CHAPTER II

STATISTICALLY CONVERGENT SEQUENCES OF FUZZY REAL NUMBERS

2.1 INTRODUCTION

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [26] and Schoenberg [80] independently. It was also found in Zygmund [110]. Later on it was studied from sequence space point of view and linked with summability by Fridy and Orhan [33], Šalát [74], Maddox [52], Tripathy ([87], [95]), Rath and Tripathy ([68], [69]) and many others.

The idea depends on certain density of the subsets of the set $N$ of natural numbers. A subset $E$ of $N$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists, where $\chi_E$ is the characteristic function of $E$. Clearly all finite subsets of $N$ have zero natural density and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$.

A classical sequence $(x_k)$ is said to be statistically convergent to $L$ if, for every $\varepsilon > 0$, $\delta(\{ k \in N : |x_k - L| \geq \varepsilon \}) = 0$. We write $x_k \xrightarrow{\text{stat}} L$ or stat-$\lim x_k = L$.

This chapter has been accepted for publication in Journal of Fuzzy Mathematics, (please see Tripathy and Das [96] in the bibliography).
Let \((x_k)\) and \((y_k)\) be two sequences, then we say that \(x_k = y_k\) for almost all \(k\) (in short \(a.a.k\)) if \(\mathcal{S}(\{ k \in N : x_k \neq y_k \}) = 0\).

Let \(D\) denote the set of all closed and bounded intervals \(X = [a_1, a_2]\) on \(R\), the real line. For \(X, Y \in D\) we define

\[
d(X, Y) = \max (|a_1 - b_1|, |a_2 - b_2|),
\]

where \(X = [a_1, a_2]\) and \(Y = [b_1, b_2]\). It is known that \((D, d)\) is a complete metric space.

A fuzzy real number \(X\) is a fuzzy set on \(R\), i.e. a mapping \(X : R \to \mathbb{I} (= [0,1])\) associating each real number \(t\) with its grade of membership \(X(t)\).

The \(\alpha\)-cut or \(\alpha\)-level set \([X]^\alpha\) of the fuzzy real number \(X\), for \(0 < \alpha \leq 1\), defined by \([X]^\alpha = \{ t \in R : X(t) \geq \alpha \}\); for \(\alpha = 0\), it is the closure of the strong 0-cut (that is, closure of the set \(\{ t \in R : X(t) > 0 \}\)).

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by \(R(I)\).

Let \(\overline{d} : R(I) \times R(I) \to R\) be defined by

\[
\overline{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).
\]

Then \(\overline{d}\) defines a metric on \(R(I)\). It is well known that \(R(I)\) is complete with respect to \(\overline{d}\).

The additive identity and multiplicative identity in \(R(I)\) are denoted by \(\overline{0}\) and \(\overline{1}\) respectively.
2.2 DEFINITIONS AND PRELIMINARIES

For the sake of completeness, we shall procure some definitions of Chapter I.

After the introduction of \( R(I) \), different classes of sequences were introduced and studied by Nanda [59], Tripathy and Nanda [100], Savaș [79], Nuray and Savaș [65], Das; Das and Choudhury [16], Das and Choudhury [15], Subrahmanyam [84] and many others.

A fuzzy real-valued sequence \((X_k)\) is said to be level convergent to the fuzzy real number \( X \) if, for each \( \alpha \in [0,1] \),

\[
\lim_{k \to \infty} a_k^\alpha = a^\alpha \quad \text{and} \quad \lim_{k \to \infty} b_k^\alpha = b^\alpha,
\]

where \([X_k]^\alpha = [a_k^\alpha, b_k^\alpha]\), for all \( k \in \mathbb{N} \) and \([X]^\alpha = [a^\alpha, b^\alpha]\).

If the convergence is uniform in \( \alpha \), then we say that \((X_k)\) converges uniformly to \( X \).

