Chapter 6

Dynamical behaviour of the soliton formation and propagation in magnetized plasma

6.1 Introduction

Further extended study, in continue to the observation of soliton dynamics in multicomponent plasmas [chapters 2 to 5], this chapter deals the model of a magnetized multicomponent plasmas. Using the reductive perturbation technique, we derive the Zakharov-Kuznetsov (Z-K) equations from basic set of equations. The plasma model is based on the actual observations wherein an appreciable percentage of negative ions are found in laboratory as well as in space plasmas [Goeler(1966), Teichmann(1966), Das and Uberoi(1972), Das(1975), Lonngren(1983)]. Actually few plasmas could be found with a single type of ions. Laboratory plasma is invariably contaminated with multiple ions of different kinds and are also found in ionosphere, magnetosphere and planetary space plasmas. Plasma, in the dynamical system creates the density cavity in which some of the electrons are found to be trapped. Due to which multitemperature electrons are formed in plasmas. This might be common in high intensity laser irradiated plasma, hot cathode discharges as well as in astrophysical plasmas such as in solar corona, earth auroral zone and Van Allen radiation

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belt. Further, evidence shows about the effective role of additional negative charges in plasma waves and the conclusions on the plasma waves without taking the proper account of negative ions will have a serious error [Teichmann(1966), Das and Uberoi(1972), Das(1975), Jones et al.(1975), Nishia and Nagasawa(1986)]. Hence, to relate our present motivation especially in planetary space plasmas as well as in the surrounding atmospheres, we have considered a generalized multicomponent plasma consisting of different kinds of ions and multitemperature electrons. The nonlinear acoustic wave has been studied in such multicomponent magnetized plasma and highlighted the nonlinear wave interaction in soliton formation similar to those believed to occur in laboratory and space plasmas as well as in many other related problems.

The main emphasis has been given to employ the hyperbolic-typed method for finding the soliton features in relation to the laboratory and space plasma environments. In case of failure of the proposed method or to show an alternate method, the study has been furthered to yield the soliton propagation. The features of the nonlinear plasma acoustic waves, which depend on the plasma composition, affect the coexistence of compressive and rarefactive solitary waves. Later, allowing for the higher order nonlinearity in the dynamics leads to further different solitary waves along with double layers. The main aim of the present study is to use a new formalism for finding the soliton propagation from the nonlinear wave equation with strong as well as weak, nonlinearity. The coexistence of different nonlinear acoustic modes due to the interaction of multiple charges in plasma has been shown. Moreover, the theoretical observations revealed many other soliton-like structures, which could be similar to the dip and hump solitons observed by the Freja scientific satellite [Wu et al.(1996)] and the collapsed solitons found in the propagation of solar flares as well as in the interplanetary space plasmas and could be interesting to explain many inherent features of the wave.
6.2 Basic equations and derivation of Z-K equation

To study the soliton characteristics in a generalized multicomponent plasma, we have considered a collisionless magnetized plasma consist of hot and warm electrons and multiple ionic species of different charges. We further, without loss of generality consider an uniform magnetic field along Z-direction i.e. $B_0 = M B_0$. The basic equations governing the plasma, in fluid approximation, are the equations of continuity and motion for ions while the electrons are modified, with $T_e >> T_\alpha$, to the Boltzmannian distribution. Under these broad assumptions, the basic equations are described by the following normalized equations [Baishya et al.(1999)] as :

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha u_\alpha) = 0 \quad (6.1)$$

$$\frac{\partial u_{ax}}{\partial t} + u_{ax} \frac{\partial u_{ax}}{\partial x} + u_{ay} \frac{\partial u_{ax}}{\partial y} + u_{az} \frac{\partial u_{ax}}{\partial z} = q_\alpha \gamma_\alpha \frac{\partial \phi}{\partial x} + u_{ay} \quad (6.2)$$

$$\frac{\partial u_{ay}}{\partial t} + u_{ax} \frac{\partial u_{ay}}{\partial x} + u_{ay} \frac{\partial u_{ay}}{\partial y} + u_{az} \frac{\partial u_{ay}}{\partial z} = q_\alpha \gamma_\alpha \frac{\partial \phi}{\partial y} - u_{az} \quad (6.3)$$

$$\frac{\partial u_{az}}{\partial t} + u_{ax} \frac{\partial u_{az}}{\partial x} + u_{ay} \frac{\partial u_{az}}{\partial y} + u_{az} \frac{\partial u_{az}}{\partial z} = q_\alpha \gamma_\alpha \frac{\partial \phi}{\partial z} \quad (6.4)$$

where $\gamma_\alpha = \frac{m_\alpha}{m_\alpha}$ and $E = - \nabla \phi$ (with usual notations) are defined.

