SUMMARY: This chapter is devoted to the estimation of the system reliability in a two-component series set-up under a stress-strength (SS) model. The system performs its intended task effectively if the strengths $X_1$ and $X_2$, say, of the two components exceed the respective stresses $Y_1$ and $Y_2$. It is assumed that the strengths jointly follow the bivariate exponential distribution of Marshall and Olkin (BEDMO), while the stresses follow independent gamma distributions. This extends a result of Ebrahimi (1982) who assumed the stresses to be independent exponential random variables.

The procedure of obtaining the bootstrap estimate of the standard error (s.e.) is also given.

The problem of estimation of $R$ where the stresses follow independent Weibull distributions is also discussed. An approximate expression for the MLE of $R$ is given.

This chapter also considers the estimation of $R = P(X_1 < Y_1 < Z_1, X_2 < Y_2 < Z_2)$ which arises in an extension of the two-component series system to the case where the stress $Y_i$ is assumed to lie between the two critical values of the strength, say, $X_i$ and $Z_i$, $i = 1,2$, for efficient performance of the system. The estimates are obtained under such distributions as the exponential, normal and the BEDMO. The problem of obtaining the s.e. is also discussed.
1. INTRODUCTION

In Chapters 7 through 9, we have considered the reliability of a single component under a stress-strength (SS) model. In this chapter, we consider a two-component series system whose survival depends on the strengths of the two components. The system works as long as the strengths $X_1$ and $X_2$, say, of the two components exceed the respective stresses $Y_1$ and $Y_2$, say. As in the case of a single component set-up, here also the strengths depend on the material properties of the components while the stresses are functions of the environment to which the system is subjected. The system reliability under such a model is defined to be the probability $R = P(X_1 > Y_1, X_2 > Y_2)$.

Literatures are abundant on the estimation of the reliability of a single component under an SS-model. But the case of a two-component series set-up remains mostly untouched. Bhattacharyya and Johnson (1973) have discussed the system reliability under the assumption of an $s$-out-of-$k$ SS-model with all the components having subjected to a common stress. Ebrahimi (1982), on the other hand, has assumed a two-component series set-up where the strengths jointly follow the bivariate exponential distribution of Marshall and Olkin (1967) while the stresses follow independent exponential distributions.

In the first part of this chapter, we extend the works of Ebrahimi (1982) to the case where the stresses are independent gamma random variables. The estimate of the system reliability is obtained and
the procedures of obtaining the bootstrap estimate of the standard error (s.e.) are also discussed.

In the second part, the estimation of R is considered under the assumption that $X_1$ and $X_2$ follow independent exponential distributions while $Y_1$ and $Y_2$ follow independent Weibull distributions. The Weibull distribution is known to arise in many areas of reliability study. An approximate expression for the maximum likelihood estimator of R is suggested in this case, which is seen to be simple while the exact expression is very complicated.

In the last part of this chapter, we extend the two-component series set-up discussed above to the case where, for efficient performance of the system, each of the stresses must lie in between two critical values (also called breaking threshold) of the strength. The reliability of the resulting system will then be given by $R = P(\min(X_1, Y_1) < Z_1, \min(X_2, Y_2) < Z_2)$. Such a model arises, for example, in a textile industry where the system works only if both the temperature and humidity lie within specified limits which may be supposed to be random. Singh (1980) and Dutta and Sriwastava (1986) have considered this model with only one component. In our study, we obtain estimates of R by considering such distributions as the exponential, normal and the bivariate exponential distribution of Marshall and Olkin (1967). The problem of estimating the s.e. is also discussed.
2. MAIN RESULTS

We here derive a theorem which is used in the following section to obtain the reliability of a two-component series system under a stress-strength set-up.

