CHAPTER 9

CONFIDENCE BOUNDS AND TESTING OF HYPOTHESIS
CONCERNING R = P(X > Y) BASED ON BIVARIATE NORMAL
SAMPLES

SUMMARY: In the context of reliability under a stress-strength model, the problem of constructing confidence bounds for R = P(X > Y) frequently arises. Many authors have considered this problem under both parametric and non-parametric frameworks. While earlier authors have taken X and Y to be independent, we in this chapter assume that the random vector (X, Y) follows a bivariate normal distribution. The problem of obtaining the sample size for specified confidence length and confidence coefficient is also considered.

The problem of testing hypothesis concerning R is discussed. In a recent study, Weerahandi and Johnson (1992) have considered this problem assuming X and Y to be independent normal variables with unequal variances. In this chapter, we consider the situation where the vector (X, Y) follows a bivariate normal distribution with equal variances. A numerical illustration is also given.

In the last part, we consider the problem of testing hypothesis $H_0: R = 0.50$ against $H_1: R \neq 0.50$, say. This arises in situations where average reliability is of interest. Three different procedures of testing are discussed and their optimum properties studied.
In the preceding two chapters, we have discussed the problem of estimation of the reliability \( R = P(X > Y) \) under different assumptions regarding the distributions of \( X \) and \( Y \). The problem of obtaining the approximate confidence intervals of \( R \) by the delta and the bootstrap methods is also discussed.

In the present chapter, the problem of inference concerning \( R \) is considered under the assumption that the vector \((X, Y)\) follows a bivariate normal distribution. The contents of this chapter can be grouped into three parts. In the first part, we discuss the problem of constructing the large-sample confidence bounds for \( R \). The problem of sample size determination to ensure a preassigned width of the confidence interval as well as preassigned confidence coefficient is also discussed.

The problem of constructing confidence bounds for \( R \) has been studied by several researchers. Birnbaum and McCarty (1958) have obtained the distribution-free upper confidence bound for \( R \) based on independent samples. Owen et al. (1964) have treated the case of paired observations and tabulated the sample sizes needed for specified confidence bounds and confidence coefficients. Sen (1967) has obtained asymptotically distribution-free confidence bounds for \( R \) based on independent samples. Govindarajulu (1967) has considered the problem assuming a bivariate normal distribution for the vector \((X, Y)\), while Govindarajulu (1968) has obtained the distribution-free confidence bounds for \( R \).
Guttman et al (1988), assuming the availability of information on some auxiliary variables, have obtained confidence limits for R. In their study, for equal residual variances, an exact solution is obtained, but for the unequal variance case, an approximate solution is developed.

The choice of optimum sample size for experiments concerned with the inference on R has been the topic of interest of many researchers. Reiser and Guttman (1989) and Reiser and Faraggi (1992) have considered the problem in an acceptance-sampling-theory framework. In such an approach, the client, though unwilling to make assumptions on the size of the variances of X and Y, is able to give information on the producer's and consumer's risks. Nandi and Aich (1994a) have obtained confidence bounds for R and also tabulated the sample size needed to ensure the given width of the confidence interval and given confidence coefficient. They have assumed a bivariate normal distribution, with known correlation coefficient, for their study.

In the second part of Chapter 9, the problem of testing hypothesis on \( R = P(X > Y) \) is considered under a bivariate normal set-up. In a recent article, Weerahandi and Johnson (1992) have considered the problem of testing hypothesis on R under the assumption that X and Y follow independent normal distributions with possibly unequal variances. We, in this study, relax the assumption of independence, but assume equal and unknown variance for X and Y.
In the last part of this chapter, we consider the problem of testing $R = 0.50$ against, say, $R \neq 0.50$. Such a problem may arise in situations where reliability of a moderate size is of interest. The problem of testing in this case is shown to be that of testing the equality of two normal means. It is assumed that the variances of $X$ and $Y$ are equal but unknown, while the correlation coefficient between $X$ and $Y$ is known.

Three different methods of testing have been discussed: Method I is based on the reduction of data to two independent samples, Method II considers the likelihood ratio approach while Method III uses a Fisher-$t$ type statistic. The optimum properties of these methods have also been studied.

The problem of testing $R = 0.50$, with incomplete data on both $X$ and $Y$, may arise in some situations. For example, the entire process of taking observations may be destructive in nature, so that observing an $X$ precludes the observing of the corresponding value of $Y$, and vice versa. A typical data set in this case would look like $(X_1, \ast), (X_2, \ast), \ldots, (X_n, \ast)$ and $(\ast, Y_1), (\ast, Y_2), \ldots, (\ast, Y_n)$. The equivalent problem of testing equality of means with incomplete data has been considered, among others, by Mehta and Gur-Land (1969) and Lin and Stivers (1975).

It is shown that the Method III discussed above can be used in cases where observations on both the variables are missing. This is an improvement over the earlier methods which are mostly useful in cases with incomplete data on only one variable.
9a. CONFIDENCE BOUNDS FOR THE RELIABILITY

9a.1 The exact distribution of a Fisher-t type statistic

We here obtain the exact distribution of a Fisher-t type statistic. This distribution will then be used to obtain the confidence bounds for \( R \) and also to test hypothesis concerning \( R \).

