_{i+1}^j$, $1 \leq i \leq n-1, 1 \leq j \leq t$;

$\begin{align*}
f(v_i^j v_{i+1}^j) + f(v_i^j v_{i+1}^j) + f(v_i^j u_j^1) &= i+4nt-2n(j-1)-2(i-1)+n(j-1)+i+1 \\
&= 4nt+3 = \lambda_2 \text{(say)}.
\end{align*}$

Also for the edges $v_i^j u_j^1$, $1 \leq i \leq n-1, 1 \leq j \leq t$;

$\begin{align*}
f(v_i^j v_{i+1}^j) + f(v_i^j u_j^1) + f(u_j^1) &= i+4nt-2n(j-1)-2(i-1)+nt+n(j-1)+i \\
&= 5nt+1 = \lambda_3 \text{(say)}.
\end{align*}$

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the trimagic constants $\lambda_1 = 4nt-n+3, \lambda_2 = 4nt+3$ and $\lambda_3 = 5nt+1$.

Thus the $t$ copies of a sun graph, $tS_n$ admits an edge trimagic total labeling.
20) Example: A super edge trimagic total labeling of 3 copies of sun $S_5$ is given in fig. 6.

![Graph 3S_5](image)

Fig. 6 Graph $3S_5$ with $\lambda_1 = 58$, $\lambda_2 = 63$ and $\lambda_3 = 76$.

21) Theorem: For $n \geq 3$ and $t \geq 1$, the $t$ copies the sun graph $tS_n$ is a super edge trimagic total.

Proof: We have proved that the graph $tS_n$ admits an edge trimagic total labeling. The labeling given in the proof of the Theorem 19, the vertices get labels $f(u_i) = n(j-1)+i$, $f(v_i) = nt+n(j-1)+i$, $1 \leq i \leq n-1$; $1 \leq j \leq t$. Clearly the graph $tS_n$ has $2nt$ vertices and get labels $1, 2, \ldots, 2nt$.

Hence the graph $tS_n$ is a super edge trimagic total graph

22) Theorem: The graph $nC_4$ admits an edge trimagic total labeling.

Proof: Let $V = \{u_i, v_i, w_i, x_i / 1 \leq i \leq n\}$ be the vertex set and $E = \{u_iv_i, v_iw_i, w_ix_i, x_iu_i / 1 \leq i \leq n\}$ be the edge set of the disconnected graph $nC_4$. The graph $nC_4$ has $4n$ vertices and $4n$ edges.

Define a bijection $f: V \cup E \to \{1, 2, \ldots, 8n\}$ such that $f(u_i) = i$, $f(v_i) = n+i$, $f(w_i) = 3n+i$, $f(x_i) = 2n+i$ for all $1 \leq i \leq n$; $f(u_iv_i) = 8n-2i+2$, $f(v_iw_i) = 6n-2i+1$, $f(w_ix_i) = 6n-2i+2$ and $f(x_iu_i) = 8n-2i+1$ for all $1 \leq i \leq n$.

Now we prove that the graph $nC_4$ admits an edge trimagic total labeling.

For the edges $u_iv_i$, $1 \leq i \leq n$;

$f(u_i)+f(u_iv_i)+f(v_i) = i+8n-2i+2+n+i = 9n+2 = \lambda_1$(say).

For the edges $v_iw_i$, $1 \leq i \leq n$;

$f(v_i)+f(v_iw_i)+f(w_i) = n+i+6n-2i+1+3n+i = 10n+1 = \lambda_2$(say).

For the edges $w_ix_i$, $1 \leq i \leq n$;

$f(w_i)+f(w_ix_i)+f(x_i) = 3n+i+6n-2i+2+2n+i = 11n+2 = \lambda_3$(say).
For the edges u_i, 1 ≤ i ≤ n;

f(x_i) + f(x_i u_i) + f(u_i) = 2n + i + 8n - 2i + 1 + i = 10n + 1 = λ_2.

Hence for each edge uv ∈ E, f(u) + f(uv) + f(v) yields any one of the trimagic constant λ_1 = 9n + 2, λ_2 = 10n + 1 and λ_3 = 11n + 2. Hence the disconnected graph nC_4 admits an edge trimagic total labeling.

23) Theorem: The graph nC_4 admits a super edge trimagic total labeling.

Proof: We have proved that the graph nC_4 has an edge trimagic total labeling. The labeling given in the proof of the theorem 22, the vertices get labels f(u_i) = i, f(v_i) = 3n + i, f(w_i) = 2n + i for all 1 ≤ i ≤ n. Clearly the graph has 4n vertices and get labels 1, 2, ..., 4n. Hence the graph nC_4 admits a super edge trimagic total labeling.