Theorem 2.1: Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two non-negative and independent random vectors having joint pdf's \(g_1(x_1, x_2)\) and \(g_2(y_1, y_2)\), respectively. Then

\[
P(X_1 > Y_1, X_2 > Y_2) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty g_1(v_1, y_1, v_2, y_2) \, dv_1 \, dy_1 \, dv_2 \, dy_2.
\]

(2.1)

Proof: We first define

\[
V_1 = X_1 / Y_1, \quad V_2 = X_2 / Y_2.
\]

We then obtain the joint pdf of \(V_1\) and \(V_2\). The joint cdf of \(V_1\) and \(V_2\) is, clearly,

\[
F(v_1, v_2) = P(V_1 \leq v_1, V_2 \leq v_2) = \int_0^v \int_0^w G_1(v_1, y_1, v_2, y_2) \, dy_1 \, dy_2, \quad (2.2)
\]

where \(G_1(x_1, x_2)\) is the joint cdf of \(X_1\) and \(X_2\). The joint pdf of \(V_1\) and \(V_2\), if it exists, may now be given by

\[
f(v_1, v_2) = \left( \frac{\partial^2}{\partial v_1 \partial v_2} \right) F(v_1, v_2) = \int_0^\infty \int_0^\infty g_1(v_1, y_1, v_2, y_2) \, dy_1 \, dy_2, \quad (2.3)
\]

assuming the validity of the change of order of integration and differentiation.
Finally,

$$P(X_1 > Y_1, X_2 > Y_2) = P(V_1 > 1, V_2 > 1)$$

$$= \int_0^\infty \int_0^\infty y_1 y_2 \ g_2 (y_1, y_2) \int_1^\infty g_1 (v_1, v_2, y_1, y_2) dv_1 \ dv_2 \ dy_1 \ dy_2.$$

(2.4)

This proves the theorem.

Corollary 1: Suppose $X_1$ and $X_2$, the strengths of the first and the second component respectively, are independent exponential random variables with parameters $\lambda_1$ and $\lambda_2$. Then we have

$$g_1(x_1, x_2) = \lambda_1 \lambda_2 \exp(- (\lambda_1 x_1 + \lambda_2 x_2)), x_1, x_2 > 0; \ \lambda_1, \lambda_2 > 0.$$

In this case, the system reliability is obtained, from (2.4), as

$$P(X_1 > Y_1, X_2 > Y_2) = \int_0^\infty \int_0^\infty \exp(- (\lambda_1 y_1 + \lambda_2 y_2)) g_2 (y_1, y_2) dy_1 dy_2.$$

(2.5)

If, further, $Y_1$ and $Y_2$ are independent with respective pdf's $g_{21}(y_1)$ and $g_{22}(y_2)$, then we have, from (2.5),

$$P(X_1 > Y_1, X_2 > Y_2) = \int_0^\infty \exp(- \lambda_1 y_1) g_{21}(y_1) dy_1 \int_0^\infty \exp(- \lambda_2 y_2) g_{22}(y_2) dy_2.$$

(2.6)

Clearly, (2.6) is the product of Laplace Transforms of the densities $g_{21}(y_1)$ and $g_{22}(y_2)$, the parameters of the transformations being $\lambda_1$ and $\lambda_2$, respectively.

Corollary 2: Suppose that the distribution of the vector $(X_1, X_2)$ is the bivariate exponential distribution of Marshall and Olkin (1967) which is

$$P(X_1 > x_1, X_2 > x_2) = \exp(- (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 \ \max(x_1, x_2))),$$

$$x_1, x_2 > 0; \ \lambda_1, \lambda_2, \lambda_3 > 0.$$

(2.7)
Now, on putting $v = y$, we have, from (2.4),

$$P(X_1 > Y_1, X_2 > Y_2) = \int_0^\infty \int_0^\infty g_2(y_1, y_2) d\gamma_1 dy_2$$

$$= \int_0^\infty \int_0^\infty g_2(y_1, y_2) \exp \left(-\left(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 \max(y_1, y_2)\right)\right) d\gamma_1 dy_2 \tag{2.8}$$

Next we list some useful results (Erdélyi et al. 1954) that are needed in this study:

1. $\Gamma_p(q, a_1, a_2, \ldots, a_p; c_1, c_2, \ldots, c_q; x) = \sum_{m=0}^\infty \frac{(a_1)_m \cdots (a_p)_m (x + m)}{(c_1)_m \cdots (c_q)_m m!}$, where $(x) = x(x+1)(x+2)\ldots(x+m-1)$.