Let \((X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)\) be a random sample from the bivariate normal distribution \(N_2(\mu_1, \mu_2; \sigma^2, \sigma^2; \rho)\), where \(\sigma^2\) is unknown while \(\rho\) is known. The Fisher-t type statistic is then defined to be

\[
T = \sqrt{m} \left( \frac{\bar{X} - \bar{Y}}{S}\right), \tag{9a.1.1}
\]

where \(\bar{X}\) and \(\bar{Y}\) are the sample means and 
\[
S^2 = m(S_x^2 + S_y^2),
\]

with
\[
S_x^2 = m^{-1} \sum_{i=1}^{m} (X_i - \bar{X})^2, \quad S_y^2 = m^{-1} \sum_{i=1}^{m} (Y_i - \bar{Y})^2,
\]

and 
\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - Y_i)^2.
\]

Clearly, for \(\rho = 0\), (9a.1.1) reduces to the well-known Fisher-t statistic for the two-sample problem.

We now obtain the distribution of \(T\) which is given in the following theorem. In the discussion below, \(h(.)\) would denote a generic density function.

In the following discussion, \(\delta\) would mean

\[
\delta = \frac{1}{2} \left( \frac{m (\mu_1 - \mu_2)}{\sigma (2(1 - \rho))^{1/2}} \right).
\]
Theorem 9a.1

The probability density function of the statistic \( T = \sigma T \), where \( \sigma = \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{1}{2}} \), is given by

\[
h(t^*) = \sum_{i=0}^{\infty} q_i(\varepsilon) f_{n+2i}(t^*/\delta), \quad (9a.1.2)
\]

where \( q_i(\varepsilon) = \binom{m-1}{i} \left( \frac{2\varepsilon}{1 - \varepsilon} \right)^i \sigma^{m-1}, \quad (9a.1.3)\]

and

\[
f_{n+2i}(t^*/\delta) = \frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi i(n/2+i)}} \sum_{j=0}^{\infty} \frac{(\sqrt{2}\delta)^j}{(n+j+1/2)^{n+j/2}} \frac{\Gamma((n+j+1)/2+1)}{(1+t^2/n)^{(n+j+1)/2+1)}}, \quad (9a.1.4)
\]

with \( \sum_{i=0}^{\infty} q_i(\varepsilon) = 1 \).

Proof:

We first obtain the distribution of the statistic \( T' \) which is related to \( T \) by

\[
T' = T/(\sigma (n(1 - \varepsilon))^{\frac{1}{2}}). \quad (9a.1.5)
\]

It is now easy to see that \( T' \) admits the representation

\[
T' = (U + \delta)/Z^{\frac{1}{3}}, \quad (9a.1.6)
\]

where

\[
Z = m(\frac{1}{2}S_x^2 + \frac{1}{2}S_y^2)
\]

and

\[
U = m^{\frac{1}{3}}((\bar{X} - \bar{Y}) - (\mu_1 - \mu_2))/(\sigma(2(1 - \varepsilon))^{\frac{1}{2}}).
\]

Clearly, \( U \) follows \( N(0,1) \). Also, \( U \) and \( Z \) are independent.
Patil and Liao (1970) have obtained the probability density function (pdf) of $Z$ as

$$
\mathcal{G}(z) = \sum_{i=0}^{\infty} q_i(\varphi) \frac{1}{\sigma^2} \frac{1}{2(1+\varphi)} \frac{1}{\sqrt{\pi/(1+\varphi)}} \frac{1}{\Gamma(m+i-1)}, \quad (9a.1.7)
$$

where

$$
\mathcal{P}(z; a, b) = \frac{a^b}{\Gamma(b)} e^{-az} z^{b-1}, \quad a, b > 0, \quad z > 0.
$$

Now, let $H(t')$ denote the cumulative distribution function (cdf) of $T$. Then, we have

$$
H(t') = \mathbb{P}(U + \delta \leq t' \sqrt{\varepsilon})
$$

$$
= \int_0^\infty \Phi(t' \sqrt{\varepsilon} - \delta) \phi(z) \, dz,
$$

and the pdf of $T'$ as

$$
h(t') = \int_0^\infty \Phi(t' \sqrt{\varepsilon} - \delta) \phi(z) \, dz,
$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cdf and pdf of a standard normal variable, respectively.

The expression (9a.1.9), on simplification, gives

$$
h(t') = \frac{e^{-\delta^2/2}}{\sqrt{\pi/(1+\varphi)}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q_i(\varphi) (\sqrt{2} \delta t')^j}{\Gamma(m+i-1) (m+i+j-1)/2}.
$$

$$
\frac{((1+\varphi) \sigma^2)^{j/2}}{\Gamma(m+i+(j-1)/2)} (1+(1+\varphi) \sigma^2)^{m+i+(j-1)/2},
$$

$$
(-\infty < t' < \infty). \quad (9a.1.10)
$$
The pdf of $T$ then follows on using the relation (9a.1.5). Finally, the pdf of $T^*$ is obtained from that of $T$ by using $T^* = cT$. This completes the proof of the theorem.