A sequence \((X_k)\) of fuzzy real numbers is said to be convergent (uniformly) to the fuzzy real number \( X_0 \) if, for every \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( d(X_k, X_0) < \varepsilon \), for all \( k \geq n_0 \).

A fuzzy real number sequence \((X_k)\) is said to be bounded if there exists a \( \mu \in \mathbb{R}^+(I) \), the set of all non-negative fuzzy real numbers, such that \( |X_k| \leq \mu \), for all \( k \in \mathbb{N} \).

Throughout the thesis \( E^F \) denote a class of sequences of fuzzy real numbers.

A class of sequences \( E^F \) is said to be normal (or solid) if \((Y_k)\in E^F\), whenever \( |Y_k| \leq |X_k| \), for all \( k \in \mathbb{N} \) and \((X_k)\in E^F\).
Let $K = \{ k_1 < k_2 < k_3 \ldots \} \subseteq N$ and $E^F$ be a class of sequences. A $K$-step set of $E^F$ is a set of sequences $\lambda^F_k = \{(X_{k_i}) \in w^F : (X_n) \in E^F\}$.

A canonical pre-image of a sequence $(X_{k_i}) \in \lambda^F_k$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step set $\lambda^F_k$ is a set of canonical pre-images of all elements in $\lambda^F_k$, i.e., $Y$ is in canonical pre-image $\lambda^F_k$ if and only if $Y$ is canonical pre-image of some $X \in \lambda^F_k$.

A class of sequences $E^F$ is said to be monotone if $E^F$ contains the canonical pre-images of all its step sets.

From the above definitions we have the following remark.

**Remark 2.2.1** A sequence space $E^F$ is solid $\Rightarrow$ $E^F$ is monotone.

A class of sequences $E^F$ is said to be symmetric if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where $\pi$ is a permutation of $N$.

A class of sequences $E^F$ is said to be sequence algebra if $(X_k \otimes Y_k) \in E^F$, whenever $(X_k), (Y_k) \in E^F$.

A class of sequences $E^F$ is said to be convergence free if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ and $X_k = 0$ implies $Y_k = 0$.

Nuray and Savaş [65] defined the notion of statistical convergence for sequences of fuzzy real numbers and studied some properties.
A fuzzy real number sequence \((X_k)\) is said to be statistically convergent to the fuzzy real number \(X_0\) if, for every \(\varepsilon > 0\),
\[
\delta \left( \left\{ k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon \right\} \right) = 0.
\]

Throughout the chapter, \(w^F, \ell^F, c^F, c_0^F, c^F, m^F, c_0 m^F\) and \(m_0^F\) denote the class of all, bounded, convergent, null, statistically convergent, bounded statistically convergent, statistically null and bounded statistically null fuzzy real number sequences respectively.

2.3 MAIN RESULTS

**THEOREM 2.3.1** \(m^F = c^F \cap \ell^F\) and \(m_0^F = c_0^F \cap \ell^F\) are closed subspaces of the complete metric space \(\ell^F\) with the metric \(\rho\) defined by
\[
\rho(X, Y) = \sup_k \bar{d}(X_k, Y_k),
\]
where \(X = (X_k)\) and \(Y = (Y_k)\) are in \(m^F\) or \(m_0^F\).

**THEOREM 2.3.2** (i) The classes of sequences \(c_0^F\) and \(m_0^F\) are solid and as such are monotone.

(ii) The classes of sequences \(c^F\) and \(m^F\) are neither monotone nor solid.

**THEOREM 2.3.3** The classes of sequences \(c^F, m^F, c_0^F\) and \(m_0^F\) are not symmetric.

**THEOREM 2.3.4** The classes of sequences \(c^F, m^F, c_0^F\) and \(m_0^F\) are sequence algebra.
THEOREM 2.3.5 The classes of sequences $c^F$, $m^F$, $c_0^F$ and $m_0^F$ are not convergence free.

THEOREM 2.3.6 The classes of sequences $m_0^F$ and $m^F$ are nowhere dense subsets in $\ell^F$.