2. $\gamma(a, x) = \int_0^x a \exp(-t) dt = a x F(a; a+1; -x)$.

3. $\Gamma(a, x) = \int_0^\infty a \exp(-t) dt = x^{(a-1)/2} \exp(-x/2) W(a-1/2, a/2)(x)$,

where $W(x)$ is the Whittaker function with arguments $a$ and $b$.

4. $\int_0^\infty x^{s-1} \exp(-ax) F(b; c; \lambda x) dx$

$$= \Gamma(s) F(b; s; \lambda) \gamma(a - 1), \quad a > \max(0, \lambda), s > 0.$$  

5. $\int_0^\infty x^{s-1} \exp(-ax) W_{k, \mu}(b x) dx$

$$= \left[\frac{b^{\mu+1/2}}{\Gamma(\mu + s + 1/2) \Gamma(-\mu + s + 1/2)} \right] F(\mu+s+1/2, \mu-k+1/2; s-k+1; \Gamma(s-k+1) (a + b/2))^{\mu+s+1/2/2} \left(2a-b)/(2a+b)\right),$ 

$$(a + b/2 > 0, s > \mu + 1/2).$$
3. THE STRESSES FOLLOW INDEPENDENT GAMMA DISTRIBUTIONS

Here we assume that the vector \((X_1, X_2)\) follows the distribution given in (2.7) while the \(Y_i, i=1,2\), follow independent gamma distributions having the pdf

\[
    g_{2i}(y_i; \theta_i) = \left(\frac{\theta_i^p}{\Gamma(p)}\right) y_i^{p-1} e^{-\theta_i y_i}, \quad y_i > 0; \theta_i, p > 0. \tag{3.1}
\]

It is also assumed that the pairs \((X_1, X_2)\) and \((Y_1, Y_2)\) are independently distributed. We then have, from (2.8), the system reliability as

\[
    R = \frac{1}{R_2} \int_0^\infty y_1^{p-1} \exp\left(-\left(\lambda_1 + \Theta_1\right) y_1\right) \left\{ \int_0^\infty y_2^{p-1} \exp\left(-\left(\lambda_2 + \Theta_2 + \lambda_{12}\right) y_2\right) dy_2 \right\} dy_1. \tag{3.2}
\]

Now,

\[
    \int_0^\infty y_2^{p-1} \exp\left(-\left(\lambda_2 + \Theta_2\right) y_2\right) dy_2 = y_2^{-1} \frac{p-1}{p+1} F\left(p, p+1; -\left(\Theta_2 + \lambda_2\right) y_2\right) \quad \text{for } y_2 > 0. \tag{3.3}
\]

and

\[
    \int_0^\infty y_2^{p-1} \exp\left(-\left(\Theta_2 + \lambda_2 + \lambda_{12}\right) y_2\right) dy_2 = y_2^{-1} \frac{p-1}{p+1} F\left(p, p+1; -\left(\Theta_2 + \lambda_2 + \lambda_{12}\right) y_2\right),
\]

Using (3.2) through (3.4), we have

\[
    R = \frac{1}{R_2} \left\{ p \int_0^\infty y_1^{p-1} \exp\left(-\left(\lambda_1 + \Theta_1 + \lambda_{12}\right) y_1\right) F\left(p, p+1; -\left(\Theta_2 + \lambda_2\right) y_1\right) dy_1 \right. \right. 
\]

\[
    + \left(\Theta_2 + \lambda_2 + \lambda_{12}\right)^{-1} \int_0^\infty y_1^{p-1} \exp\left(-\left(\Theta_1 + \lambda_1 + \left(\Theta_2 + \lambda_2 + \lambda_{12}\right)/2\right) y_1\right) F\left(p, p+1; -\left(\Theta_2 + \lambda_2 + \lambda_{12}\right) y_1\right) dy_1 \right\}. \tag{3.5}
\]
Let $I_1$ and $I_2$ be, respectively, the first and the second integral within brackets on the right hand side of (3.5). Then we have

