SUMMARY: In this chapter, the reliability function $P(X > Y)$ under a stress-strength (SS) model is estimated when $X$ and $Y$ follow, respectively, the exponential and the log-normal distribution. In deriving the uniformly minimum variance unbiased estimator (UMVUE), an interesting relationship between the jackknife estimator and the UMVUE is utilized. The maximum likelihood estimator (MLE) and the Bayes estimator of $R$ are also considered. It is shown that, in some situations, the Bayes estimators of $R$ do not exist. Finally, it is observed that for large sample size the Bayes estimators become close to the MLE's.
In Chapter 7, the problem of estimation of $R = P(X > Y)$ is considered for some useful distributions which have not been covered in the literature so far. The problem of constructing the confidence interval for $R$ is also taken up.

In this chapter, we consider the problem of estimating $R$ when the strength $X$ follows an exponential distribution, while the stress $Y$ follows a log-normal distribution.

The use of the log-normal distribution in reliability study has a long history (see, e.g., Zellner (1971), Padgett and Wei (1977) and Martz and Waller (1972)). The distribution has been used both in the Bayesian and non-Bayesian analysis. It is shown, in this chapter, that there are situations where the Bayes point estimators of $R$ do not exist. The same has also been reported by other researchers working in this area.

In Section 1 of this chapter, the UMVU (uniformly minimum variance unbiased) estimator of $R$ is obtained under the assumption that the distribution of the strength $X$ is known completely. The UMVU estimator is obtained by using some interesting relationships between the jackknife estimator and the UMVU estimator (see Gray et al (1973)).
The assumption of a known strength or stress distribution is rather common in the literature. For example, Church and Harris (1970) have assumed that $X$ and $Y$ are independent normal random variables, with the distribution of $Y$ being known. Similarly, Downton (1973) has obtained an approximate UMVU estimator of $R$ assuming that the distribution of $Y$ is known. Reiser and Guttman (1987) have compared, through a simulation study, three point estimators for $R$ in the normal case. These three estimators of $R$ are the maximum likelihood estimator, UMVU estimator and a predictive estimator calculated from the Behrens-Fisher distribution.

The Bayesian approach, as in the case of other areas of reliability, has found importance in studying the stress-strength relationships. This approach has been used fruitfully to provide estimates for a number of different problems. In particular, Enis and Geisser (1971) have discussed predictive estimators with $X$ and $Y$ both exponentially distributed or both normally distributed with equal variances. Reiser and Guttman (1987) have extended the Enis and Geisser predictive estimator to the case of $X$ and $Y$ independent normally distributed variates with unequal variances.

In Section 2, the maximum likelihood estimators of $R$ are obtained under different considerations. Finally, in Section 3, the Bayes estimators of $R$ are considered for some useful prior distributions on the parameter space. It is shown that, for large sample, the MLE's and the Bayes estimators are identical.
SECTION 1

Suppose that the pdf's of X and Y are given, respectively, by

\[ g_1(x) = \lambda \exp(-\lambda x), \quad (x > 0; \lambda > 0) \quad (1.1) \]

and

\[ g_2(y) = \exp\left(-\frac{\ln y - \xi^2}{2\sigma^2}\right)/(y\sigma\sqrt{2\pi}), \quad (y > 0; -\infty < \xi < \infty, \sigma > 0). \quad (1.2) \]

If, now, the X and Y are assumed to be independent, then we have

\[ P(X > Y) = \int_0^\infty \exp\left(-\lambda y - \frac{(\ln y - \xi)^2}{2\sigma^2}\right)/(y\sigma\sqrt{2\pi}) \, dy \]

\[ = \sum_{j=0}^{\infty} (-\lambda)^j \exp\left(j \frac{\sigma^2}{2} + j\xi\right) / \Gamma(j+1). \quad (1.3) \]

Let \( Y_1, Y_2, \ldots, Y_n \) be a random sample from \( g_2(y) \). If we define \( Z_i = \ln Y_i, \quad i = 1, 2, \ldots, n \), then \( Z_i \) follows \( N(\xi, \sigma^2) \). Also, it will be assumed, in this Section, that the strength distribution is known, so that \( \lambda \) is also known.

Case 1: \( \xi \) is unknown but \( \sigma^2 \) is known.

From (1.3), we have

\[ P(X > Y) = \sum_{j=0}^{\infty} a_j \exp\left(j\xi\right), \quad (1.4) \]
where  \( a_j = (-\lambda)^j \exp\left( j^2 \sigma^2 / 2 \right) / \sqrt{j} \).