THEOREM 2.3.7 The following are equivalent:

(a) $(X_k) \in (c)^F$ and $\text{stat-lim} X_k = L, L \in R(I)$.

(b) there exists $(Y_k) \in c^F$ such that $X_k = Y_k$, for a. a. $k$.

(c) there exists a subset $K = \{k_n, i \in N\}$ of $N$ such that $\delta(K) = 1$ and

$$\lim_{i \to \infty} X_{k_i} = L.$$ 

2.4 PROOF OF THE THEOREMS OF SECTION 2.3

PROOF OF THEOREM 2.3.1 We prove the result for the case of $m^F$. The other one can be established by similar technique.

Let $(X^{(n)})$ be a Cauchy sequence in $m^F$. Then $(X^{(n)})$ be a Cauchy sequence in $\ell^F$. Since $\ell^F$ is a complete metric space (please refer to Theorem 1 of Nanda [59]), so $X^{(n)} \to X$ in $\ell^F$.

We shall show that

$$X \in m^F.$$ 

Since $X^{(n)} = (X_k^{(n)}) = (X_1^{(n)}, X_2^{(n)}, X_3^{(n)}, \ldots) \in m^F$, so for each $n \in N$ there exists $A_n \in R(I)$ such that

$$\text{stat-lim} X_k^{(n)} = A_n.$$
We prove the followings:

(i) \( \lim_{n \to \infty} A_n = A \).

(ii) stat-lim \( X^* = A \).

(i). Since \( (X^{(n)}) \) is a convergent sequence, so for a given \( \varepsilon > 0 \), there exists such a \( n_0 \in \mathbb{N} \) that for each \( m, n > n_0 \) we have

\[
\rho(X^{(m)}, X^{(n)}) = \sup_k \bar{d}(X^{(m)}_k, X^{(n)}_k) < \frac{\varepsilon}{3}.
\]

\[
\Rightarrow \bar{d}(X^{(m)}_k, X^{(n)}_k) < \frac{\varepsilon}{3}.
\] 

\[
(2.4.1)
\]

Again, since \( X^{(n)} = (X^{(n)}_k) \in \mathcal{M}^E \), so for a given \( \varepsilon > 0 \), we have

\[
\bar{d}(X^{(m)}_k, A_n) < \frac{\varepsilon}{3}, \text{ for a. a. } k.
\]

\[
(2.4.2)
\]

and

\[
\bar{d}(X^{(n)}_k, A_n) < \frac{\varepsilon}{3}, \text{ for a. a. } k.
\]

\[
(2.4.3)
\]

Now for each \( m, n > n_0 \in \mathbb{N} \) and from the inequalities (2.4.1), (2.4.2) and (2.4.3), we get

\[
\bar{d}(A_m, A_n) \leq \bar{d}(A_m, X^{(m)}_k) + \bar{d}(X^{(m)}_k, X^{(n)}_k) + \bar{d}(X^{(n)}_k, A_n), \text{ for a. a. } k.
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Thus \( (A_n) \) is a Cauchy sequence in \( R(I) \). Since \( R(I) \) is complete, so there exists such a number \( A \in R(I) \) such that

\[
\lim_{n \to \infty} A_n = A.
\]

(ii). We have

\[
X^{(n)} \to X.
\]

For a given \( \lambda > 0 \), there exists such a \( q \in \mathbb{N} \) that

\[
\sup_{k} \bar{d}(X^{(q)}_k, X_k) < \frac{\lambda}{3}.
\]
The number \( q \) can be chosen in such a way that together with (2.4.4), we get
\[
\overline{d}(A_q, A) < \frac{\lambda}{3}.
\]

Since, \( \text{stat-lim} X^{(q)}_k = A_q \).

