CHAPTER 2

JACKKNIFING A MULTIVARIATE RATIO ESTIMATOR
UNDER A SUPER-POPULATION MODEL

SUMMARY: This chapter deals with the jackknifing of a multivariate ratio estimator due to Srivastava (1967). The case of jackknifing a ratio estimator involving only a single auxiliary variable is discussed in Chapter 1 as well as in Nandi and Aich (1994). It is shown that the jackknifed estimator is quite good compared to its competitors, including the Olkin's multivariate ratio estimator (1958). The jackknifed estimator is then examined under a model which is an extension of Durbin's model (1959) to the multi-auxiliary situation. It is shown that any increase in variance due to jackknifing is of the order $1/n^2$, $n$ being the sample size. Finally, the conditions under which jackknifing reduces bias as well as variance are identified. A numerical illustration is also given by taking real-life data.
1. INTRODUCTION

In Chapter 1, we have dealt only with the case of a single auxiliary variable. A modified ratio estimator incorporating a single auxiliary character is jackknifed, and the resulting estimator is shown to have better precision under Durbin's model, in addition to having reduced bias.

Olkin (1958) has extended the classical ratio estimator involving one auxiliary variable to the case of a multi-auxiliary situation. He has considered a weighted average of several uni-auxiliary ratio estimators, the weights being determined optimally by minimizing the variance. Raj (1965), on the other hand, has considered a weighted average of several difference estimators. His estimator, though computationally simpler, is no more efficient than Olkin's estimator.

The classical ratio estimator has been extended in a number of ways. Srivastava (1967, 1971) has defined a class of estimators which contains both the ratio and product estimators. As is well-known, the ratio estimator is useful when the auxiliary variable is positively correlated with the study variable, otherwise a product estimator should be chosen. The multivariate ratio estimator of Olkin can be shown to be a member of Srivastava's class of estimators.
In a recent article, Rao (1991) has compared some commonly used estimators using auxiliary information for four natural populations and found no merit in considering modifications of the classical estimators, e.g., Srivastava's modified ratio estimator. Nandi and Aich (1994), however, have observed that if the modified estimators are jackknifed, then a gain in precision can be achieved in some cases, in addition to having reduced biases. They have considered the case of a single auxiliary variable only in their discussion.

The first idea about the jackknife technique, which has originated outside the field of sample survey, is due to Quenouille (1956) who has used the technique to reduce bias of an estimator in an infinite population context. For finite populations, the jackknife technique is first used by Durbin (1959). The literature of jackknifing in problems of sample survey has grown rapidly since then.

The present chapter is concerned with the problem of jackknifing the Srivastava's version of multivariate ratio estimator. The proposed jackknifed estimator is shown to be unbiased up to the order 1/n, n being the sample size. Also, this estimator is seen to be the minimum variance unbiased estimator (MVUE) in a class of estimators, in some sense. The expression of the minimum variance is obtained. The proposed estimator is then compared with its competitors by using real-life data and found to be better.
Finally, the proposed estimator is examined under Durbin's super-population model. This model is an extension to the multi-auxiliary situation of a model considered by Durbin (1959) for the uni-auxiliary case. In the multi-auxiliary case, it is assumed that each regression, taken individually, is linear passing through the origin while the residual variance is a constant. Also, the auxiliary variables jointly follow a version of a multivariate gamma distribution. The expressions of the bias and the variance of the proposed estimator, under the above model, are obtained. Also, the conditions for the superiority of the proposed jackknifed estimator compared to the unjackknifed version have been identified.

2. NOTATIONS AND PRELIMINARIES

Let \( y \) denote the character whose population mean \( \bar{Y} \) is to be estimated using information on \( p \)-auxiliary characters which may be denoted by \( x_1, x_2, \ldots, x_p \), respectively. Also, let the vector \( (Y_i, X_{i1}, X_{i2}, \ldots, X_{ip}) \) denote the values of the \( i \)-th unit of the population, \( i = 1, 2, \ldots, N \), which we assume to be positive. It is also assumed that the population means of the \( p \)-auxiliary characters, namely, \( \bar{X}_1, \bar{X}_2, \ldots, \bar{X}_p \), are known.