$$I_1 = \left( \theta_1 + \lambda_1 + \lambda_{12} \right)^{-2p} \frac{\Gamma(2p)}{\Gamma(p) \Gamma(2p)} \frac{\Gamma(p+2; \lambda_1; -\frac{2}{\theta_1 + \lambda_1 + \lambda_{12}})}{\Gamma(p+1; \theta_1 + \theta_2 + \lambda_1 + \lambda_{12})^2}$$

$$I_2 = \left[ \frac{\left( \theta_2 + \lambda_2 + \lambda_{12} \right)^{p+1}}{\Gamma(p+1) \Gamma(p) \Gamma(2p)} \right] \frac{\Gamma(p+2; \lambda_1; -\frac{2}{\theta_1 + \lambda_1 + \lambda_{12}})}{\Gamma(p+1; \theta_1 + \theta_2 + \lambda_1 + \lambda_{12})^2}$$

On putting these values in (3.5) and simplifying, we have

$$R = \frac{\left( \theta_1, \theta_2 \right)^p}{p \Gamma^2(p)} \left\{ \frac{\Gamma(2p; p+1; -\left( \theta_2 + \lambda_2 \right)/\left( \theta_1 + \lambda_1 + \lambda_{12} \right))}{\left( \theta_1 + \lambda_1 + \lambda_{12} \right)^2p} \right\}$$

$$+ \frac{\Gamma(2p+1; p+1; \left( \theta_1 + \lambda_1 \right)/\left( \theta_1 + \theta_2 + \lambda_1 + \lambda_{12} \right))}{\left( \theta_1 + \theta_2 + \lambda_1 + \lambda_{12} \right)^2p}$$

(3.6)

In case $Y_1$ and $Y_2$ are independent exponential random variables with parameters $\theta_1$ and $\theta_2$, respectively, then $p$ equals unity so that (3.6) reduces to

$$R = \left( \theta_1, \theta_2 \right)/\left( \theta_1 + \theta_2 + \lambda_1 + \lambda_{12} \right) \left\{ \left( \theta_1 + \lambda_1 + \lambda_{12} \right)^{-1} + \left( \theta_2 + \lambda_1 + \lambda_{12} \right)^{-1} \right\}$$

(3.7)

which agrees with the result of Ebrahimi (1982).
4. THE ESTIMATION OF SYSTEM RELIABILITY

Suppose \((Y_1, Y_2, \ldots, Y_n)\) is a random sample of size \(n\) from \(g_{Y_i}(Y_i, \Theta_i)\). Then, the maximum likelihood estimator (MLE) of \(\Theta_i\), assuming \(p\) to be known, is

\[
\hat{\Theta}_i = \frac{p}{\bar{Y}_i}, \quad \text{where} \quad \bar{Y}_i = \frac{\sum_{j=1}^{n} Y_{ij}}{n}, \quad (i = 1, 2). \tag{4.1}
\]

To estimate \(\lambda_1\), \(\lambda_2\) and \(\lambda_{12}\), we consider a random sample of \(n\) pairs, namely, \((X_{11}, X_{21}), (X_{12}, X_{22}), \ldots, (X_{1n}, X_{2n})\) from the distribution given in (2.7). It is, however, not possible to obtain explicit expressions for the MLE's of these parameters. We, therefore, consider some ad hoc estimators called the "INT" estimators of Proschan and Sullo (1976). These estimators are easy to obtain and have very high asymptotic relative efficiency compared with the MLE's.