For a given \( \lambda > 0, \)
\[
\overline{d}(X^{(q)}_k, A_q) < \frac{\lambda}{3}, \quad \text{for a. a. } k.
\]

Now, \( \overline{d}(X_k, A) \leq \overline{d}(X_k, X^{(q)}_k) + \overline{d}(X^{(q)}_k, A_q) + \overline{d}(A_q, A), \) for a. a. \( k. \)
\[
< \frac{\lambda}{3} + \frac{\lambda}{3} + \frac{\lambda}{3} = \lambda.
\]

Hence \( \text{stat-lim} X_k = A \). This proves the result.

**PROOF OF THEOREM 2.3.2 (i)** Consider two sequences \((X_k)\) and \((Y_k)\) such that
\[
|Y_k| \leq |X_k|, \quad \text{for all } k \in \mathbb{N}.
\]

Then for a given \( \varepsilon > 0, \) we have
\[
\left\{ k \in \mathbb{N} : \overline{d}(X_k, 0) \geq \varepsilon \right\} \supseteq \left\{ k \in \mathbb{N} : \overline{d}(Y_k, 0) \geq \varepsilon \right\}.
\]

Since \((X_k) \subseteq m^F_0 \subseteq c^F_0, \) so \( \Delta \left( \left\{ k \in \mathbb{N} : \overline{d}(X_k, 0) \geq \varepsilon \right\} \right) = 0 \)
and hence \( \Delta \left( \left\{ k \in \mathbb{N} : \overline{d}(Y_k, 0) \geq \varepsilon \right\} \right) = 0. \)

Thus \((Y_k) \subseteq m^F_0 \subseteq c^F_0 \) and the classes \( c^F_0 \) and \( m^F_0 \) are solid.

The classes of sequences \( c^F_0 \) and \( m^F_0 \) are monotone follows from remark 2.2.1.
(ii) The result follows from the following example.

**Example 2.4.1** Let us consider the sequence \((X_k) \in m^F\) (shown in Fig. 2.4.1), defined as follows:

For \(k = n^2, n \in \mathbb{N}\),

\[
X_k(t) = \begin{cases} 
  t-2, & \text{for } 2 \leq t \leq 3, \\
  4-t, & \text{for } 3 < t \leq 4, \\
  0, & \text{otherwise}. 
\end{cases}
\]

Otherwise,

\[
X_k(t) = \begin{cases} 
  1-k(t-2^{-1}), & \text{for } 2^{-1} \leq t \leq 2^{-1} + k^{-1}, \\
  0, & \text{otherwise}. 
\end{cases}
\]

Now for \(\alpha\) we get,

\[
[X_1]^\alpha = \begin{cases} 
  [2+\alpha, 4-\alpha], & \text{for } k = n^2, n \in \mathbb{N}, \\
  [2^{-1}, 2^{-1} + k^{-1}(1-\alpha)], & \text{otherwise}. 
\end{cases}
\]

Thus \((X_k) \in m^F \subseteq c^F\)
Let \( J = \{ k \in \mathbb{N} : k = 2i, \ i \in \mathbb{N} \} \) be a subset of \( \mathbb{N} \) and let \( (m^F)_J \) be the canonical pre-image of a \( J \)-step set \( (m^F)_J \) of \( m^F \), defined as follows:

\[
(Y_k) \in (m^F)_J \text{ is the canonical pre-image of } (X_k) \in m^F \implies Y_k = \begin{cases} X_k, & \text{for } k \in J, \\ 0, & \text{for } k \notin J. \end{cases}
\]

Now for \( \alpha \) we have,

\[
[Y_k] = \begin{cases} [2 + \alpha, 4 - \alpha], & \text{for } k \in J \text{ and } k = n^2, n \in \mathbb{N}, \\ [2^{-1}, 2^{-1} + k^{-1}(1 - \alpha)], & \text{for } k \in J \text{ and } k \neq n^2, \text{ for any } n \in \mathbb{N}, \\ [0, 0], & \text{for } k \notin J. \end{cases}
\]

Thus \((Y_k) \notin \overline{c}^F (\ni m^F)\). Hence \( \overline{c}^F \) \( \text{ and } m^F \text{ are not monotone.} \)

The classes \( \overline{c}^F \text{ and } m^F \) are not solid follows from the remark 2.2.1.