Let us now consider a random sample \( (y_i, x_{i1}, x_{i2}, \ldots, x_{ip}) \), \( i = 1, 2, \ldots, n \), of size \( n \) drawn from the above population without replacement. Also, let \( \bar{y}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p \) be the sample means of \( y \) and the \( p \)-auxiliary variables, respectively.
The following population characteristics are then defined:

\[
\begin{align*}
S_0 &= (N - 1)^{-1} \sum_{i=1}^{N} (Y_i - \overline{Y})^2, \\
S_j &= (N - 1)^{-1} \sum_{i=1}^{N} (x_{ji} - \overline{x_j})^2, \\
C_o &= S_0 / \overline{Y}^2, \\
C_j &= S_j / \overline{x_j}^2, \quad j = 1, 2, \ldots, p, \\
f &= (N - n)/N, \text{ the finite population correction (fpc)}.
\end{align*}
\]

Again, let \( \rho_{ij} \) denote the correlation coefficient between the characters \( x_j \) and \( x_j' \) (\( j \neq j' \)), and \( \rho_{oj} \) that between \( y \) and \( x_j \).

To obtain the large sample approximations for the mean and the variance of the proposed estimator, we shall use the Taylor's series method which is traditionally employed in the context of ratio and regression methods of estimation. Because of the complicated form of the terms of order \( 1/n^2 \) and their dubious values (see Olkin (1958)), only terms of order \( 1/n \), \( n \) being the sample size, will be considered under the following assumptions:

\[
\begin{align*}
Y_i &= \overline{Y} + \epsilon_i, \\
x_{ji} &= \overline{x_j} + \epsilon_{ji}, \\
( i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, p )
\end{align*}
\]

with

\[
\begin{align*}
E (\epsilon_i) &= 0, \quad E (\epsilon_{ji}) = 0 \\
\text{and } \quad V (\bar{\epsilon}) &= (f/n) S_0^2, \\
V (\bar{\epsilon}_j) &= (f/n) S_j^2, \\
\text{Cov} (\bar{\epsilon}, \bar{\epsilon}_j) &= (f/n) \rho_{oj} S_0 S_j.
\end{align*}
\]

where \( \bar{\epsilon} \) and \( \bar{\epsilon}_j \) are the sample means of the respective errors.
For the Taylor's series expansion, it will further be assumed that
\[ |(\bar{\epsilon} / \bar{Y})_j| < 1, \quad |(\bar{\epsilon}_j^2 / \bar{X}_j^2)| < 1, \text{ for each } j. \quad (2.2a) \]

Finally, the proposed estimator is examined under the following extension of Durbin's model (1959):

(i) Each regression, taken individually, is a straight line passing through the origin. In other words, we have
\[ E(y / x_j) = \beta_j x_j, \quad j = 1, 2, \ldots, p. \]
(ii) The conditional variance of \( y \), for fixed \( x_j \), is a constant free of \( j \). That is,
\[ V(y / x_j) = \sigma^2, \quad j = 1, 2, \ldots, p. \]

Also, the vector \( (X_1, X_2, \ldots, X_p) \) follows a version of a multivariate gamma distribution (see Johnson and Kotz (1972), p. 217) having the \( p+1 \) parameters \( \theta_0, \theta_1, \theta_2, \ldots, \theta_p \) which are all positive. Again, the marginal distribution of \( X_j \) is a standard gamma distribution having the parameter \( \theta_0 + \theta_j = \lambda_j \) (say). The covariance between \( X_j \) and \( X_j' \) is \( \theta_j \), while the correlation coefficient between \( X_j \) and \( X_j' \) is \( \theta_j / \sqrt{\lambda_j \lambda_j'} \). Similarly, the coefficient of variation (cv) of \( X_j \) is \( 1/\sqrt{\lambda_j} \).

