CHAPTER IV

DIFFERENCE FUZZY DOUBLE SEQUENCE SPACES

4.1 INTRODUCTION

The notion of difference sequence space for complex terms was introduced by Kizmaz [34], defined as

$$Z(\Delta) = \{ (x_k) : (\Delta x_k) \in Z \}, \text{ for } Z = \ell_\infty, c \text{ and } c_0$$

where \( \Delta x_k = x_k - x_{k+1} \), for all \( k \in \mathbb{N} \).

It has been proved that the above spaces are Banach spaces under the norm given by

$$\| (x_k) \| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$

Savas [58] studied some properties of the fuzzy real-valued difference sequence spaces \( c_r(\Delta) \) and \( (\ell_\infty)_r(\Delta) \).

Let \( D \) be the set of all closed bounded intervals \( X = [X^l, X^r] \). Then we write

\( X \leq Y \) if and only if \( X^l \leq Y^l \) and \( X^r \leq Y^r \)

and

$$d(X, Y) = \max \{ |X^l - Y^l|, |X^r - Y^r| \}.$$ 

Then clearly \( (D, d) \) is a complete metric space.


**4.2 DEFINITION AND PRELIMINARIES**

We recall some of the definitions given earlier in introduction for completeness:

**Definition 4.2.1.** A fuzzy real-valued double sequence \(< X_{mn}>\) is said to be *convergent in Pringsheim's sense* to the fuzzy real number \(X\), if for every \(\varepsilon > 0\), there exists \(m_0 = m_0(\varepsilon)\), \(n_0 = n_0(\varepsilon)\), such that \(\bar{d}(X_{mn}, X) < \varepsilon\), for all \(m \geq m_0\) and \(n \geq n_0\).

**Definition 4.2.2.** A fuzzy real-valued double sequence \(< X_{mn}>\) is said to be *bounded* if

\[
\sup_{m,n} \bar{d}(X_{mn}, 0) < \infty,
\]
equivalently, if there exists \(\mu \in R(I)^*\), such that \(|X_{mn}| \leq \mu\) for all \(m, n \in N\), where \(R(I)^*\) denotes the set of all positive fuzzy real numbers.

**Definition 4.2.3.** A fuzzy real-valued double sequence \(< X_{mn}>\) is said to be *regularly convergent* if it is convergent in Pringsheim's sense and the followings hold:

For any \(\varepsilon > 0\), there exists \(m_1 = m_1(\varepsilon, n)\) and \(n_1 = n_1(\varepsilon, m)\) such that

\[
\bar{d}(X_{mn}, L_n) < \varepsilon, \quad \text{for all } m \geq m_1, \text{ for some } L_n \in R(I), \text{ for each } n \in N,
\]
and

\[
\bar{d}(X_{mn}, M_m) < \varepsilon, \quad \text{for all } n \geq n_1, \text{ for some } M_m \in R(I), \text{ for each } m \in N.
\]

**Definition 4.2.4.** A fuzzy real-valued double sequence space \(E_F\) is said to be *normal (or solid)* if \(< X_{mn}> \in E_F\), whenever \(|X_{mn}| \leq |Y_{mn}|\) for all \(m, n \in N\) and \(< Y_{mn}> \in E_F\).
Now we introduce the notion of step spaces for double sequences of fuzzy real numbers as follows:

**Definition 4.2.5.** Let $K = \{(n, k) : i \in N; n_i < m_2 < n_1 < - - - \text{ and } k_1 < k_2 < k_3 < - - - \} \subseteq N \times N$ and $E_F$ be a fuzzy double sequence space. A $K$-step space of $E_F$ is a sequence space $A_K^F = \{ < X_{n, k} > \in \omega_F : < X_{nk} > \in E_F \}$.

A canonical pre-image of a sequence $< X_{nk} > \in E_F$ is a sequence $< Y_{nk} > \in \omega_F$ defined as follows:

\[
Y_{nk} = \begin{cases} X_{nk}, & \text{if } (n, k) \in K, \\ 0, & \text{otherwise.} \end{cases}
\]

**Definition 4.2.6.** A fuzzy real-valued double sequence space $E_F$ is said to be monotone if $E_F$ contains the canonical pre-images of all its step spaces.

**Definition 4.2.7.** A fuzzy real-valued double sequence space $E_F$ is said to be symmetric if $< X_{\pi(n), \pi(n)} > \in E_F$, whenever $< X_{nn} > \in E_F$, where $\pi$ is a permutation of $N$.