**PROOF OF THEOREM 2.3.3** The result follows from the following example.

**Example 2.4.2** Consider the sequence \((X_k)\) defined in example 2.4.1.

Here we have,

\((X_k) \in Z, \text{ for } Z = c^F, m^F, c_0^F \text{ and } m_0^F.\)

Let \((Y_k)\) be a rearrangement of the sequence \((X_k)\), defined as follows:

\((Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, X_7, ... )\)

Then for \( \alpha \) we get,

\[
[Y_k] = \begin{cases} [2 + \alpha, 4 - \alpha], & \text{for } k \text{ odd,} \\ [2^{-1}, 2^{-1} + k^{-1}(1 - \alpha)], & \text{for } k \text{ even.} \end{cases}
\]

Thus \((Y_k) \notin Z, \text{ for } Z = c^F, m^F, c_0^F \text{ and } m_0^F.\)

Therefore the classes of sequences \( c^F, m^F, c_0^F \text{ and } m_0^F \) are not symmetric.
PROOF OF THEOREM 2.3.4 We prove it for the class $m_0^F$ and the other classes can be proved by similar method.

Let $0<\varepsilon<1$ be given. Suppose $(X_k), (Y_k) \in m_0^F$.

Then we have,

$$\{k \in N : \tilde{d}(X_k \otimes Y_k, 0) < \varepsilon\} \supseteq \{k \in N : \tilde{d}(X_k, 0) < \varepsilon\} \cap \{k \in N : \tilde{d}(Y_k, 0) < \varepsilon\}.$$ 

Since $\delta(\{k \in N : \tilde{d}(X_k, 0) < \varepsilon\}) = 1$ and $\delta(\{k \in N : \tilde{d}(Y_k, 0) < \varepsilon\}) = 1$.

So, $\delta(\{k \in N : \tilde{d}(X_k \otimes Y_k, 0) < \varepsilon\}) = 1$.

Thus $(X_k \otimes Y_k) \in m_0^F$. Hence the class $m_0^F$ is a sequence algebra.

PROOF OF THEOREM 2.3.5 The result follows from the following example.

**Example 2.4.3** Consider the sequence $(X_k) \in Z$, for $Z = c^F, m^F, c_0^F$ and $m_0^F$, defined as follows:

For $k = n^2, n \in N$, $X_k = \bar{0}$.

Otherwise, $X_k(t) = \begin{cases} 1 + 3^{-1}kt, & \text{for } -3k^{-1} \leq t \leq 0, \\ 1 - 3^{-1}kt, & \text{for } 0 < t \leq 3k^{-1}, \\ 0, & \text{otherwise}. \end{cases}$

(Shown in Fig. 2.4.2.)
Then for \( \alpha \) we have,

\[
[X_k]^\alpha = \begin{cases} 
[0, 0], & \text{for } k = n^2, n \in N, \\
[3(\alpha - 1)k^{-1}, 3(1-\alpha)k^{-1}], & \text{otherwise}.
\end{cases}
\]

Thus \((X_k) \in Z\), for \( Z = c^F, m^F, c_0^F \) and \( m_0^F \).

Let the sequence \((Y_k)\) be defined as follows:

For \( k = n^2, n \in N \), \( Y_k = 0 \).

Otherwise,

\[
Y_k(t) = \begin{cases} 
1, & \text{for } k \leq t \leq k + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Then for \( \alpha \) we have,

\[
[Y_k]^\alpha = \begin{cases} 
[0, 0], & \text{for } k = n^2, n \in N, \\
[k, k + 1], & \text{otherwise}.
\end{cases}
\]

Thus \((Y_k) \notin Z\), for \( Z = c^F, m^F, c_0^F \) and \( m_0^F \).