Suppose we define \(n_1\), \(n_2\) and \(n_o\) as

\[
n_1 = \text{number of pairs such that } X_{ij} < X_{2j},
\]

\[
n_2 = \text{number of pairs such that } X_{ij} > X_{2j},
\]

and \(n_o = \text{number of pairs such that } X_{ij} = X_{2j}, \quad (j = 1, 2, \ldots, n). \tag{4.2}
\]

The "INT" estimators are then given by

\[
\hat{\lambda}_1 = \frac{(n_1 + n_2)}{((n_o + n_1) \sum_{j=1}^{n} X_{1j})},
\]

\[
\hat{\lambda}_2 = \frac{(n_1 + n_2)}{((n_o + n_2) \sum_{j=1}^{n} X_{2j})},
\]

\[
\hat{\lambda}_{12} = \frac{n_o}{((n_1 + n_2) + n_2/ (n_o + n_1)) \sum_{j=1}^{n} \max(X_{1j}, X_{2j})}. \tag{4.3}
\]

The estimator \(\hat{R}\), say, of \(R\) is finally obtained on putting the above estimates in (3.6).
5. THE BOOTSTRAP ESTIMATE OF THE STANDARD ERROR

The exact expressions for the standard error (s.e.) of $\hat{R}$ and its estimate are complicated. Approximate expressions for these can be obtained by using delta method. We shall, however, give here the bootstrap estimate of the standard error of $\hat{R}$.

Let $\hat{\theta}_1, \hat{\theta}_2, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_{12}$ be the estimates as described in Section 4. Then, as in the case of a parametric bootstrap (Efron (1992), p. 28-29), we replace the densities of $Y_1, Y_2$ and $(X_1, X_2)$ by $\hat{g}_{21}, \hat{g}_{22}$ and $\hat{g}_1$, respectively, where

$$\hat{g}_{21} = g_{21}(y_1, \hat{\theta}_1),$$  \hspace{1cm} (5.1)
$$\hat{g}_{22} = g_{22}(y_2, \hat{\theta}_2),$$  \hspace{1cm} (5.2)
and
$$\hat{g}_1 = g_1(x_1, x_2; \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_{12}).$$ \hspace{1cm} (5.3)

Now, let $Y_{11}^{\ast}, Y_{12}^{\ast}, \ldots, Y_{1n}^{\ast}$ and $Y_{21}^{\ast}, Y_{22}^{\ast}, \ldots, Y_{2n}^{\ast}, b = 1, 2, \ldots, B,$
be $B$-bootstrap resamples drawn from the distributions (5.1) and (5.2), respectively. Similarly, $(X_{11}^{\ast}, X_{21}^{\ast}), (X_{12}^{\ast}, X_{22}^{\ast}), \ldots, (X_{1n}^{\ast}, X_{2n}^{\ast}), b = 1, 2, \ldots, B,$
are the $B$-bootstrap resamples drawn from (5.3).

For each $b$, we then obtain the estimates $\hat{\theta}_1^{\ast b}, \hat{\theta}_2^{\ast b}, \hat{\lambda}_1^{\ast b}, \hat{\lambda}_2^{\ast b}$ and $\hat{\lambda}_{12}^{\ast b}$
as described in Section 4. These estimates when substituted in (3.6)
give an estimate $\hat{R}^{\ast b}$, say, of $R$. Finally, the bootstrap estimate of the standard error of $\hat{R}$ is given by

$$\text{s.e.}(\hat{R}) = \sqrt{\frac{1}{(B - 1)} \sum_{b=1}^{B} \left( \hat{R}^{\ast b} - \hat{R}^{\ast 0} \right)^2},$$ \hspace{1cm} (5.4)

where $R^{\ast 0} = (1/B) \sum_{b=1}^{B} \hat{R}^{\ast b}$.
The procedure of obtaining the standard error of $\hat{R}$ discussed above is efficient for large values of $B$, say, 10,000 or more. Thus, the procedure though simple, will need massive computation.

We conclude this part of the Chapter by noting that the assumption of common shape parameter for the stresses is rather strong, though simpler mathematically. The case of unequal shape parameters, however, merits investigation.