24) Example: A super edge trimagic total labeling of 4C_4 is given in fig. 7

![Graph 4C_4 with λ_1 = 38, λ_2 = 41 and λ_3 = 46.](image)

25) Theorem: The graph nC_6 admits an edge trimagic total labeling.

Proof: Let V = {u_1, v_1, w_1, x_1, y_1, z_1 / 1 ≤ i ≤ n} be the vertex set and E = {u_i v_i, v_i w_i, w_i x_i, x_i y_i, y_i z_i, z_i u_i / 1 ≤ i ≤ n} be the edge set of the disconnected graph nC_6. The graph nC_6 has 6n vertices and 6n edges.

Define a bijection f: V ∪ E → {1, 2, ..., 12n} such that f(u_i) = i, f(v_i) = 3n + i, f(w_i) = n + i, f(x_i) = 4n + i, f(y_i) = 2n + i, f(z_i) = 5n + i for all 1 ≤ i ≤ n; f(u_i v_i) = 12n - 2i + 2, f(v_i w_i) = 12n - 2i + 1, f(w_i x_i) = 8n - 2i + 1, f(x_i y_i) = 10n - 2i + 1, f(y_i z_i) = 8n - 2i + 2 and f(z_i u_i) = 10n - 2i + 2 for all 1 ≤ i ≤ n.

Now we prove that the graph nC_6 admits an edge trimagic total labeling.

For the edges u_i v_i, 1 ≤ i ≤ n;

f(u_i) + f(u_i v_i) + f(v_i) = i + 12n - 2i + 2 + 3n + i = 15n + 2 = λ_1 (say).

For the edges v_i w_i, 1 ≤ i ≤ n;

f(v_i) + f(v_i w_i) + f(w_i) = 3n + i + 12n - 2i + 1 + n + i = 16n + 1 = λ_2 (say).

For the edges w_i x_i, 1 ≤ i ≤ n;

f(w_i) + f(w_i x_i) + f(x_i) = n + i + 8n - 2i + 1 + 4n + i = 13n + 1 = λ_3 (say).

For the edges x_i y_i, 1 ≤ i ≤ n;

f(x_i) + f(x_i y_i) + f(y_i) = 4n + i + 10n - 2i + 1 + 2n + i = 16n + 1 = λ_2.

For the edges y_i z_i, 1 ≤ i ≤ n;

f(y_i) + f(y_i z_i) + f(z_i) = 2n + i + 8n - 2i + 2 + 5n + i = 15n + 2 = λ_3.
For the edges $z_iu_i$, $1 \leq i \leq n$;

$$f(z_i)+f(z_iu_i)+f(u_i) = 5n+i+10n-2i+2+i = 15n+2 = \lambda_1.$$  

Hence for each edge $uv \in E$, $f(u)+f(uv)+f(v)$ yields any one of the trimagic constant $\lambda_1 = 15n+2$, $\lambda_2 = 16n+1$ and $\lambda_3 = 13n+1$. Hence the disconnected graph $nC_6$ admits an edge trimagic total labeling.

26) **Theorem:** The graph $nC_6$ admits a super edge trimagic total labeling.

**Proof:** We have proved that the graph $nC_6$ has an edge trimagic total labeling. The labeling given in the proof of the theorem 25, the vertices get labels $f(u_i) = i$, $f(v_i) = 3n+i$, $f(w_i) = n+i$, $f(x_i) = 4n+i$, $f(y_i) = 2n+i$, $f(z_i) = 5n+i$ for all $1 \leq i \leq n$. Clearly the graph $nC_6$ has 6n vertices and get labels 1, 2, …, 6n. Hence the graph $nC_6$ admits a super edge trimagic total labeling.

27) **Example:** A super edge trimagic total labeling of $3C_6$ is given in fig. 8.

![Fig. 8 Graph 3C_6 with \( \lambda_1 = 47 \), \( \lambda_2 = 49 \) and \( \lambda_3 = 40 \).](image)

### III. Conclusions

In this paper, we proved that the disconnected graphs $K_{1,p} \cup K_{1,q} \cup K_{1,r}$ ($p \leq q \leq r$), $nP_2 \cup K_{1,n+1}$, $t$ copies of the sun graph $S_{3n}$, $nC_4$ and $nC_6$ are super edge trimagic total and $nP_3$ is superior edge trimagic total. There may be many interesting trimagic graphs can be constructed also in future.

### References