It is observed that, for $\ell \neq 0$, the pdf of $T$ reduces to the non-central t-distribution with non-centrality parameter $\delta$ and degrees of freedom $n = 2m - 2$. Further, if $\delta = 0$, then the distribution reduces to the central t-distribution with $n$ degrees of freedom.

Again, the distribution of the statistic $\frac{((\bar{X} - \bar{Y}) - (\mu_1 - \mu_2))}{(s^{2m})}$ for given $\ell$ can be obtained from that of $T$ on putting $\delta = 0$. This gives the pdf as

$$h(t) = \frac{((1+\ell)/(1-\ell))^{\frac{1}{2}}}{(n\pi)^{\frac{1}{2}}} \sum_{i=0}^{\infty} \frac{q_i(\ell)}{\Gamma(n/2+i)} \frac{\Gamma((n+1)/2+i)}{(1 + \frac{t^2(1+\ell)}{n(1-\ell)})^{(n+1)/2+i}} \left( -\infty < t < \infty \right). \quad (9a.1.11)$$

The distribution (9a.1.11) is seen to be symmetrical about zero. This distribution can be used to set confidence interval for the difference $\mu_1 - \mu_2$, in case $X$ and $Y$ are dependent with correlation coefficient $\ell$, which is assumed to be known.

Finally, on putting $\ell = 0$ in (9a.1.11) and noting that $q_0(0) = 1$ while $q_i(0) = 0$ for $i > 1$, we get the pdf of a central t-distribution with $n$ degrees of freedom.
9a.2 The large-sample distribution of the statistic $T$

We now obtain an asymptotic distribution of $T$ assuming the sample size $n$ to be large. Thus, for large $n$, we have

$$
\frac{\Gamma((n+j+1)/2+1)}{\Gamma(n/2+1)} = (n/2)^{(j+1)/2}, \quad (9a.2.1)
$$

and

$$
(1+t^2(1+\rho)/(n(1-\rho)))^{((n+j+1)/2+1)} = \exp(t^2(1+\rho)/(2(1-\rho))). \quad (9a.2.2)
$$

Under the above considerations, and also on using the Scheffé's theorem (1947) on convergence of densities, it is seen that $T$ is asymptotically normal with asymptotic mean $\delta$ and asymptotic variance unity. In other words, the statistic $T$ is also asymptotically normal with asymptotic mean $\delta((1-\rho)/(1+\rho))^{1/2}$ and asymptotic variance $(1-\rho)/(1+\rho)$.

The large-sample distribution of $T$ is, therefore, given by

$$
h(t) = \frac{((1+\rho)/(1-\rho))^{1/2}}{\sqrt{2\pi}} \exp(-(1+\rho)/(2(1-\rho))(t-\delta (1-\rho)/(1+\rho))^2), \quad ( -\infty < t < \infty), \quad (9a.2.3)
$$

Clearly, under $H_0: \mu_1 = \mu_2$, $T$ follows $N(0, (1-\rho)/(1+\rho))$, asymptotically. This is in contrast to the case of two independent samples where, under $H_0$, $T$ is known to follow a standard normal distribution for large sample.
9a.3 Confidence bounds for $R$ based on correlated variables

Here, we consider the problem of setting confidence bounds for $R$ with a given confidence coefficient $1 - \alpha$, when $m$ independent observations $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$ are available from a bivariate normal population $N_2(\mu_1, \mu_2; \sigma^2, \sigma^2; \rho)$.

It is easy to see that

$$R = P(X > Y) = \Phi((\mu_1 - \mu_2)/(\sigma \sqrt{2(1 - \rho)})). \quad (9a.3.1)$$

We now seek two numbers "a" and "b" (a < b) such that, for given $\alpha$, we have

$$P(a < R < b) = 1 - \alpha. \quad (9a.3.2)$$

Since $\Phi(x)$ is a monotonically increasing function of $x$ and also since $R$ admits the representation $\frac{\Phi'(\delta/m^{\frac{1}{2}})}{\Phi'(m^{\frac{1}{2}})}$, we have from (9a.3.2),

$$P\left(m^{\frac{1}{2}} \Phi^{-1}(a) \leq \delta \leq m^{\frac{1}{2}} \Phi^{-1}(b)\right) = 1 - \alpha. \quad (9a.3.3)$$

It is seen in the preceding Section that $T$ is asymptotically normal with mean $\delta((1 - \rho)/(1 + \rho))^{\frac{1}{2}}$ and variance $(1 - \rho)/(1 + \rho)$. Thus, for given $\alpha$, it is possible to choose a number $\zeta_{\alpha/2}$ such that

$$P(-\zeta_{\alpha/2} \leq (T - \delta((1 - \rho)/(1 + \rho))^{\frac{1}{2}})/(1 - \rho)/(1 + \rho) \leq \zeta_{\alpha/2}) = 1 - \alpha. \quad (9a.3.4)$$
The expression (9a.3·4) can equivalently be written as

\[ P(-\frac{r_{\alpha/2}}{\sqrt{m}} + T \frac{1+\epsilon}{1-\epsilon} \leq \delta \leq \frac{r_{\alpha/2}}{\sqrt{m}} + T \frac{1+\epsilon}{1-\epsilon} ) = 1 - \alpha. \quad (9a.3·5) \]