The above equations are supplemented by the Poisson equation

$$\nabla^2 \phi = n_{ei} + n_{eh} + n_j - n_i \quad (6.5)$$

where $\alpha = i, j$ with $q_i = 1$ and $q_j = -1$ respectively for positive and negative ions. $m_\alpha$ is the mass of the ionic species moving with velocity, $u_\alpha(= u_{ax}, u_{ay}, u_{az})$ and the density is $n_\alpha$. The overall charge neutrality condition is maintained throughout the plasma. All other notations are defined elsewhere [chapter-1]
where \( A \) is the phase velocity of the acoustic mode to be determined later in a self consistent manner. The configuration of the propagation is very similar to the dynamics of plasma in many of the astrophysical problems. Actually, it could relate the problems of solar flares where the electron streams are found to propagate as an electromagnetic wave along the solar wind field lines and yield a nature of electrostatic Langmuir wave propagation \[Gurnett(1995)\]. Under certain plasma configurations it expects because of higher intensifying the electric field the solar radio bursts as a nature of soliton structures \[Gurnett(1995)\]. Moreover, it will be seen that it could be a basis for describing other nonlinear wave phenomena explored in space by the scientific satellites as well as by other man-made spacecrafts.

Further, for the use of reductive perturbation technique, the plasma parameters are expanded in terms of power series \[Washimi and Taniuti(1966), Jeffrey and Kawahara(1982)\] in \( \epsilon \) as:

\[
\begin{align*}
n_{el} &= \mu \exp\left(\frac{\phi}{\mu + \nu \beta}\right) \\
n_{eh} &= \nu \exp\left(\frac{\beta \phi}{\mu + \nu \beta}\right)
\end{align*}
\]

with \( \beta = \frac{T_e}{T_{eh}}, \) along with \( \mu \) and \( \nu \); the electron densities at equilibrium state normalized by that of ions (i.e. \( \mu + \nu = 1 \)).

The nonlinear Z-K equation can be best described by using the reductive perturbation technique and needs the following stretched time and space variables

\[
\xi = \epsilon^{\frac{1}{2}} x, \quad \eta = \epsilon^{\frac{1}{2}} y, \quad \zeta = \epsilon^{\frac{1}{2}} (x - \lambda t), \quad \tau = \epsilon^{\frac{3}{2}} t
\]

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\[ n_\alpha = n_\alpha^{(0)} + \varepsilon n_\alpha^{(1)} + \varepsilon^2 n_\alpha^{(2)} + \varepsilon^3 n_\alpha^{(3)} \ldots \]

\[ u_{\alpha r} = \varepsilon u_{\alpha r}^{(1)} + \varepsilon^2 u_{\alpha r}^{(2)} + \varepsilon^3 u_{\alpha r}^{(3)} \ldots ; \quad \tau \equiv (x, y, z) \]

\[ \phi = \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \ldots \quad (6.8) \]

Now following the usual procedure of reductive perturbation technique, the simple mathematical manipulation with the Eqs.(6.1) - (6.6), derives the Z-K equation in the following form

\[ \frac{\partial^2 \phi^{(1)}}{\partial \tau^2} + A \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial}{\partial \zeta} \left[ \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right] = 0 \quad (6.9) \]

where

\[ A = \frac{\lambda}{2} \left[ 3 \sum \frac{\gamma^2 n_\alpha^{(0)} q_\alpha}{\chi^4} - \frac{\mu + \nu \beta^2}{(\mu + \nu \beta)^2} \right], \]

\[ B = \frac{\lambda}{2}, \quad \lambda^2 = \sum \gamma^2 n_\alpha^{(0)} \]