It is, however, possible to express \( P(X > Y) \) as a power series in \( \xi \). Thus, we have

\[
P(X > Y) = \sum_{j=0}^{\infty} a_j \left( \sum_{r=0}^{\infty} \frac{(j \xi)^r}{r!} \right) / |r|
\]

\[
= \sum_{r=0}^{\infty} b_r \xi^r
\]

\[
= f(\xi), \text{ say,} \quad (1.5)
\]

where  \( b_r = \left( \sum_{j=0}^{\infty} j^r (-\lambda)^j \exp(j \sigma^2 / 2) / \sqrt{j} / |r| \right) \).

Now, using the Theorem 5 of Gray et al (1973), we have the UMVU estimator of \( R = P(X > I) \) as

\[
\hat{R}_i = \sum_{r=0}^{\infty} b_r \bar{Z}^r + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)}{(2k)} f(\bar{Z}) \left( \sigma^2 / (2n) \right)^k / |k|
\]

where \( \bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i \), and \( f(\bar{Z}) \) is the 2k-th order derivative of \( f(\xi) \), with respect to \( \xi \), and evaluated at \( \xi = \bar{Z} \).

It is easy to see that the 2k-th order derivative of \( f(\xi) \), with respect to \( \xi \), is

\[
d^{2k}f(\xi)/d\xi^{2k} = \sum_{j=0}^{\infty} b_{2k+j} \xi^j \frac{((2k+j))}{j!} \quad (1.7)
\]
The UMVU estimator of $R$ is finally obtained, on using (1.7) in (1.6), as

$$
\hat{R}_1 = f(Z) + \sum_{k=1}^{\infty} (-1)^k \left( \frac{\sigma^2}{2n} \right)^k \left( \frac{2k}{|k|} \right)(1 - \frac{|Z|}{E})^{-|k|} \log b_2k,
$$

(1.8)

where $E$ is the operator defined by $E b_{2k} = b_{2k+1}$.

Case 2: $\xi$ is known but $\sigma^2$ is unknown. Without any loss of generality, let us take $\xi = 0$.

In this case, we have

$$
R = P(X > Y)
= \sum_{r=0}^{\infty} b_r' (\sigma^2)^r = f'(\sigma^2), \text{ say,}
$$

(1.8a)

where $b_r' = \left( \sum_{j=0}^{\infty} a_j' j^{2r} \right)/(2^r |r|)$, and $a_j' = (-1)^j \lambda_j^r / |j|$.

Now, using the Theorem 8 of Gray et al (1973), we have the UMVU estimator of $R$ as

$$
\hat{R}_2 = \sum_{j=0}^{\infty} \left( \frac{S_0^2}{j} \right)^{(n/2)} \left( \frac{\lambda_j^r}{j} \right) f(0)/(\left( \frac{n}{2} \right)_j |j|),
$$

(1.9)
where \( f^{(0)} \) is the \( j \)-th order derivative of \( f(\sigma^2) \), with respect to \( \sigma^2 \) and evaluated at \( \sigma^2 = 0 \), and \( \left( \frac{n}{2} \right)_j \) is defined as
\[
\left( \frac{n}{2} \right)_j = \frac{n}{2} \left( \frac{n}{2} + 1 \right) \ldots \left( \frac{n}{2} + j - 1 \right),
\]
and, also, \( S_o^2 = \sum_{i=1}^{n} \frac{z_i^2}{2} \).

It is easy to see that
\[
\left( \frac{d^j f(\sigma^2)}{d(\sigma^2)^j} \right)_{\sigma^2 = 0} = b^j_j.
\]

Hence, the UMVU estimator of \( R \) is finally obtained as
\[
\hat{R}_o = \sum_{j=0}^{\infty} \left( b^j_j (n/2)^j \right) \frac{S_o^2}{\left( \frac{n}{2} \right)_j}.
\]  \hspace{1cm} (1.10)

Case 3: The parameters \( \xi \) and \( \sigma^2 \) both are unknown.

In this case, we have
\[
R = P(X > Y)
\]
\[
= \sum_{j=0}^{\infty} c_j f_j(\xi, \sigma^2) = g(\xi, \sigma^2), \text{ say},
\]  \hspace{1cm} (1.11)

where \( c_j = (-\lambda)^j |j| \), and \( f_j(\xi, \sigma^2) = \exp(j^2 \sigma^2/2 + j \xi) \).
Now, using the Theorem 10 of Gray et al (1973), we have the UMVU estimator of $R$ as

$$
\hat R_j = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^s \left( S_z^2 \right)^{s+r} n^r r \left( \frac{n-1}{2} \right) g(2s,r)(\bar z,0)/
$$

$$
(2s+r) r'((2s+2r+n-1)/2)), \quad (1.12)
$$

where $\bar z = n^{-1} \sum_{i=1}^{n} z_i, \quad S_z^2 = n^{-1} \sum_{i=1}^{n} (z_i - \bar z)^2$

and $g^{(2s,r)}(\bar z,0) = (\partial / \partial \xi)^{2s} (\partial / \partial \sigma^2)^r g(\xi,\sigma^2)$, being evaluated at $\xi = \bar z, \sigma^2 = 0$.