Finally, comparing the relations (9a.3·3) and (9a.3·5), we have

\[ a = \Phi \left( m \left( -\frac{r_{\alpha/2}}{\sqrt{m}} + T \frac{1+\epsilon}{1-\epsilon} \right) \right), \quad (9a.3·6) \]
\[ b = \Phi \left( m \left( \frac{r_{\alpha/2}}{\sqrt{m}} + T \frac{1+\epsilon}{1-\epsilon} \right) \right). \quad (9a.3·7) \]

9a.4 Determination of the sample size ensuring a preassigned width of the confidence interval and preassigned confidence coefficient

Here, our problem is to obtain the adequate sample size \(m\) that ensures a confidence interval of width \(2\epsilon\) and a confidence coefficient \(1 - \alpha\). To this end, we consider the equation

\[ P( |\hat{R} - \hat{R}| < \epsilon ) = P(\hat{R} - \epsilon \leq \hat{R} \leq \hat{R} + \epsilon ) \geq 1 - \alpha , \quad (9a.4·1) \]

where \(\hat{R}\) is a suitable estimate of \(R\) which we take to be

\[ \hat{R} = \frac{\Phi((\bar{X} - \bar{Y})/(s(2(1 - \epsilon))^{\frac{1}{2}}))}{\Phi(T / (m(1 - \epsilon))^{\frac{1}{2}})}. \quad (9a.4·2) \]
Using (9a.4.2), we write (9a.4.1) as

\[ P\left((m(1-\xi))^{1/2} \Phi^{-1}(R-\xi) \leq T \leq (m(1-\xi))^{1/2} \Phi^{-1}(R+\xi)\right) \geq 1 - \alpha. \]

(9a.4.3)

Now let \( F(t) \) denote the cumulative distribution function of \( T \). Then, from (9a.4.3), we have

\[ F((m(1-\xi))^{1/2} \Phi^{-1}(R+\xi)) - F((m(1-\xi))^{1/2} \Phi^{-1}(R-\xi)) \geq 1 - \alpha. \]

(9a.4.4)

To solve the equation (9a.4.4) for \( m \), we utilize the asymptotic distribution of \( T \). Thus, we have

\[ F(t) = P(T \leq t) = \bar{\Phi}(\delta + t((1+\xi)/(1-\xi))^{1/2}). \]

(9a.4.5)

Putting (9a.4.5) in (9a.4.4), we have, on simplification,

\[ \bar{\Phi}(\delta + (m(1+\xi))^{1/2} \Phi^{-1}(R+\xi)) - \bar{\Phi}(\delta + (m(1+\xi))^{1/2} \Phi^{-1}(R-\xi)) \geq 1 - \alpha. \]

(9a.4.6)

Unfortunately, a solution for \( m \) in (9a.4.6), for given \( \xi \) and \( \alpha \), depends on the unknown \( R \). Now, following the approach of Govindarajulu (1967), we set \( R = \frac{1}{2} \) to get a lower bound for \( m \). It is to be noted that \( R = \frac{1}{2} \) implies \( \xi = 0 \).
Thus, on using \( \Phi^{-1}(\frac{1}{2} - \xi) = -\Phi^{-1}(\frac{1}{2} + \xi) \) in (9a.4.6), we have the lower bound as

\[
m \geq \left( \frac{\Phi^{-1}(\alpha/2)}{\Phi^{-1}(\frac{1}{2} + \xi)} \right)^2 \cdot (1 + \rho)^{-1}, \tag{9a.4.7}
\]

which reduces to the expression (2.5) of Govindarajulu (1967), when \( \rho \) equals zero. It is thus seen that, in the dependent case, the correlation coefficient has got a significant role in the determination of the sample size.

It would be of interest to compare the performance of the proposed method with the case when \( \sigma_x^2 = \sigma_y^2 = \sigma^2 \) (say) and \( \theta \) all are unknown. In this case, it is shown by Lehmann (1959, p. 208) that, on multiplication by a suitable constant, the statistic

\[
\frac{\bar{x} - \bar{y}}{\left( S_x^2 + S_y^2 - S_{xy} \right)^{1/2}}, \text{ with } S_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{m},
\]

has a non-central t-distribution with \( m - 1 \) degrees of freedom. One can then proceed, as in Govindarajulu (1967), to obtain a lower bound for the sample size \( m \).