Now, we quote the following asymptotic expansion of the gamma function for later use in our study:
\[ \Gamma(n a + b + 1) / \Gamma(n a) \overset{\rightarrow}{=} (n a)^{b+1}. \quad (2.3) \]
3. JACKKNIFING SRIVASTAVA'S MULTIVARIATE RATIO ESTIMATOR

Srivastava (1967) has defined a class of estimators which can be given by

$$\{ \overline{y}_S : -\infty < \alpha_j < \infty, \quad j = 1, 2, \ldots, p \}, \quad (3.1)$$

where

$$\overline{y}_S = \frac{1}{p} \sum_{j=1}^{p} w_j \overline{x}_j, \quad \text{and} \quad \overline{x}_j = \overline{y}(\overline{x}_j/\overline{x}_j)\alpha_j.$$

The weights $w_j$ are determined optimally by minimizing the variance of $\overline{y}_S$ under the restriction that $\sum_j w_j = 1$. As in Srivastava (1967), the $\alpha_j$ are known and taken to be equal to $\frac{-c}{C_j} g_j / c_j$, $j = 1, 2, \ldots, p$.

It is easy to see that the Olkin's multivariate ratio estimator, written, say, as $\overline{y}_{OL}$, is a member of (3.1) with $\alpha_j = -1$, for all $j$.

Our aim here is to jackknife $\overline{y}_S$ so as to reduce its bias. As the jackknifed estimator $\overline{y}_{JS}$, we define the weighted average

$$\overline{y}_{JS} = \sum_{j=1}^{p} w_j^* \overline{y}_j^* \quad (3.2),$$

where $\overline{y}_j^*$ is the jackknifed version of $\overline{y}_j$. It should be noted that the weights $w^* = (w_1^*, w_2^*, \ldots, w_p^*)$ are different from those in (3.1) and to be determined by minimizing the variance of $\overline{y}_{JS}$.

The procedure of calculating $\overline{y}_j^*$ for large sample is shown below.
It is to be noted that the quantity $\bar{y}_{\alpha_j}^{*}$ is based on the n-pairs of observations, namely, $(y_i, x_{ij}^*)$, $i = 1, 2, \ldots, n$. Next, we obtain $\bar{y}_{\alpha_j}^{(i)}$ which has the same functional form as $\bar{y}_{\alpha_j}^{*}$, but based only on the data that remain after omitting the pair $(y_i, x_{ij}^*)$. The "pseudovalues" $\bar{y}_{\alpha_j}^{(i)}$ can now be defined as

$$\bar{y}_{\alpha_j}^{(i)*} = n \bar{y}_{\alpha_j}^{(i)} - (n - 1) \bar{y}_{\alpha_j}^{(i)}, \quad i = 1, 2, \ldots, n. \quad (3.3)$$

Finally, the jackknifed version $\bar{y}_{\alpha_j}^{*}$ of $\bar{y}_{\alpha_j}^{*}$ is obtained as the arithmetic mean of the pseudovalues and is given by

$$\bar{y}_{\alpha_j}^{*} = (1/n) \sum_{i=1}^{n} \bar{y}_{\alpha_j}^{(i)*} = n \bar{y}_{\alpha_j} - (n - 1)(1/n) \sum_{i=1}^{n} \bar{y}_{\alpha_j}^{(i)}. \quad (3.4)$$

If the sample size n is large, then the actual computation of $\bar{y}_{\alpha_j}^{*}$ from (3.4) is cumbersome. However, in this case an asymptotic expression for $\bar{y}_{\alpha_j}^{*}$ can be obtained under some suitable assumptions.