Hence the classes \( c^F, m^F, c_0^F \) and \( m_0^F \) are not convergence free.
PROOF OF THEOREM 2.3.6 We have $m_0^F$ and $m^F$ are closed subsets of the complete metric space $\ell^F_w$. Also $m_0^F$ and $m^F$ are proper subspaces of $\ell^F_w$ which follows from the following example.

Example 2.4.4 Consider the sequence $(X_k)$ as follows:

For $k$ odd, \[ X_k(t) = \begin{cases} 1, & \text{for } -(1+k^{-1}) \leq t \leq 0, \\ 0, & \text{otherwise} \end{cases} \]

and for $k$ even, \[ X_k(t) = \begin{cases} 1, & \text{for } 0 < t \leq 1+k^{-1}, \\ 0, & \text{otherwise.} \end{cases} \]

Then for $\alpha$ we have,

\[ [X_k]^\alpha = \begin{cases} [-(-1+k^{-1}), 0], & \text{for } k \text{ odd}, \\ [0, 1+k^{-1}], & \text{for } k \text{ even}. \end{cases} \]

Thus $(X_k) \in \ell^F_w$, but $(X_k) \not\in m^F (\Rightarrow (m_0)^F)$. Hence the result.

PROOF OF THEOREM 2.3.7

$(a) \Rightarrow (b)$.

Let $(X_k) \in \ell^F_w$ and let stat-lim $X_k = L$. Suppose $(n_j)$ be an increasing sequence of natural numbers such that for all $n > n_j$,

\[ \frac{1}{n} \left| \left\{ k \leq n : \overline{d} (X_k, L) \geq \frac{1}{j} \right\} \right| < \frac{1}{j}, \]

where $|A|$ denotes the cardinality of the set $A$.

Let us defined the sequence $(Y_k)$ of the elements from $R(I)$ as $Y_k = X_k$ for all $k \leq n_1$; for $n_j < k \leq n_{j+1}$, $j \in N$, let $Y_k = X_k$, if $\overline{d} (X_k, L) < \frac{1}{j}$, otherwise $Y_k = L$. 

It follows from the above construction that \((Y_k) \in c^F\).

Further, for a given \(\epsilon > 0\), it follows from the inclusion

\[
\{ k \leq n : X_k \neq Y_k \} \subseteq \{ k \leq n : \overline{d}(X_k, L) \geq \epsilon \}
\]

that \(X_k = Y_k\) for \(a.a. k\).

\((b) \Rightarrow (c)\).

For \((X_k) \in \overset{\sim}{c}^F\), let there exists \((Y_k) \in c^F\) such that \(X_k = Y_k\) for \(a.a. k\).

Let \(M = \{ k \in N : X_k \neq Y_k \}\), then \(\delta(M) = 0\)

and \(K = N - M = \{ k_i : k_i < k_{i+1}, i \in N \}\), then \(\delta(K) = 1\)

Hence \(\lim\limits_{i \to \infty} X_{k_i} = L = \lim\limits_{i \to \infty} Y_k = \text{stat-lim} X_k\)

\((c) \Rightarrow (a)\).

Let there exists \(K = \{ k_i : k_i < k_{i+1}, i \in N \}\subset N\) be such that

\[\lim_{i \to \infty} X_{k_i} = L.\]

Then for any \(\epsilon > 0\),

\[
\{ k \in N : \overline{d}(X_k, L) \geq \epsilon \} \subseteq K^c \cup \{ k \in K : \overline{d}(X_k, L) \geq \epsilon \}.
\]

Thus \(\delta\left( \{ k \in N : \overline{d}(X_k, L) \geq \epsilon \} \right) = 0\), since \(\delta(K^c) = 0\) and the second set on the right hand side is finite.

Hence \((X_k) \in \overset{\sim}{c}^F\). This completes the proof of the theorem.