6. STRENGTHS FOLLOW EXPONENTIAL DISTRIBUTIONS WHILE STRESSES FOLLOW WEIBULL DISTRIBUTIONS

In this part of the Chapter, we assume that the strengths $X_1$ and $X_2$ follow independent exponential distributions having the parameters $\lambda_1$ and $\lambda_2$, respectively, while the stresses $Y_1$ and $Y_2$ follow independent Weibull distributions having the respective probability density function (pdf) as

$$g_i(y_i) = \theta_i \beta y_i^{\beta - 1} \exp(-\theta_i y_i^\beta), \quad \theta_i, \beta > 0, \quad y_i > 0,$$

(6.1)

The situation where the shape parameter $\beta$ equals 2 arises in many areas of reliability study. For this case, the system reliability $R$ is obtained as

$$R = P( X_1 > Y_1, X_2 > Y_2 ) = \Phi_1 \Phi_2 I_1 I_2,$$

(6.2)
where
\[ I_i = \left( \exp \left( \frac{\lambda_i}{(8 \sigma_i)} \right) D_{-2} \left( \frac{\lambda_i}{(2 \sigma_i)^{\frac{1}{2}}} \right) \right)^{(i - 1) \mathbb{C}} / (2 \sigma_i), \] 
wherein \( D_a(x) \) is a parabolic cylinder function and is given by
\[ D_a(x) = 2^{a/2+1/4 - x^2} e^{-\frac{1}{2}, a/2+1/4, \frac{1}{4} \left( \frac{x}{2} \right)}. \]  

Finally, on simplification we have,
\[ R = \exp \left( \frac{\lambda_1}{(3 \sigma_1)} + \frac{\lambda_2}{(3 \sigma_2)} \right) D_{-2} \left( \frac{\lambda_1}{(2 \sigma_1)^{\frac{1}{2}}} \right) D_{-2} \left( \frac{\lambda_2}{(2 \sigma_2)^{\frac{1}{2}}} \right). \]  

An approximate expression for \( R \) can now be obtained under the assumption that the variability of \( Y_i, i = 1, 2 \), being determined by the extremely unpredictable environment, is much larger compared to that of the strength \( X_i \), so that we may take \( V(Y_i)/V(X_i) = 0.2146 \left( \frac{\lambda_i}{\theta_i} \right) \) to be large. Under these considerations and also on using
\[ D_a(x) = x^a \exp \left( - \frac{x^2}{4} \right), \text{ for large } x, \]  
we have the system reliability \( R \), approximately, as
\[ R = 4 \sigma_1 \sigma_2 / (\lambda_1 \lambda_2)^2. \] 

The maximum likelihood estimator of \( R \):

For a given sample \( Y_{11}, Y_{12}, \ldots, Y_{1n} \) from the distribution of \( Y_1 \), the MLE of \( \theta_1 \) is obtained as \( \hat{\theta}_1 = n / \sum_{j=1}^{n} Y_{1j}^2 \). Similarly, the MLE of \( \theta_2 \) is obtained as \( \hat{\theta}_2 = n / \sum_{j=1}^{n} Y_{2j}^2 \).
Also, given a random sample \( X_{1i}, X_{12}, \ldots, X_{1n} \) from the distribution of \( X_1 \), the MLE of \( \lambda_1 \) is obtained as \( \hat{\lambda}_1 = n/\sum_{j=1}^{n} X_{1j} \).

Similarly, \( \hat{\lambda}_2 = n/\sum_{j=1}^{n} X_{2j} \).

An approximate MLE of \( R \) is then obtained on putting these estimates in (6.5). This follows from the invariance of the maximum likelihood estimation (Zehna (1966)).

It is finally noted that the exact expression of the standard error (s.e.) of \( \hat{R} \) is very complicated. However, approximate expression can be obtained by using the delta or the bootstrap method, as shown in Chapter 7 as well as in Section 5 of the present chapter.

7. ESTIMATION OF THE SYSTEM RELIABILITY \( P(X_1 < Y_1 < Z_1, X_2 < Y_2 < Z_2) \)

Here, we consider the estimation of \( R = P(X_1 < Y_1 < Z_1, X_2 < Y_2 < Z_2) \) for some useful distributions.