The following table gives the lower bound for \( m \), as obtained in (9a.4.7), for different choices of \( \xi \) and \( \alpha \).
Table 1. Sample Sizes for Setting Two-sided Confidence Limits on the Probability that One Component of a Bivariate Normal Vector is Stochastically Larger than the Other

(2$\xi$ = fixed length of the confidence interval, and $1-\alpha$ = least possible value of the confidence coefficient)

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\xi$</th>
<th>-0.25</th>
<th>-0.50</th>
<th>0.25</th>
<th>0.50</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.05</td>
<td>140</td>
<td>35</td>
<td>8</td>
<td>210</td>
<td>52</td>
</tr>
<tr>
<td>0.10</td>
<td>0.05</td>
<td>229</td>
<td>56</td>
<td>13</td>
<td>344</td>
<td>84</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
<td>327</td>
<td>80</td>
<td>19</td>
<td>492</td>
<td>120</td>
</tr>
<tr>
<td>0.01</td>
<td>0.05</td>
<td>565</td>
<td>137</td>
<td>33</td>
<td>850</td>
<td>206</td>
</tr>
<tr>
<td>0.005</td>
<td>0.05</td>
<td>673</td>
<td>164</td>
<td>39</td>
<td>1012</td>
<td>246</td>
</tr>
</tbody>
</table>
9b. TESTING HYPOTHESIS ON R = P(X > Y) WITH MISSING DATA

Here, we consider the problem of testing hypothesis on R = P(X > Y) based on incomplete data on both the variables X and Y. A typical data set in this case would look like (X_1, *), (X_2, *), ..., (X_n, *), (*, Y_1), (*, Y_2), ..., (*, Y_n), where '*' means missing observation. Such a problem may arise, for example, in cases where the process of taking observations is destructive in nature, so that observing the strength X precludes the observing of the corresponding value of Y, and vice versa.

In a different context, the problem of testing hypothesis concerning the equality of means with incomplete data on only one of the variables is considered, among others, by Mehta and Currin (1969) and Lin and Stivers (1975). Thus, the typical data set in their case was of the form (X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n), (X_{n+1}, *), ..., (X_N, *), with N - n missing observations on Y. The proposed tests were also compared to the paired t-test, and found to have better power in some cases. Lin and Stivers (1975) gave a Monte Carlo study of the different testing procedures. In all these studies, it is assumed that the vector (X, Y) follows a bivariate normal distribution with means \( \mu_1 \) and \( \mu_2 \), correlation coefficient \( \rho \) and the common variance \( \sigma^2 \).

In the present study, we shall assume a bivariate normal set-up following the earlier works in this context. However, the correlation coefficient \( \rho \) will be assumed to be known while common variance \( \sigma^2 \) unknown.
9b.1 Testing \( H_0 : R = \rho_0 \) against \( H_1 : R > \rho_0 \)

Let \((X, Y)\) follow the bivariate normal distribution given by \( N_2(\mu_1, \mu_2; \sigma^2, \sigma^2; \rho) \), with \( \rho \) being known while \( \sigma^2 \) unknown. Also, let \((X_1^*, Y_1^*), (X_2^*, Y_2^*), \ldots, (X_m^*, Y_m^*)\) and \((*, Y_1^*), (*, Y_2^*), \ldots, (*, Y_m^*)\) be the incomplete data obtained from this population. Here, for the sake of balance and convenience, we have taken the observations on \( X \) and \( Y \) to be of the same size. Our problem here is to test the hypothesis \( H_0 : R = \rho_0 \) against, say, \( H_1 : R > \rho_0 \).

It is clear that the problem defined above is a two-sample problem with dependent samples which is in contrast to the classical two-sample case where the samples are independent. The latter problem is usually treated by the Fisher's t-statistic while for the former we shall use the statistic \( T \), defined in (9a.1.1), whose distribution is given in Theorem 9a.1. It is to be noted that owing to the incomplete pairing of data, the above problem cannot be reduced to the independent case by using suitable transformations.

It is easy to see that in our problem the reliability \( R \) can be written as

\[
R = \Phi \left( \frac{\mu_1 - \mu_2}{\sigma \sqrt{2(1 - \rho)}} \right),
\]

where \( \Phi(x) \) is the cumulative distribution function of the standard normal variable. Since \( \Phi(x) \) is a monotonically increasing function of \( x \), our hypotheses can be rewritten as

\[
H_0 : \Theta = \Theta_0 \text{ against } H_1 : \Theta > \Theta_0,
\]

(9b.1.2)
where $\Theta = (\mu_1 - \mu_2)/\sigma \sqrt{2}$ and $\Theta_0 = (1 - \xi)^{\frac{1}{2}} \Phi^{-1}(R_0)$. (9b.1.3)

Now, as an estimator of $\Theta$, we take

$$\hat{\Theta} = (\bar{X} - \bar{Y})/(S \sqrt{2}) = T / n^{\frac{1}{2}}. \quad (9b.1.4)$$