This nonlinear Z-K equation is of interest to describe the evolution of various solitary waves in plasma and analysis of the different properties of solitary waves and double layers which expects the occurrences of such features in space and laboratory plasmas. Again the main aim is to employ a travelling wave solution technique in the form of a modified formalism, known as tanh-method [discussed in chapter-3], to solving the Z-K equation for the new findings on soliton propagation. We now use the following transformations

\[ \chi = \gamma (l \xi + m \eta + n \zeta - U \tau) \quad \text{and} \quad \phi^{(1)}(\xi, \eta, \zeta, \tau) = \psi(\chi) \quad (6.10) \]

with respect to a frame moving with the velocity \( U \) along with the direction cosines \((l, m, n)\) and width = \( \gamma^{-1} \), and the Z-K equation is then reduced to the following form

\[ n \gamma^2 \left[ B n^2 + m^2 + l^2 \right] \frac{d^2 \psi}{d \chi^2} - U \psi + \frac{1}{2} A n \psi^2 = 0 \quad (6.11) \]
This equation has been solved as an elliptic function and based on the earlier knowledge of its soliton solution as a hyperbolic function, we, in order to use the proposed method, substitute $Z = \tanh(\chi)$ and $\psi(\chi) = W(Z)$. Due to which Eq.(6.11) is modified to the form

$$n\gamma^2 \left[ Bn^2 + m^2 + l^2 \right] \left( 1 - Z^2 \right)^2 \frac{d^2W}{dZ^2}$$

$$-2n\gamma^2 \left[ Bn^2 + m^2 + l^2 \right] \left( 1 - Z^2 \right) Z \frac{dW}{dZ} - UW + \frac{1}{2} AnW^2 = 0 \quad (6.12)$$

So, ultimately the basic equations are reduced to a Fuchsian-like nonlinear ordinary differential equation. This differential equation obviously has regular singularities at $Z = \pm 1$ and thus the Frobenius method is applicable for finding the series solution as :

$$W(Z) = \sum_{r=0}^{\infty} a_r Z^{\rho+r} \quad (6.13)$$

The series expansion (6.13) into Eq.(6.12) yields the recurrence relation as :

$$n\gamma^2 \left[ Bn^2 + m^2 + l^2 \right] \left( 1 - Z^2 \right)^2 \sum_{r=0}^{\infty} a_r (\rho + r)(\rho + r - 1) Z^{\rho+r-2}$$

$$-2n\gamma^2 \left[ Bn^2 + m^2 + l^2 \right] \left( 1 - Z^2 \right) Z \sum_{r=0}^{\infty} a_r (\rho + r) Z^{\rho+r-1}$$

$$- U \sum_{r=0}^{\infty} a_r Z^{\rho+r} + \frac{1}{2} An \left[ \sum_{r=0}^{\infty} a_r Z^{\rho+r} \right]^2 = 0 \quad (6.14)$$

The solution for roots of the indicial equation and $a_r$ evaluate the nature of the wave propagation in plasma. But the evaluation follows a tedious and laborious process and thus to avoid it, a short cut procedure in finding $W(Z)$ has been followed. The series (6.13) is truncated to a finite number with $(N + 1)$ terms and $\rho = 0$. Thence, the actual number $N$ in series $W(Z)$ has been evaluated by balancing the leading order analysis i.e. balancing the order of nonlinearity with that of higher order linear term and it is easy to
get $N = 2$. Thus the series solution represents $W(Z) = a_0 + a_1 Z + a_2 Z^2$. The substitution of $W(Z)$ in Eq.(6.12) evaluates the coefficients of the series as:

$$
a_0 = \frac{3U}{An}, \quad a_1 = 0, \quad a_2 = -a_0 \quad \text{and} \quad \gamma = \frac{1}{2} \left[ \frac{U}{n(Bn^2 + m^2 + \mu)} \right]^{1/2}
$$

and consequently the solution of Z-K equation derives

$$
\phi(\xi, \eta, \zeta, \tau) = \frac{3U}{An} \operatorname{sech}^2 \left[ \frac{l \xi + m \eta + n \zeta - U \tau}{\delta} \right] \quad (6.15)
$$