Case 1. Stress and strength follow exponential distribution

Let \( X_i, Y_i \) and \( Z_i \), \( i = 1, 2 \), be all independent exponential random variables having means \( 1/\lambda_i, 1/\mu_i \) and \( 1/\nu_i \), respectively. Then, we have
As in the last Section, the MLE of $\lambda_1$ can be obtained as the reciprocal of the sample mean. Similarly, the MLE's of $\lambda_2$, $\mu_1$, $\gamma_1$, $\mu_2$, and $\gamma_2$ are obtained as the reciprocals of the sample means of the samples drawn from the respective distributions of $X_2$, $Y_1$, $Z_1$, $Y_2$ and $Z_2$. Finally, the MLE of $R$ is obtained by invariance, on putting these estimators in (7.1).

Case 2. Stresses follow exponential distribution while strengths follow the bivariate exponential distribution of Marshall and Olkin

Let $Y_1$ follow the exponential distribution having the mean $1/\mu_i$, $i = 1, 2$. Also, we suppose that they are independent. Again, the vectors $(X_1, Z_1)$ and $(X_2, Z_2)$ are assumed to be independent, each having the bivariate exponential distribution of Marshall and Olkin (1967) which is defined by

$$
P(X_1 > x_1, Z_1 > z_1) = \exp(-\lambda_{11}x_1 - \lambda_{12}z_1 - \lambda_{10} \max(x_1, z_1)),
$$

\[ (x_1, z_1 > 0). \]  

(7.2)

In this case, $X_1$ and $Z_1$ follow, marginally, univariate exponential distributions having means $(\lambda_{11} - \lambda_{10})^{-1}$ and $(\lambda_{12} + \lambda_{10})^{-1}$, $i = 1, 2$, respectively.
Now, assuming the vector \((Y_1, Y_2)\) to be independent of \((X_1, Z_1)\), 
\[i = 1, 2,\] we have after simplification,

\[
R = \mu_1 \mu_2 \lambda_{11} \lambda_{21} (\mu_2^{*} + \lambda_{12}^{*} + \lambda_{11}^{*})^{-1} (\mu_1^{*} + \lambda_{21}^{*})^{-1} (\mu_1^{*} + \lambda_{12}^{*})^{-1} (\mu_2^{*} + \lambda_{12}^{*} + \lambda_{21}^{*})^{-1}.
\]

(Ebrahimi (1987) has obtained explicitly the MLE's of \(\lambda_{11}\) and 
\(\lambda_{12}\) assuming that \(\lambda_{10}\) is known. He has also considered the case where \(\lambda_{1i} = \lambda_{i2}\), \(i = 1, 2\). In the general case, however, it is not possible to give explicit expressions for the MLE's. We, therefore, give some simple estimators, called the "INT estimators" of Proschan and Sullo (1976).

Given random samples \((X_{i1}, Z_{i1}), (X_{i2}, Z_{i2}), \ldots, (X_{in}, Z_{in}),\) 
\(i = 1, 2,\) from \((7.2)\), we define the following quantities:

Let 
\[n_{i1} = \# \text{ of pairs such that } X_{ij} < Z_{ij},\]
\[n_{i2} = \# \text{ of pairs such that } X_{ij} > Z_{ij},\]
\[n_{i0} = \# \text{ of pairs such that } X_{ij} = Z_{ij}, (j = 1, 2, \ldots, n),\]

The estimators are then obtained as

\[\hat{\lambda}_{i1} = \left(\frac{n_{i1}}{n_{i1} + n_{i0}}\right) \sum_{j=1}^{n} X_{ij},\]
\[\hat{\lambda}_{i2} = \left(\frac{n_{i2}}{n_{i1} + n_{i0}}\right) \sum_{j=1}^{n} Z_{ij},\]

and

\[\hat{\lambda}_{i0} = \frac{n_{i0}}{1 + \frac{n_{i1}}{n_{i2} + n_{i0}} + \frac{n_{i2}}{n_{i1} + n_{i0}}} \left( \sum_{j=1}^{n} \max(X_{ij}, Z_{ij}) \right)^{-1}.\]
On putting these estimators as well as the MLE of $\mu_i$, $i=1,2$, which is the reciprocal of the sample mean of the sample drawn from the distribution of $Y_i$, in (7.3), we get an estimator of $R$. Since the "IMT estimators" were derived by using the first iterate in solving the maximum likelihood equations by an iterative procedure, the estimator of $R$ obtained above should be close to the MLE of $R$.