It is seen that the null distribution of the statistic $T$ can be obtained from (9a.1.2) on putting $\delta = n \Phi^{-1}(R_0) = \delta_0$ (say). Also, the alternative hypothesis being one-sided, the critical region for a size-$\alpha$ test is given by

$$W \equiv (t_{\alpha}; n, \infty), \quad (9b.1.5)$$

where the critical point $t_{\alpha}; n$ is obtained from

$$P(T > t_{\alpha}; n / H_0) = \alpha. \quad (9b.1.6)$$

The equation (9b.1.6) can be written as

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \left( q_1(\xi) \sqrt{2} \delta_0 \right)^j \frac{\Gamma((j+1)/2))/\Gamma(j))}{2^{(j+1)/2)} \right).$$

$$I_1 = a(n/2+1, (j+1)/2) = 2 \alpha \exp(\delta_0^2/2), \quad (9b.1.7)$$

where

$$a = (t_{\alpha}; n(1+\xi)/(n(1-\xi)))/(1 + (t_{\alpha}; n(1+\xi)/(n(1-\xi))))$$

wherein

$$I_1(p, q) = \int_0^x y^{p-1} (1 - y)^{q-1} dy \quad \beta(p, q),$$

$I_1(p, q)$ being the incomplete beta function (Pearson (1948)).
The equation (9b.1.7) can be solved for \( t \) and hence for \( t_{\alpha;n} \) by the method of iteration. If \( n = 2m - 2 \) is moderately large, then \( \zeta_\alpha \) will be a good initial root for \( t_{\alpha;n} \), where \( \zeta_\alpha \) is given by \( P( Z > \zeta_\alpha ) = \alpha \), \( Z \) having a \( N(0,1) \) distribution.

The power \( \xi(\delta) \) of the above test is clearly given by

\[
\xi(\delta) = \left( \frac{\exp(-\delta^2/2)/(2\sqrt{\pi})}{\Gamma((j+1)/2)/\Gamma(j)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (q_i(\epsilon)(\sqrt{2}\delta)^j) \right)^i \omega_{\alpha}(n/2+i,(j+1)/2). \tag{9b.1.8}
\]

The large sample test of \( H_0 \): A large sample test of \( H_0 \) can be developed by noting that, under \( H_0 \), \( T \) is asymptotically normal with mean \( \delta_0 \left( \frac{1-\epsilon}{1+\epsilon} \right)^{\frac{3}{2}} \) and variance \( \frac{1-\epsilon}{1+\epsilon} \cdot \frac{(1-\epsilon)}{1+\epsilon} \cdot \frac{1}{2} \). For a size-\( \alpha \) test, the critical point \( t_0 \), say, is defined by the relation \( P( T > t_0 \mid H_0 ) = \alpha \) and is given by

\[
t_0 = \left( \frac{1-\epsilon}{1+\epsilon} \right)^{\frac{1}{2}} \left( \zeta_\alpha + \delta_0 \right). \tag{9b.1.9}
\]

The power of the test in the large sample case is similarly obtained as

\[
\xi(\delta) = 1 - \Phi(\zeta_\alpha + \delta_0 - \delta), \delta > \delta_0. \tag{9b.1.10}
\]

The power curve \( (\delta, \xi(\delta)), \delta > \delta_0 \), can also be plotted to know the effectiveness of the proposed test.

We give below a numerical illustration of the theory that has been developed.
9b.2 A NUMERICAL ILLUSTRATION

We wish here to test $H_0 : R = 0.52$ against $H_1 : R > 0.52$. Let us take $\zeta = 0.005$, $\alpha = 0.01$ and $m = 5$, so that $n = 8$. Our problem is then to solve the equation (9b.1.7) to obtain the critical point $t_{\alpha;n}$.

Here, $\psi_0 = 0.1118$. Since $\zeta$ and $\psi_0$ are close to zero, in solving (9b.1.7) we retain only the linear terms in $\zeta$ and $\psi_0$. Thus, putting $q_0 = 1$, $q_1 = -0.0203$ in (9b.1.7) and also writing $a^* = 1 - a$, we have

$$1.7725 I_a(4, \frac{1}{2}) + 0.1581 I_a(4, 1) - 0.036 I_a(5, \frac{1}{2})$$

$$-0.0032 I_a(5, 1) = 0.036.$$  

To solve for $a^*$ from the above equation, we start with the initial root $t_{\alpha;n} = 2.58$. This gives the initial root of $a^*$ as 0.54. An iteration method then gives $a^*$ to be approximately equal to 0.46, which finally gives the critical point as $t_{\alpha;n} = 3.0487$. The hypothesis $H_0$ is, therefore, rejected at 1% level of significance if the observed value of $T$ exceeds 3.0487, and accepted otherwise.

In case $\zeta$ and $\psi_0$ are not small, the solution of (9b.1.7) for the critical point $t_{\alpha;n}$ should not be a problem on modern computers.
9c. TESTING $H_0 : R = 0.50$ AGAINST $H_1 : R \neq 0.50$

The problem of testing $H_0 : R = 0.50$ against the two-sided alternative $H_1 : R \neq 0.50$ arises in situations where reliability of a moderate size is of interest. Under the assumption of a bivariate normal set-up as in the preceding Sections, it is then easy to see that the above hypotheses can be rewritten as $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$. This follows from the fact that $\Phi^{-1}(0.50)$ is equal to zero.