**Definition 4.2.8.** A fuzzy real-valued double sequence space $E_F$ is said to be convergence free if $< X_{nn} > \in E_F$ whenever $< Y_{nn} > \in E_F$ and $Y_{nn} = 0$ implies $X_{nn} = 0$.

Here we introduce the notion of fuzzy real-valued difference double sequence spaces as follows:

\[
Z(\Delta) = \{ < X_{nk} > : < \Delta X_{nk} > \in Z \},
\]

for $Z = (\bar{2} e_{\omega} F), (2 c_0)_{\bar{F}}, (2 c_0)_{\bar{F}}^{BR}, (2 c_0)_{F}^{BR}, (2 c_0)_{F}^{BR}, (2 c_0)_{F}^{BR}, 2 c_0^P, 2 c_0^P, 2 c_0^P, 2 c_0^P, 2 c_0^P, 2 c_0^P, 2 c_0^P, 2 c_0^P, 2 c_0^P$.

where $\Delta X_{nk} = X_{nk} - X_{n+1,k} - X_{n,k+1} + X_{n+1,k+1}$, for all $n, k \in N$. 
4.3. MAIN RESULTS

THEOREM 4.3.1. The spaces $Z(A)$, for $Z = (\ell_\infty)_F$, $(c_0)_F$, $(c_0)^{BR}_F$,

$(c_0)^R_F$, $(c_0)^{BR}_F$, $(c_0)^b_F$, $(c_0)^{BR}_F$ are complete metric space defined by the metric $h(X, Y) = \sup_n d(X_{ni}, Y_{ni}) + \sup_k d(X_{ik}, Y_{ik}) + \sup_{n,k} d(\Delta X_{nk}, \Delta Y_{nk})$.

THEOREM 4.3.2. The spaces $Z(A)$, for $Z = (\ell_\infty)_F$, $(c_0)_F$, $(c_0)^{BR}_F$,

$(c_0)^R_F$, $(c_0)^{BR}_F$, $(c_0)^b_F$, $(c_0)^{BR}_F$ are not symmetric.

THEOREM 4.3.3. The spaces $Z(A)$, for $Z = (\ell_\infty)_F$, $(c_0)_F$, $(c_0)^{BR}_F$,

$(c_0)^R_F$, $(c_0)^{BR}_F$, $(c_0)^b_F$, $(c_0)^{BR}_F$ are not convergence free.

THEOREM 4.3.4. The spaces $Z(A)$, for $Z = (\ell_\infty)_F$, $(c_0)_F$, $(c_0)^{BR}_F$,

$(c_0)^R_F$, $(c_0)^{BR}_F$, $(c_0)^b_F$, $(c_0)^{BR}_F$ are not solid.

PROPOSITION 4.3.5. (i) $Z \subset Z(A)$, for $Z = (\ell_\infty)_F$, $(c_0)^R_F$, $(c_0)^{BR}_F$,

$(c_0)^{BR}_F$, $(c_0)^R_F$, $(c_0)^b_F$, $(c_0)^{BR}_F$, $(c_0)^R_F$, and the inclusions are strict.

(ii) $(c_0)^F \subset (c_0)^F(A)$. The inclusion is strict.

4.4 PROOF OF THE RESULTS OF SECTION 4.3

PROOF OF THE THEOREM 4.3.1. We shall prove it for $(\ell_\infty)_F(A)$.

Let $<X^i>$ be a Cauchy sequence in $(\ell_\infty)_F(A)$. Then for a given $\varepsilon > 0$

there exists $n_0$ such that

$h(X^i, X^j) = \sup_n d(X^i_{ni}, X^j_{ni}) + \sup_k d(X^i_{ik}, X^j_{ik}) + \sup_{n,k} d(\Delta X^i_{nk}, \Delta X^j_{nk})$

$< \varepsilon$, for all $i, j \geq n_0$. 

\[ (4.3.1) \]
\[ \sup_{n} d(X'_{nl}, X'_{nl'}) < \varepsilon, \text{ for all } i, j \geq n_0 \]

\[ \Rightarrow (X'_{nl}) \text{ is a Cauchy sequence in } R(l), \text{ for all } n \in N. \]

Since \( R(l) \) is complete metric space by the metric \( d \), so \( (X'_{nl})_{n=1}^{\infty} \) converges for each \( n \in N \). Let \( \lim_{i \to \infty} X'_{nl} = X_{nl} \), for each \( n \in N \), and similarly from (4.3.1) we have \( (X'_{lk}) \) is a Cauchy sequence and hence convergent. Let \( \lim_{j \to \infty} X'_{lk} = X_{lk} \), for each \( k \in N \). Next from (4.3.1) we have \( (\Delta X'_{nk})_{n=1}^{\infty} \) is a Cauchy sequence for each \( n, k \in N \). Hence \( (\Delta X'_{nk}) \) converges for each \( n, k \in N \).