where $\delta = 2 \left[ \frac{(Bn^2 + m^2 + \mu) n}{U} \right]^{1/2}$

This is nothing but the soliton solution of a nonlinear wave equation and depends on nonlinear effect $A$; a function of plasma parameters that analyses the behaviour of solitons. The solution yields a compressive soliton profile until $A > 0$, otherwise it gives, in the region wherein $A < 0$, a rarefactive soliton propagation. These results could be known either dip or hump soliton. A similar observation stimulated in space plasmas and confirmed by the Freja scientific satellite [Wu et al.(1996)]. The findings have been formulated in a plasma model contaminated with additional negative charges and the characteristic behaviour depends on plasma composition. As the negative charge-concentration increases, the plasma density depression cause a smaller effect on the nonlinearity and produces the growth of soliton profile. Due to this, the solitary wave becomes too large at the neighbor of the critical density introduces by the presence of an additional negative charge. Thus, because of the depression of plasma density, the solitary wave either collapses or explodes, as it approaches the region of critical density at which the nonlinearity goes to zero. To execute such phenomena in experiments, the plasma composition needs to be controlled to yield the desired soliton while the observation in astrophysical plasma is of the natural phenomena produced by the plasma density depression. Again, both collapse and explosion in solitons are of the same nature showing large amplitude wave propagation but difference, in collapsed or explosive solitons, follows from either by the energy conservation or not. However, it shows a singularity at $A = 0$, which appears mainly because of the interaction of additional negative charges in soliton propagation.

To avoid the singularity, we have modified the plasma-acoustic pulse with higher amplitude and that allows us to take higher order nonlinearity and expect new findings, as will
be seen, different from the observations made by the Freja scientific satellite. It should be noted also that the higher order nonlinear effects in acoustic mode might be comparable with other important effects in the dynamical system. But our interest is to find the nonlinear effect in isolation and consequently derive the soliton behaviour in the dynamical system. Following the usual technique, a modified Z-K(mZ-K) equation is derived as:

\[
\frac{\partial \phi^{(1)}}{\partial \tau} + \left[A \phi^{(1)} + D \left(\phi^{(1)}\right)^2\right] \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + \frac{\partial}{\partial \zeta} \left[\frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2}\right] = 0 \quad (6.16)
\]

where

\[
D = \frac{\lambda}{4} \left[\frac{15(\sum_{m=1}^{2} \mu_m \eta_{m}^2)}{\lambda^3} - \frac{\mu + \nu \beta^3}{(\mu + \nu \beta)^3}\right].
\]

In contrast to the earlier Z-K equation Eq.(6.9), Eq.(6.16) has the nonlinearity as a combination of cubic and quadratic forms. We differ from the earlier procedure for finding the soliton solution and use another algorithm [Gao and Tian(1996), Tian et al.(1997)] of tanh-typed method to evaluate soliton features. Having in mind that \(sech\Psi\) and \(tanh\Psi\) are the only two hyperbolic functions not diverging for a real \(\Psi \to \pm \infty\), we assume the exact solitonic solution for the Eq.(6.16) as:

\[
\phi^{(1)}(\xi, \eta, \zeta, \tau) = \sum_{i=0}^{L} A_i(\xi, \eta, \zeta, \tau) \cdot tanh[\Psi(\xi, \eta, \zeta, \tau)] + \sum_{j=0}^{J} B_j(\xi, \eta, \zeta, \tau) \cdot sech[\Psi(\xi, \eta, \zeta, \tau)] \cdot tanh[\Psi(\xi, \eta, \zeta, \tau)] \quad (6.17)
\]

where \(L\) and \(J\) are the integers to be determined later, while \(A_i\)'s, \(B_j\)'s and \(\Psi\) are all differentiable functions. Then, we balance the highest-order contributions from the linear and nonlinear terms of Eq.(6.16), so as to get the values of \(L\) and \(J\) for Ansatz (6.17). Next is to substitute Ansatz (6.17) into Eq.(6.16), and then equate to zero all the coefficients of like powers of \(sech(\sim)\) and \(tanh(\sim)\) to obtain a set of partial differential equations, from which we would end up with the explicit expressions for \(A_i\)'s, \(B_j\)'s, \(\Psi\) and/or the constraints among them. As the calculation goes on, in order to drastically simplify the work, we choose the travelling-wave format for \(\Psi\).