With this in mind, let us write $\bar{y}_{\alpha_j}^{(i)}$ as

$$\bar{y}_{\alpha_j}^{(i)} = (1 - 1/n)^{\alpha_j + 1} \frac{(1 - y_i/(n\bar{y})) (1 - x_{ij}/(n\bar{x}_{ij}))}{\bar{y}_{\alpha_j}}, \quad (3.5)$$

where

$$\bar{y} = (1/n) \sum_{i=1}^{n} y_i, \quad \bar{x}_{ij} = (1/n) \sum_{i=1}^{n} x_{ij}.$$ 

Next, we put (3.5) in (3.4) and simplify the resulting expression under the assumptions that $|(y_i/(n\bar{y}))| < 1$ and $|(x_{ij}/(n\bar{x}_{ij}))| < 1$. 


Thus, up to the order $1/n$, we have

$$y^{*} = \left( 1 + \alpha_j (x_j + 1) / (2n) \right) y$$

(3.6)

The jackknifed version of the Srivastava's multivariate ratio estimator can then be obtained, up to the same order, as

$$\overline{y}_{JS}^{*} = \sum_{j=1}^{p} w_{j} \overline{y}_{x_j}^{*} = \overline{y}_{S}^{*} + \left(1/ (2n) \right) \sum_{j=1}^{p} w_{j} \alpha_j (x_j + 1) \overline{y}_{x_j}$$

(3.7)

Since the estimator $\overline{y}_{x_j}^{*}$ is unbiased up to the order $1/n$ (see Section 2 of Chapter 1), and also $\sum_{j=1}^{p} w_{j} = 1$, it follows from (3.2) that the jackknifed estimator $\overline{y}_{JS}^{*}$ is unbiased up to the order $1/n$. If, further, the weights $w_{j}$ are so determined as to minimize the variance of $\overline{y}_{JS}$, then the resulting $\overline{y}_{JS}$ with these optimum weights can be treated as the minimum variance unbiased estimator (MVUE) for the population mean $\overline{Y}$, in some sense.

4. OPTIMUM CHOICE FOR THE WEIGHTS

The criterion for optimality of the weight vector $w$, defined as $w = (w_1, w_2, \ldots, w_p)$, is the minimization of the variance of $\overline{y}_{JS}$, defined in (3.7), subject to the constraint $\sum_{j=1}^{p} w_{j} = 1$. To obtain the optimum weights, we make use of the Lagrangian multiplier method.

Now, it can be shown by Taylor's expansion and also under the assumptions of Section 2, that
\[ V \left( \bar{y}_{x_j} \right) = (f/n) \bar{Y}^2 \left( \alpha_j^2 + \alpha_j^2 c_j^2 + 2 \alpha_j \ell_j c_j c_j \right), \]

\[ \text{Cov}( \bar{y}_{x_j}, \bar{y}_{x_j'} ) = (f/n) \bar{Y}^2 \left( \alpha_j^2 + \alpha_j^2 \ell_j c_j c_j + \alpha_j \ell_j c_j c_j + \alpha_j \ell_j c_j c_j \right). \]

The variance \( V(\bar{y}_{x_j}) \) and Cov(\( \bar{y}_{x_j}, \bar{y}_{x_j'} \)) then follow from (3.6) and (4.1), respectively.

Then
\[ V \left( \bar{y}_{x_j} \right) = V \left( \sum_{j=1}^{p} w_j \bar{y}_{x_j} \right) = \left( \frac{\bar{Y}^2}{n} \right) \tilde{w} A \tilde{w}, \]

where
\[ A = \left( \begin{array}{cccc}
A_{j,j'} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{p,p}
\end{array} \right)_{p \times p} \]

wherein
\[ a_{j,j'} = (1 + (1/(2n))(\alpha_j \gamma (\alpha_j \gamma + 1) + \alpha_j \gamma (\alpha_j \gamma + 1))) \]
\[ \left( c_j^2 + \alpha_j \ell_j c_j c_j + \alpha_j \ell_j c_j c_j + \alpha_j \ell_j c_j c_j \right), \]
\[ (j, j' = 1, 2, \ldots, p). \]

It will be assumed that \( A \) is positive definite, so that \( A^{-1} \) exists. Let \( \bar{e} = (1, 1, \ldots, 1)' \) be a vector of \( p \)-elements.