Let us assume that a random sample of size $m$, namely, $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m)$ is available from the bivariate normal population $N_2(\mu_1, \mu_2; \sigma^2, \sigma^2; \rho)$, with $\rho$ being known while the common variance $\sigma^2$ unknown. Our problem is thus the problem of testing the equality of two normal means in the presence of correlation. For $\rho = 0$, the problem becomes the classical two-sample problem and can be solved by using the Fisher's $t$-statistic. However, for the present problem with $\rho \neq 0$, we give three testing procedures as described below.

Method I is based on the reduction of data to two independent samples by using suitable transformations. Method II considers the likelihood ratio approach while Method III uses a Fisher-$t$ type statistic, namely, the statistic $T$ defined in (9a.1.1).

As is already explained in Section 9b, Method III can be used in cases of incomplete data on both the variables, or when the pairing of the observations is lost due to an accident or mishandling of data.
9c.1 Method I: Use of transformations to reduce the problem to the independent case

Let us define the transformations

\[ U = X, \]
\[ V = \frac{(y - \phi X)}{\sqrt{1 - \phi^2}}. \]  

(9c.1.1)

Clearly, \( U \) and \( V \) are independent with \( U \) having \( N(\mu_1, \sigma^2) \) and \( V \) having \( N((\mu_2 - \phi \mu_1)/\sqrt(1 - \phi^2), \sigma^2) \) distributions. Also, under (9c.1.1), the null hypothesis \( H_0 : \mu_1 = \mu_2 \) reduces to \( H'_0 : \mu_1 = c \mu_2 \), where \( \mu_2' = (\mu_2 - \phi \mu_1)/\sqrt(1 - \phi^2), \ c = \sqrt{(1+\phi)/(1 - \phi)} \). The reduced problem is, therefore, the classical two-sample problem with equal but unknown variance.

We now choose the statistic \( T_1 \) defined by

\[ T_1 = \frac{(\bar{U} - c \bar{V})}{(\hat{\sigma} \sqrt{(1 + c^2)/m})}, \]  

(9c.1.2)

where \( \bar{U} \) and \( \bar{V} \) are the sample means while \( \hat{\sigma}^2 \) is given by

\[ \hat{\sigma}^2 = (1/n) \left( \sum_{i=1}^{m} (U_i - \bar{U})^2 + \sum_{i=1}^{m} (V_i - \bar{V})^2 \right), \]  

(9c.1.3)

with \( n = 2m - 2 \).

Then, under \( H_0' : \mu_1 = c \mu_2 \), \( T_1 \) follows a t-distribution (central) with \( n \) degrees of freedom while under \( H'_1 : \mu_1 \neq c \mu_2 \), it follows a non-central t-distribution with the same degrees of freedom and non-centrality parameter \( \delta = (\Delta/\sigma) \cdot (m / (2(1 - \phi)))^{1/2} \), where \( \Delta = \mu_1 - c \mu_2 \) (see Lehmann (1959), p.172).
The critical region of the UKP unbiased test of $H_0'$ is given by

$$W = \left\{ |T_1| > t_{\alpha/2; n} \right\},$$  \hspace{1cm} (9c.1.4)

where $t_{\alpha/2; n}$, for given $\alpha$, is obtained from $P(W/H_0') = \alpha$.

Finally, the power $\beta(\delta)$ of the test is obtained as the probability of $W$ under $H_1'$ and given by

$$\beta(\delta) = \left( \exp(-\delta^2/2) \right) \sum_{j=0}^{\infty} \frac{\delta^{2j}}{j!} b(j),$$  \hspace{1cm} (9c.1.5)

where $b(j) = \left( \frac{2^j}{\Gamma(j+\frac{1}{2})} \right) \Gamma(j+\frac{1}{2}) I_{-a} (n/2, j+\frac{1}{2})$,

wherein $a = t_{\alpha/2; n}^2 / (n + t_{\alpha/2; n}^2)$.

9c.2 Method II : Likelihood ratio approach

Let us write $\Theta = \left\{ (\mu_1, \mu_2, \sigma^2) : -\infty < \mu_1, \mu_2 < \infty; \sigma^2 > 0 \right\}$ and $\Theta_0 = \left\{ (\mu, \sigma^2) : -\infty < \mu < \infty; \sigma^2 > 0 \right\}$, $\mu$ being the common (and unknown) value of $\mu_1$ and $\mu_2$ specified by the null hypothesis. The likelihood ratio statistic is then obtained as

$$T_2 = \max_{\Theta} L(\mu, \sigma^2) / \max_{\Theta_0} L(\mu_1, \mu_2; \sigma^2)$$

$$= \left( 1 + (1+\xi)(\bar{X}-\bar{Y})^2 / (2(S_x^2 + S_y^2 - 2\xi S_{xy})) \right)^{-m},$$  \hspace{1cm} (9c.2.1)

where $L(\cdot)$ denotes the likelihood function and $S_{xy}$ is the sample covariance.
It is now easy to see that
\[ T_1/n = \left(\frac{1+\nu}{2}\right) \left(\frac{\bar{X}-\bar{Y}}{S_x+S_y-2\rho S_{xy}}\right)^2, \] (9c.2.2)
which follows a \( \beta(\frac{\nu}{2}, \frac{n-\nu}{2}) \) distribution under \( H_0 \). Again, since \( T_2 = \left(1 + \frac{T_1^2}{n}\right)^{-\frac{n}{2}} \), it follows that the Methods I and II lead to the same testing procedures, and have the same optimum properties.