It is easy to see that

$$
g^{(2s,r)}(\bar z,0) = (\sum_{j=0}^{\infty} c_j j^{2s+2r} \exp(j\bar z))/2^r. \quad (1.13)
$$

Finally, we have from (1.12),

$$
\hat R_j = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} (-1)^s \left( S_z^2 \right)^{s+r} n^r r \left( \frac{n-1}{2} \right) g(2s,r)(\bar z,0)/
$$

$$
(2s+r) r'((s+r+(n-1)/2)))), (\sum_{j=0}^{\infty} c_j j^{2s+2r} \exp(j\bar z))). \quad (1.14)
$$
The above expression of the UMVU estimator of $R$ can be simplified further as shown below:

Let us write $\hat{R}_3$ as

$$\hat{R}_3 = \sum_{j=0}^{\infty} c_j \exp( j \bar{z}) \left( \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \left( j S_z \right)^2 (s+r)^{2(s+r)} \right).$$

Putting $s + r = m$, as

$$\sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{(-1)^{m-r}}{(m-r)!} \frac{2^m}{m!} \frac{r!}{(m-2r)!} \frac{1}{\Gamma(m+(n-1)/2)).}$$

where $\Gamma(\alpha) = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)}$, $\Gamma(\cdot)$ being the gamma function.

An alternative expression for $\hat{R}_3$ can be obtained as

$$\hat{R}_3 = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-3}{2}\right)} \frac{2}{\left((n-1)^{\frac{1}{2}} S_z\right)^{n-3}/2} \sum_{j=0}^{\infty} \left( c_j \exp(j \bar{z}) / j^{(n-3)/2} \right).$$

$$I_{(n-3)/2} \left((n-1)^{\frac{j}{2}} j S_z\right),$$

(1.16)
where $I_\nu(Z)$ is the modified Bessel function and given by

$$I_\nu(Z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k + \nu}}{\Gamma(k + 1)\Gamma(k + \nu + 1)}.$$  \hfill (1.16a)

SECTION 2

In this Section, we consider the maximum likelihood estimator of $R = P(X > Y)$ under different situations.

Case 1: The parameter $\lambda$ is known. Also, $\sigma^2$ is known while $\xi$ is unknown.

In this case, we have

$$R = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} p_r \xi^r \exp(-\lambda \sigma^2 / 2) / j.$$  \hfill (2.1)

Then, by invariance, the MLE of $R$ is obtained as

$$\hat{R} = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} p_r \bar{z}^r,$$

where $\bar{z} = \sum_{i=1}^{n} z_i$.

Case 2: The parameter $\lambda$ is known. Also, $\xi$ is known while $\sigma^2$ is unknown. Without any loss of generality, we take $\xi = 0$. 

In this case, we have

\[ R = \sum_{r=0}^{\infty} q_r (\sigma^2)^r, \quad (2.3) \]

where

\[ q_r = (\sum_{j=0}^{\infty} j^r (-\lambda)^j / j! / (2^r \lambda^r)). \]

Since, the MLE of \( \sigma^2 \), in this case, is \( S_0 = n^{-1} \sum_{i=1}^{n} z_i^2 \), the MLE of \( R \), by invariance, is

\[ \hat{R} = \sum_{r=0}^{\infty} q_r s_0^{2r}. \quad (2.4) \]

Case 3. The parameters \( \lambda, \xi \) and \( \sigma^2 \) are all unknown.

It is to be remembered that the expression for \( R \) is (see Section 1) given by

\[ R = \sum_{j=0}^{\infty} (-\lambda)^j \exp(j\xi + j^2 \sigma^2 / 2) / j! . \]

Now, let \( X_1, X_2, \ldots, X_n \) be a random sample from the strength distribution \( g_1(x) \). Then, the MLE of \( \lambda \) is obtained as

\[ \hat{\lambda} = n \left( \sum_{i=1}^{n} x_i \right)^{-1}. \]

Similarly, the MLE's of \( \xi \) and \( \sigma^2 \) are given by

\[ \hat{\xi} = n^{-1} \sum_{i=1}^{n} z_i, \quad \text{and} \quad \hat{\sigma}^2 = n^{-1} \sum_{i=1}^{n} (z_i - \bar{z})^2. \]
Finally, the MLE of the reliability $R$ is obtained on putting these estimates in the expression of $R$ given above. This also follows from the invariance of the maximum likelihood estimation (see Zehna (1966)).

SECTION 3

In this Section, we consider the Bayes estimators of $R$ under different choices of the prior distribution. In all these cases, the parameter is assumed to be known.