Our problem is then as follows:

\[ \text{Minimize } \tilde{w}' A \tilde{w} \]
\[ \text{subject to } \bar{e}' \tilde{w} = 1. \] (4.3a)

Then, using the Lagrangian multiplier method, the optimum weights are obtained as
\[ \tilde{w} = \left( A^{-1} \bar{e} \right) / \left( \bar{e}' A^{-1} \bar{e} \right). \] (4.4)
It then follows from (1.4) that
\[ s_j = \frac{\text{sum of the elements in the } j\text{-th row of } A^{-1}}{\text{sum of all the } p \text{ elements of } A^{-1}}. \] (4.5)

The variance of the jackknifed estimator, having these optimum weights, can be obtained as
\[ V(\bar{y}_{JS}) = \frac{\bar{y}^2}{n} (\bar{e}'A^{-1}e). \] (4.6)

A particular case: To facilitate the comparison of the jackknifed estimator \( \bar{y}_{JS} \) with its competitors, we consider the following situation:

Let
\[ c_1 = c_2 = \ldots = c_p = c, \]
\[ e_1 = e_2 = \ldots = e_p = e, \] (4.7)
\[ e_{j'} = e, \text{ for all } j \neq j', \]
so that \( \alpha_j = -\frac{e_{j'}}{c_j} c_0 / c_j = -\frac{e_0}{e_0} c_0 / c = \alpha \) (say).

The optimum weights, under the assumptions of (4.7), can be obtained as \( w_j = 1/p \), for all \( j \). Thus, in this case the optimum weights are uniform.

Now, ignoring the fpc, we have the variance of \( \bar{y}_{JS} \), under the above considerations, as
\[ V(\bar{y}_{JS}) = \frac{\bar{y}^2}{n} (1 + \alpha(\alpha + 1)/n) \cdot \frac{2}{\alpha} e_0^2 c_0 c + \frac{2}{\alpha} e_0^2 c_0 c + (\frac{2}{\alpha} e_0^2 / p)(1 + e(p-1)). \] (4.7a)
The asymptotic relative efficiency of the Jackknifed estimator compared to the Srivastava's estimator:

Let us define the asymptotic relative efficiency of the jackknifed estimator $\tilde{y}_{JS}$ compared to the Srivastava's multivariate ratio estimator $\bar{y}_{S}$ as $\varv\left(\tilde{y}_{JS}\right) / \varv\left(\bar{y}_{S}\right)$, which we denote by $\text{ARE}\left(\tilde{y}_{JS}\right)$. Now, from (4.7a) and also using the expression of the variance of $\bar{y}_{S}$, as given in Srivastava (1967), we have

$$\text{ARE}\left(\tilde{y}_{JS}\right) = \left(1 + \alpha(\alpha + 1) / n\right)^{-1}. \quad (4.8)$$

In many practical problems, $C_0$ equals $C$, so that we have $\alpha = -C_0$. This gives

$$\text{ARE}\left(\tilde{y}_{JS}\right) = \left(1 - C_0(1 - C_0) / n\right)^{-1}.$$

$$= \sum_{r=0}^{\infty} C_0^r (1 - C_0)^r / n^r, \text{ for } |C_0(1 - C_0)| < n. \quad (4.8a)$$

From (4.8a), it is clear that the $\text{ARE}\left(\tilde{y}_{JS}\right)$ is greater than 100% if $C_0$ lies in the interval $(0, 1)$. In other words, the jackknifed estimator is more efficient than the unjackknifed estimator if the variable under study is positively correlated with each of the auxiliary variables. We have thus identified a situation where jackknifing reduces not only the bias but also the variance of an estimator. However, the estimators $\bar{y}_{S}$ and $\tilde{y}_{JS}$ become equally efficient if $n$ is large, or if $C_0$ is close to either zero or one.
5. A NUMERICAL ILLUSTRATION

The following data, taken from Olkin (1958), relate to the number of inhabitants in the 200 largest U.S. cities in 1930, with \( Y = 1950 \), \( X_1 = 1940 \) and \( X_2 = 1930 \) values (in thousands).