9c.3 Method III: Use of a Fisher-t type statistic

Let the vector \((X,Y)\) follow a bivariate normal distribution with means \(\mu_1\) and \(\mu_2\), common variance \(\sigma^2\) and correlation coefficient \(\rho\). Also, let \(\rho\) be known while \(\sigma^2\) unknown. Our problem here is to test \(H_0: \mu_1 = \mu_2\) against \(H_1: \mu_1 \neq \mu_2\) on the basis of the incomplete data which may be given by
\[ (X_1,*) , (X_2,*) , \ldots , (X_n,*) \]
and \[ (*,Y_1) , (*,Y_2) , \ldots , (*,Y_m) . \] (9c.3.1)

Such a problem may arise, for example, in cases where the inspection process is destructive, so that observing an \(X\) precludes the observing of the corresponding value of \(Y\), and vice versa. Thus, the pairs are all incomplete. It is clear that neither the Methods I & II, nor the paired-t statistic are applicable in this problem.

One can, however, use the Fisher-t type statistic \(T\), defined in (9a.1.1), to solve the present problem.
The power of the test based on $T$: As the test-statistic, we choose $T^*$ which is related to $T$ by the relation $T^* = cT$, where $c = ((1+\varepsilon)/(1-\varepsilon))^{1/2}$. Then, the distribution of $T^*$ under $H_0$ is obtained from Theorem 9a.1 on putting $\delta=0$. The null distribution of $T^*$ being symmetrical about zero, we take the critical region to be

$$w^* = \left\{ |T^*| > t^*0 \right\}, \quad (9c.3.2)$$

where $t^*0$, for given $\alpha$, is obtained from $P(W^*/H_0) = \alpha$.

The equation determining $t^*0$ can be simplified to obtain

$$\sum_{i=0}^{\infty} q_i(\varepsilon) I_{1-a^*}(n/2+i, \frac{1}{2}) = \alpha, \quad (9c.3.3)$$

where $a^* = t^*0^2/(n + t^*0^2)$.

The equation (9c.3.3) can be solved first to obtain $a^*$ and then $t^*0$, using numerical methods.

The power of the test is defined to be $P(W^*/H_1)$ and is given by

$$f^*(\delta) = \left(\exp(-\delta^2/2)/\sqrt{\pi}\right) \sum_{j=0}^{\infty} \delta^{2j} b^*(j), \quad (9c.3.4)$$

with $b^*(j) = \frac{2^j}{\Gamma(j+\frac{1}{2})} \sum_{i=0}^{\infty} q_i(\varepsilon) I_{1-a^*}(n/2+i, j+\frac{1}{2})$.

The comparison of powers in (9c.1.5) and (9c.3.4) at different alternatives would be of great interest.
The proposed test based on $T^*(i.e., T)$ is unbiased as well as consistent, as shown below.

(a) Unbiasedness: To prove the unbiasedness, we note that the first derivative of $\beta^*(\delta)$, with respect to $\delta$, can be written as

$$\beta^*(\delta) = (\exp(-\delta^2/2)/\pi) \sum_{j=0}^{\infty} r(j) 2^j, \quad (9.3.5)$$

where $r(j) = (2j - \delta^2 - \delta^4)/(\Gamma(j+3/2)/\Gamma(2j)) \sum_{i=0}^{\infty} q_i(r) I_{1-a^*}(n/2+1, j+\frac{1}{2})$.

To solve $\beta^*(\delta) = 0$, we note that a convergent power series which is identically zero has all the coefficients equal to zero. This gives $r(j) = 0$ for all $j$ and, hence, we have $\delta = 0$. Thus, the minimum of $\beta^*(\delta)$ is attained at $\delta = 0$ and we have

$$\beta^*(\delta) \geq \beta^*(0) = b^*(0)/\sqrt{n}. \quad (9.3.6)$$

The unbiasedness finally follows on using $b^*(0) = \alpha\sqrt{n}$.

(b) Consistency: The consistency of the test, namely, the limit $\beta^*(\delta) \rightarrow 1$ as $n \rightarrow \infty$, can be proved on using the fact that as $n \rightarrow \infty$, $a^* \rightarrow 0$, so that $I_{1-a^*}(n/2+1, j+\frac{1}{2})$ tends to unity. This gives the limiting power as

$$\beta^*(\delta) = (\exp(-\delta^2/2)/\sqrt{n}) \sum_{j=0}^{\infty} \delta^{2j} (2^j/\Gamma(2j)) \Gamma(j+3/2). \quad (9.3.7)$$

The result finally follows on using the Legendre's duplication formula which is $\Gamma(2j) = 2^{2j-1} j(j) \Gamma(j+\frac{1}{2})/\sqrt{n}$. 