Case 1: Here, $\sigma^2$ is known. Also, $\xi$ is assumed to have a non-informative prior. It is then easy to show that (see Martz et al (1982), p.438) the posterior distribution of $\xi$ is normal with mean $\bar{z}$ and variance $\sigma^2/n$, where $\bar{z}$ is as defined in the preceding Sections.

The Bayes point estimator of $R$, under squared-error loss, is then obtained as

$$E(R|\text{data}) = \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \exp \left( j \bar{z} + (1+1/n)j \sigma^2/2 \right).$$

Case 2: Here, it is again assumed that $\sigma^2$ is known. But, as the prior distribution of $\xi$ we take $N(\lambda_0, \psi_0^2)$, which is the natural conjugate prior for $\xi$. 

\[ \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \exp \left( j \bar{z} + (1+1/n)j \sigma^2/2 \right). \]
It is then easy to show that the posterior distribution of \( \xi \) is \( \mathcal{N}(\lambda^*_0, \gamma^2) \), where

\[
\lambda^*_0 = (\lambda_0 \gamma + n \overline{\gamma}) / (\tau + n \gamma), \quad \gamma^2 = (\gamma + n \gamma)^{-1},
\]

with \( \gamma_0 = \gamma_0^{-2} \) (prior precision) and \( \gamma = \sigma^{-2} \) (sampling precision).

The Bayes point estimator of \( R \), under squared-error loss, is then obtained as

\[
E(R/\text{data}) = \sum_{j=0}^{\infty} \left( (-\lambda)^j / j! \right) \exp(-\lambda^*_0 + j(\sigma^2 + \gamma^2)/2).
\]

(3.2)

Case 3: Here, both \( \xi \) and \( \sigma^2 \) (\( = C^{-1} \)) are assumed to be unknown. Also, following Raiffa and Schlaifer (1961, p. 300), we take a normal-gamma prior for \( (\xi, \gamma) \), which is the natural conjugate prior in this case. Thus, the joint prior of \( (\xi, \gamma) \) is taken to be

\[
h(\xi, \gamma) \propto \gamma^{\alpha-1} \exp(-\beta_0 \gamma - \xi \lambda_0(\xi - \lambda_0)^2/2), (-\infty < \xi < \infty, \gamma > 0).
\]

(3.3)

It is then clear that the joint posterior distribution of \( (\xi, \gamma) \) is also normal-gamma.

Finally, the posterior mean of \( R \) is seen, on simplification, to depend on the integral

\[
\int_0^\infty \gamma^{(\alpha+n)/2 - 1} \exp(-\beta \gamma + \lambda^*_0 \gamma^2 / \gamma) \, d\gamma.
\]

(3.3a)
where \( \beta = \beta_0 + \sum_{i=1}^{n} \frac{\left( z_i - \bar{z} \right)^2}{2} + n \frac{\gamma_r ( \frac{\bar{z} - \lambda_0}{2} )^2}{(2(\gamma_r + n))}, \)

and \( \beta^* = \left( 1 + \frac{\gamma_r + n}{n} \right)^{-1} \). 

As \( \beta^* > 0 \), the integral (3.3a) does not exist, so that the Bayes point estimator of \( R \), under squared-error loss, does not exist in Case 3.

Case 4: Here, it is assumed that both \( \xi \) and \( \sigma^2 \) are unknown and they have the Jeffreys' non-informative prior which is given by

\[
h(\xi, \sigma) \propto \sigma^{-1}. \tag{4.1}
\]

The joint posterior density of \( (\xi, \sigma) \) can then be obtained as

\[
h(\xi, \sigma / \text{data}) = \left( \frac{n}{2\pi} \right)^{\frac{3}{2}} \frac{2}{\Gamma(n_0/2)}^2 \frac{n_0 S_*^2}{2} \frac{n_0}{2} \sigma^{-1} \left( \frac{\xi - \bar{z}}{\sigma} \right)^2, \tag{4.2}
\]

where \( n_0 = n-1 \), and \( S_*^2 = \sum_{i=1}^{n} (z_i - \bar{z})^2 \).

The posterior mean of \( R \), in this case, is seen to depend on the integral

\[
\int_0^\infty t^{n_0 - 1} \exp \left( -t + \Theta/t \right) dt, \tag{4.3}
\]
where \( \Theta = (1 + 1/n) n_0 \frac{s_j}{\sigma^2} \).

Now, as \( \Theta > 0 \), the integral (4.3) does not exist. Hence, the Bayes point estimator of reliability \( R = P(X > Y) \), under squared-error loss, does not exist in the present case.

We have thus identified some situations where the Bayes point estimators of \( R \) do not exist. Also, in situations where they exist, e.g., in Case 1 and Case 2 of Section 3 discussed above, these estimators can be seen to be close to the MLE's for large sample size. This confirms the well-known fact that for large sample the prior gets washed out, and data at hand play the central role in making inference.