The following population characteristics are obtained from Olkin's data:

\[
\bar{Y} = 1699, \quad \bar{X}_1 = 1482, \quad \bar{X}_2 = 1420,
\]

\[
C_0 = 1.0242, \quad C_1 = 1.0479, \quad C_2 = 1.0635,
\]

\[
\rho_{01} = 0.9867, \quad \rho_{02} = 0.9695, \quad \rho_{12} = 0.9942.
\]

It is clearly seen that the conditions (4.7) are satisfied approximately by Olkin's data, so that the variance expressions, as developed in Section 4, can be used.

The relative biases and efficiencies of some estimators are obtained, ignoring the fpc, and compiled in Table 1. The sample size is taken to be 10 for computation of the gain in precision due to jackknifing.

From the Table 1 given below, it appears that the jackknifed estimator fares better compared to each of the competitors both in terms of bias and efficiency. It is also seen that the mean per unit has the minimum efficiency, though it is unbiased. As \( \rho_o \) is close to unity, the gain in efficiency of the jackknifed estimator is small compared to Srivastava's estimator.
### TABLE 1. ASYMPTOTIC RELATIVE BIASES AND EFFICIENCIES OF SOME ESTIMATORS COMPARED TO THE PROPOSED JACKKNIFED MULTIVARIATE RATIO ESTIMATOR

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Relative bias: ((\text{Bias}/ \bar{y}), 100%)</th>
<th>ARE compared to (\bar{y}_{JS}) (in percentages)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jackknifed estimator(\bar{y}_{JS})</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Srivastava's estimator(\bar{y}_S)</td>
<td>2.75</td>
<td>99.52</td>
</tr>
<tr>
<td>Clkin's estimator (\bar{y}_{CL})</td>
<td>5.81</td>
<td>94.92</td>
</tr>
<tr>
<td>Mean per unit (\bar{y})</td>
<td>0.00</td>
<td>56.11</td>
</tr>
</tbody>
</table>

**Note:** In the above Table, the ARE of an estimator compared to the jackknifed estimator \(\bar{y}_{JS}\) is defined to be the ratio \(V(\bar{y}_{JS}) / V(\text{an estimator})\), expressed as percentages. These are shown in the last column.

A simple random sample of size 50 drawn from the Clkin's data is given in the APPENDIX.
6. JACKKNIFING SRIVASTAVA'S MULTIVARIATE RATIO ESTIMATOR UNDER DURBIN'S MODEL

In this Section, we jackknife Srivastava's multivariate ratio estimator under Durbin's model considered in Section 2.

The expected value of Srivastava's estimator $\bar{y}_S$, under Durbin's model, can be shown to be

$$E(\bar{y}_S) = \sum_{j=1}^{p} w_j \bar{x}_j \beta_j \left( n \lambda_j + \alpha_j + 1 \right) \frac{1}{\sqrt{n \lambda_j}} \frac{1}{\sqrt{l}}$$

provided it exists.

If all $\alpha_j$ are equal to -1, then (6.1) reduces to $\sum_{j=1}^{p} w_j \beta_j \lambda_j$, which is the population mean of $y$. Thus, in this case, $\bar{y}_S$ is unbiased. Also, for this choice of $\alpha_j$, $\bar{y}_S$ becomes the Olkin's multivariate ratio estimator. Thus, we have the result that Olkin's estimator is unbiased under Durbin's model (see Olkin (1958)).