Let us consider \( (\Delta X'_{nk}) \). Then \( (X'_{nk})) \), \( (X'_{nk}) \) and \( (X'_{nk}) \) are convergent. Hence \( (X'_{nk}) \) converges. Let \( \lim_{i \to \infty} X'_{nk} = X_{nk} \), for all \( n, k \in N \). Taking limit as \( j \to \infty \) in (4.3.1), we have \( d(X', X) < \varepsilon \) for all \( i \geq n_0 \). Now for all \( i \geq n_0 \) we have

\[ h(X, \tilde{0}) = h(X, X') + h(X', \tilde{0}) < \infty. \]

Hence the space is complete.

**PROOF OF THE THEOREM 4.3.2.** The spaces \( Z(\Delta) \), for \( Z = (2^\ell)_{F} \), \( (2c_0)_{F}^{B}, (2c_0)_{F}^{RR}, (2c_0)_{F}^{R}, 2c_{F}^{B}, 2c_{F}^{RR}, 2c_{F}^{R} \) are not symmetric, follows from the following example.

**Example 4.3.1.** Consider the sequence \( <X_{nk}> \) defined as
For $n$ odd and for all $k \in N$,

$$X_{nk}(t) = \begin{cases} 2(1-t) & \text{for } 0 \leq t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $n$ even and for all $k \in N$,

$$X_{nk}(t) = \begin{cases} (2-t) & \text{for } 0 \leq t \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of $X_{nk}(t)$ are given below:

![Graph of $X_{nk}(t)$ for $n$ odd and $n$ even](image)

Then $<X_{nk}> \in Z(\Delta)$, for $Z = (2c_{0})_{F}^{R}, (2c_{0})_{F}^{BR}, (2c_{0})_{F}^{R}, 2c_{F}^{BR}$.

Consider $<Y_{nk}>$ a rearrangement of $<X_{nk}>$ defined as,

For all $k = j^{2}, i \in N$ and for all $n$ odd,

$$Y_{nk}(t) = \begin{cases} (2-t) & \text{for } 0 \leq t \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise

$$Y_{nk}(t) = \begin{cases} 2(1-t) & \text{for } 0 \leq t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $<Y_{nk}> \not\in Z(\Delta)$, for $Z = (2c_{0})_{F}^{R}, (2c_{0})_{F}^{BR}, (2c_{0})_{F}^{R}, 2c_{F}^{BR}$.

Hence the spaces are not symmetric.
PROOF OF THE THEOREM 4.3.3. The spaces $Z(\Delta)$, for $Z = (\ell_\infty)_F$, $(2_{C_0})_F^R$, $(2_{C_0})_F^{BR}$, $(2_{C_0})_F^R$, $(2_{C_0})_F$, $2_{C_F}$, $2_{C_F}^R$ are not convergence free, follows from the following example:

Example 4.3.2. Consider the sequence $\langle X_{nk} \rangle$ defined as,

For $(n + k)$ prime,

$$X_{nk}(t) = \begin{cases} 
1 + (n+k)t, & \text{for } \frac{1}{n+k} \leq t \leq 0, \\
1 - (n+k)t, & \text{for } 0 \leq t \leq \frac{1}{n+k}, \\
0, & \text{otherwise.}
\end{cases}$$

$X_{nk} = 0$, otherwise.

Then $\Delta X_{11} = X_{11} - X_{12} - X_{21}$; and

$$\Delta X_{nk} = X_{nk} - X_{n+1,k+1}, \text{ for } n \neq k;$$

$$= -X_{n,k+1} - X_{n+1,k}, \text{ for } n = k (n, k \geq 2).$$

Graph of $X_{nk}(t)$

$\langle X_{nk} \rangle \in Z(\Delta)$ for $Z = (\ell_\infty)_F$, $(2_{C_0})_F^R$, $(2_{C_0})_F^{BR}$, $(2_{C_0})_F^R$, $(2_{C_0})_F$, $2_{C_F}$, $2_{C_F}^R$. 
Now consider the sequence \(< Y_{nk} >\) defined as,

For \((n + k)\) prime,

\[
Y_{nk}(t) = \begin{cases} 
1 + \frac{t}{n+k}, & \text{for } -(n+k) \leq t \leq 0, \\
1 - \frac{t}{n+k}, & \text{for } 0 \leq t \leq (n+k), \\
0, & \text{otherwise.}
\end{cases}
\]

Then \(\Delta Y_{11} = Y_{11} - Y_{12} - Y_{21}\);

and \(\Delta Y_{nk} = Y_{nk} - Y_{n+1, k+1}\); for \(n \neq k,

= - Y_{n, k+1} - Y_{n+1, k}; \text{ for } n = k (n, k \geq 2).