The solitary wave solution thus found as:
\[
\phi(\xi, \eta, \zeta, \tau) = \frac{-A}{2D} \pm \sqrt{\frac{3A^2 n + 12DU}{2D^2 n}} \sech \left[ \frac{n\zeta + m\eta - U\tau}{\sqrt{\frac{n(U^2 + m^2 + Bn^2)}{4D^2 + U}}} \right]
\] (6.18)

This soliton solution depends on the nature of A and D which should make the expression, under the radical sign, positive to get the sech-typed soliton profile. The + and − signs represent respectively compressive and rarefactive soliton propagation. Otherwise, its negative value yields the shock-like structure of the plasma-acoustic wave. The root cause for observing the shock-like soliton structure is due to the depression of plasma density by the increase of additional negative charges. Such soliton behaviour might be the ideal cause in relating the path to explain the happenings in solar flares as similar to the radio burst, interaction of the solar wind with the shock causing the radio emissions [Gurnett(1995)] as well as in the surrounding astrophysical problems. The soliton propagation with further higher order nonlinear consideration might be successful in avoiding the shock-like behaviour in plasma wave dynamics. So, a higher order Z-K equation has been derived to relate the high amplitude solitary wave in the dynamical system as:

\[
\frac{\partial \phi^{(1)}}{\partial r} + \left[ A\phi^{(1)} + D (\phi^{(1)})^2 + H (\phi^{(1)})^3 \right] \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + \\
\frac{\partial}{\partial \zeta} \left[ \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right] = 0
\] (6.19)

where \( H = \frac{\lambda}{12} \left[ \frac{105}{\lambda^4} \sum_{\nu=1}^{n} \frac{\nu^4}{\nu^6} - \frac{\mu + \nu \delta^4}{(\mu + \nu \delta)^4} \right] \) is the additional higher order nonlinear contribution.

The algorithm (6.17) leads to the solitary wave solution of the further mZ-K(FmZ-K) equation (Eq.(6.19)) as:

\[
\phi(\xi, \eta, \zeta, \tau) = \frac{-D}{3H} + \left[ \frac{10(U + \frac{Dn^2}{27H^2})}{Hn} \right]^{\frac{1}{3}} \sech^{\frac{1}{3}} \left[ \frac{k\zeta + m\eta + n\xi - U\tau}{\delta} \right]
\] (6.20)
with the width of wave \( \delta = \left[ \frac{4n(Bn^2+m^2+\Omega^2)}{9(U+\frac{2nB}{2\pi})} \right]^{\frac{1}{2}} \), wherein the nonlinear parameters are to be controlled by the relation \( A = \frac{D^2}{3R} \).

In case of strong nonlinearity, where the combination of quadratic and cubic appears, the solution can not be the straightforward as will be seen next. By considering the higher order amplitude, the acoustic wave equation has been derived in a generalized form of a Z-K equation as:

\[
\frac{\partial \phi^{(1)}}{\partial t} + \sum_{r=1}^{p} A_r \left( \phi^{(1)} \right)^r \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + \frac{\partial}{\partial \zeta} \left[ \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right] = 0 \tag{6.21}
\]

where \( p \) is the integer measuring the order of the nonlinearity in the dynamical system. Following the similar procedure of using a linear transformation, Eq.(6.21) is reducible to a standard form and, later by the sech-tanh-method, the soliton solution is derived as a hyperbolic function as:

\[
\phi(\xi, \eta, \zeta, \tau) = -\frac{A_{p-1}}{pA_p} \pm \left[ \frac{(p+1)(p+2)(U-Mn)}{2A_p} \right]^{\frac{1}{2}} sech^p \left[ \frac{l\xi + m\eta + n\zeta - Ut}{\delta} \right] \tag{6.22}
\]

with \( \delta = \left[ \frac{4n(Bn^2+m^2+\Omega^2)}{9(U-Mn)p^2} \right]^{\frac{1}{2}} \), \( M = \sum_{i=1}^{p} A_i \lambda^i \) and \( \lambda = -\frac{A_{p-1}}{pA_p} \).