The confidence interval for the reliability $R$ would be of great interest. However, exact results are unattainable as the distribution of $R$, obtained above, is unknown. Some approximate results can, therefore, be developed following the delta method or the bootstrap technique discussed in Section 6 of Chapter 7.

Case 3. Stresses follow exponential distribution while strengths follow bivariate normal distribution

Here, we assume that the vectors $(Y_1,Y_2)$, $(X_1,Z_1)$ and $(X_2,Z_2)$ are independently distributed bivariates. Also, $Y_i$, $i=1,2$, follows an exponential distribution having the mean $1/\mu_i$ while $(X_i,Z_i)$ follows a bivariate normal distribution given by

$$
\phi(\mathbf{\theta}_{1i}, \mathbf{\theta}_{2i}; \sigma^2_{1i}, \sigma^2_{2i}, \theta_1), \quad i=1,2.
$$

Let us write $\Phi(x) = 1 - \Phi(x)$, where $\Phi(x)$ is the cdf of a standard normal variable, and $f(h,k;\mathbf{\theta}) = P(U > h, V > k)$, with $(U,V)$ having the distribution $N(0,0;1,1;\mathbf{\theta})$. 
Then, we have

\[ R = \mu_1 \mu_2 \int_0^\infty \int_0^\infty (\bar{\phi}((y_1 - \Theta_{12})/\sigma_{12}) - \Phi((y_1 - \Theta_{11})/\sigma_{11}) \right) 
\]

\[ \times (y_1 - \Theta_{12})/\sigma_{12} ; \phi_1 ))(\bar{\phi}((y_2 - \Theta_{22})/\sigma_{22}) - \Phi((y_2 - \Theta_{21})/\sigma_{21}) \right) 
\]

\[ \times (y_2 - \Theta_{22})/\sigma_{22} ; \phi_2 )) \exp(-\mu_1 y_1 - \mu_2 y_2) \, dy_1 \, dy_2. \] (7.5)

Finally, the MLE of \( R \) is obtained, by invariance, on putting the MLE's of \( \Theta_{11} \), \( \Theta_{12} \), \( \sigma_{11} \), \( \sigma_{12} \) and \( \phi_1 \), \( i = 1, 2 \), which are well-known, in the expression (7.5). It is, however, not possible to write the resulting expression of \( \hat{R} \) in a closed form.

A particular case: An interesting special case arises if we assume that the distribution of \( (y_1, z_1) \), \( i = 1, 2 \), is known with \( \phi_1 = 0 \), and also without any loss of generality, with \( \Theta_{11} = \Theta_{12} = 0 \), \( \sigma_{11} = \sigma_{12} = 1 \). In this case, \( L(h, k ; 0) = \Phi(h) \bar{\Phi}(k) \).

Under these considerations, we have

\[ R = \mu_1 \mu_2 \frac{2}{\int_0^\infty \Phi(y_1) \bar{\Phi}(y_1) \exp(-\mu_1 y_1) \, dy_1. \] (7.6)

The integrals on the right hand side of (7.6) can be seen to be the Laplace Transforms of \( \Phi(y_1) \bar{\Phi}(y_1) \), \( i = 1, 2 \), the parameters of the transformations being \( \mu_1 \).

The MLE of \( R \) is finally obtained, by invariance, on putting the MLE's of \( \mu_i \), \( i = 1, 2 \), in the expression (7.6). The expression for the variance and, hence, the s.e. of \( \hat{R} \) appears to be quite complicated.