Thus \(< Y_{nk} > \not\in Z(\Delta), \text{ for } Z = (2c_0)_F, (2c_0)^b_F, (2c_0)^{BR}_F, (2c_0)^b_R, 2c^{BR}_F, 2c^b_F, 2c^b_R, 2c^b_P, 2c^b_P.

Hence the spaces are not convergence free.

**PROOF OF THE THEOREM 4.3.4.** The spaces \(Z(\Delta), \text{ for } Z = (2c_0)_F,\)

\((2c_0)^b_F, (2c_0)^{BR}_F, (2c_0)^b_R, 2c^{BR}_F, 2c^b_F, 2c^b_R\) are not solid, follows from the following example:
Example 4.3.3. Consider the sequence \(<X_{nk}> \) defined as

\[
X_{nk} = \begin{cases} 
1, & \text{for } n \text{ odd, for all } k \in \mathbb{N}, \\
2, & \text{otherwise.}
\end{cases}
\]

Then \(\Delta X_{nk} = 0\), for all \(n, k \in \mathbb{N}\).

Thus \(<X_{nk}> \in Z(\Delta)\) for \(Z=(\ell_\infty)_F, (2c_0)_F, (2c_0)^{BR}_F, (2c_0)^R, 2c_F^{BR}, 2c_F^B, 2c_F^R\).

Now consider the sequence \(<Y_{nk}> \) defined as

\[
Y_{nk} = \begin{cases} 
1, & \text{for } (n+k) \text{ even for all } n, k \in \mathbb{N}, \\
-1, & \text{otherwise.}
\end{cases}
\]

Then \(\Delta Y_{nk} = 4\), for \((n+k)\) even,

\[
\Delta Y_{nk} = -4, \text{ for } (n+k) \text{ odd, for all } n, k \in \mathbb{N}.
\]

Thus \(\|Y_{nk}\| \leq \|X_{nk}\|\) and \(<Y_{nk}> \notin Z(\Delta)\), for \(Z=(\ell_\infty)_F, (2c_0)_F, (2c_0)^{BR}_F, (2c_0)^R, 2c_F^{BR}, 2c_F^B, 2c_F^R\).

Hence the spaces are not solid.

PROOF OF THE PROPOSITION 4.3.5. (i) We shall prove it for \(Z=(2c_0)_F\), other cases follow similarly.

Consider \(<X_{nk}> \in Z, \) for all \(n, k \in \mathbb{N}\), then for a given \(\varepsilon > 0\), such that

\[
d(X_{nk}, \bar{0}) < \frac{\varepsilon}{4}, \text{ for all } n, k \in \mathbb{N}.
\]
Now for all $n, k \in \mathbb{N}$.

\[
\begin{align*}
    d(\Delta X_{nk}, \bar{0}) &= d(X_{nk} - X_{n,k+1} - X_{n,k+1} + X_{n,k+1}, \bar{0}) \\
    &\leq d(X_{nk}, \bar{0}) + d(X_{n,k+1}, \bar{0}) + d(X_{n,k+1}, \bar{0}) + d(X_{n,k+1}, \bar{0}) \\
    &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
    &= \varepsilon.
\end{align*}
\]

The inclusion is strict, follows from the following example:

**Example 4.3.4.** Consider the sequence $<X_{nk}>$ defined by

\[
X_{nk} = \begin{cases} 
    1, & \text{for } n \text{ odd, } n \in \mathbb{N} \text{ for all } k \in \mathbb{N}, \\
    0, & \text{otherwise}.
\end{cases}
\]

Then $\Delta X_{nk} = \bar{0}$, for all $n, k \in \mathbb{N}$.

Thus $<X_{nk}> \in (\mathcal{L}_c)_F(\Lambda)$ whereas $<X_{nk}> \not\in (\mathcal{L}_c)_F$.

Hence the result follows.

(ii) The proof is simple and similar to the previous one. So it is omitted.