Again, if the $\alpha_j$ are different from -1, then Srivastava's estimator $\bar{y}_S$ is asymptotically unbiased, which follows after putting (2.3) in (6.1). Now, assuming $n$ to be large and also utilizing (2.3) it can be shown that

$$E(\bar{y}_{JS} - \bar{y}_S) = \left( \frac{1}{2n} \right) \sum_{j=1}^{p} w_j \alpha_j (\alpha_j + 1) \beta_j \lambda_j$$

which is of order $1/n$. 
The relation (6.2) shows that the difference of the biases of \( \bar{y}_S \) and \( \bar{y}_{JS} \) is of the order 1/n. Also, the expected value of the jackknifed estimator \( \bar{y}_{JS} \), under Durbin's model, can be obtained on using (6.1) in (6.2).

The large sample variance of the jackknifed estimator under Durbin's model:

To obtain the large sample variance of \( \bar{y}_{JS} \), we use the relation (3.6). Thus, we have the representation

\[
\bar{y}_{JS} = \sum_{j=1}^{p} w_j^* \bar{y}_{\alpha_j},
\]

where

\[
w_j^* = w_j \left( 1 + \alpha_j (\alpha_j + 1)/(2n) \right), \quad \text{with } \sum_{j=1}^{p} w_j = 1. \quad (6.3a)
\]

It then follows that upto the order 1/n,

\[
V(\bar{y}_S) = \left( \sigma^2/n \right) + \sum_{j=1}^{p} \sum_{j' \neq j} w_j w_{j'} \Phi(j, j')/(2n), \quad (6.4)
\]

and

\[
V(\bar{y}_{JS}) = \left( \sum_{j=1}^{p} w_j^* \right)^2 \left( \sigma^2/n \right) + \sum_{j=1}^{p} \sum_{j' \neq j} w_j w_{j'} \Phi(j, j')/(2n), \quad (6.5)
\]

where

\[
\Phi(j, j') = \alpha_j (\alpha_j + 1) \beta_j \beta_j' + \alpha_j' (\alpha_j' + 1) \beta_j \beta_j' + 2 \beta_j \beta_j' (\beta_j + 1) = \sqrt{\lambda_j \lambda_j'}. \quad (6.6)
\]

Finally, upto the order 1/n, we have

\[
V(\bar{y}_{JS}) = V(\bar{y}_S) + \left( \sum_{j=1}^{p} w_j^* \right)^2 \left( \sigma^2/n \right). \quad (6.7)
\]
It is easy to see that
\[
\left( \sum_{j=1}^{p} w_j' \right)^2 - 1 \approx \left( \frac{1}{n} \right) \sum_{j=1}^{p} w_j \alpha_j' \left( \alpha_j + 1 \right). \quad (6.7a)
\]

From (6.7) and (6.7a) it then follows that any increase in variance due to jackknifing the estimator \( \overline{Y} \) is of the order \( \frac{1}{n^2} \), and hence negligible in large sample.

It also follows from (6.7) that the jackknifed estimator \( \overline{Y}_j \) will have smaller variance, if
\[
\sum_{j=1}^{p} w_j \alpha_j' \left( \alpha_j + 1 \right) < 0. \quad (6.8)
\]

If, further, the \( \alpha_j \) are very small, then the condition (6.8) reduces to \( \sum_{j=1}^{p} \alpha_j w_j < 0 \). In other words, if the weighted average of the \( \alpha_j \) is negative, then jackknifing will result in a better precision.

It is to be noted that the condition (6.8) is satisfied, in particular, if \( \alpha_j \in (-1, 0) \) and \( w_j > 0 \), for all \( j \). However, if each \( \alpha_j \) is equal to the common value \( \alpha \), say, then the restriction that \( w_j > 0 \) can be dropped, and (6.8) then becomes \( \alpha (\alpha + 1) < 0 \). This is because the sum \( \sum_{j=1}^{p} w_j \) is equal to unity.

A similar result was obtained by Nandi and Aich (1994) in the case of a single auxiliary variable.