Now the observations on other modes might be possible by using the same hyperbolic-transformation method, but we are reluctant. Rather we proceed with an alternate procedure to find the soliton propagation of Eq.(6.19) for which the following transformation

\[
\chi = l\xi + m\eta + n\zeta - Ut \quad \text{with} \quad \phi^{(1)} = \psi(\chi) \tag{6.23}
\]

has been used in Eq.(6.19) due to which it has been modified as:

\[
\frac{\left[ \frac{d\psi}{d\chi} \right]^2}{\frac{d\psi}{d\chi}} = a\psi^2 - b\psi^3 - c\psi^4 - h\psi^5 \tag{6.24}
\]

\[
a = \frac{2n}{2n(Bn^2+m^2+\Omega^2)}, \quad b = \frac{An}{6n(Bn^2+m^2+\Omega^2)},
\]

\[
c = \frac{Dn}{12n(Bn^2+m^2+\Omega^2)}, \quad h = \frac{H}{20n(Bn^2+m^2+\Omega^2)}.
\]
Again, under the control of plasma parameters, Eq. (6.24) modifies to the form

$$\left[\frac{d\psi}{d\chi}\right]^2 = k\psi^2(\psi_0 - \psi)^3$$

(6.25)

where all the parameters are defined through the relations \( k = h \), \( 3bh = c^3 \), \( \phi_0 = -\frac{c}{3h} \) along with the usual boundary conditions at \( |\chi| \to \infty \) read as \( \psi \to 0 \), \( \psi' \to 0 \), \( \psi'' \to 0 \). There are various possible procedures could be adopted to find the soliton solution from Eq. (6.25) subjected to the condition of the region where \( \psi(\chi) \) satisfies the initial condition. Now Eq. (6.25) admits the soliton profiles as an hyperbolic function as derived earlier [chapter-2] and depends fully on the condition of the region which are given as:

$$\psi(\chi) = \psi_0 sech^2 \left( \pm \frac{1}{2} \sqrt{k\psi_0(\psi_0 - \psi)} \chi \right)$$

(6.26)

$$\psi(\chi) = \psi_0 cosech^2 \left( \pm \frac{1}{2} \sqrt{k\psi_0(\psi_0 - \psi)} \chi \right)$$

(6.27)

along with the possible coexistences of double layer solution as

$$\psi(\chi) = \psi_0 tanh^2 \left( \pm \frac{1}{2} \sqrt{k\psi_0(\psi_0 - \psi)} \chi \right)$$

(6.28)

These solutions describe respectively the profiles of spiky, explosive or collapse solitary waves along with double layers in plasma-acoustic wave dynamics. It concludes from the observations that the different solitary wave solutions are generally expected due to the presence of additional negative charges in the magnetized plasmas. First of all, the spikes in soliton propagation observed theoretically from the Z-K equation and its characteristics depend fully on the nonlinear coefficients. The observations though could relate the waves in space plasmas as similar to the reports by Kellogg et al. (1992) and to other phenomena in heliosphere but yet to know the structures of nonlinearity playing the role to the evolution process. The explosion or collapse of the solitary wave is related to the growth of the amplitude of wave profile, as well as to the conservation of the energy therein [chapter-4]. However, the explosion or collapse can occur because of the nonlinearity in plasma intensifying electric field [Zakharov(1972)]. The feedback from the nonlinearity.
generated the causes of collapsed soliton and meet the eyes to think whether the soliton structure could be expected as the outcome of the waves observed in association with solar flares e.g. solar radio burst, emission or radiation [Papadopoulos(1978)]. The last solution represents the double layers which have the enormous evidence in space plasmas confirmed by the satellites and other spacecrafts [Raadu(1989)].

The procedure could continue upto any order intensifying the nonlinearity and will be solvable easily if and only if the equation could be reducible to a simplified form as

$$\left[ \frac{d\psi}{dx} \right]^2 = k\psi^2 (\psi_0 - \psi)^{p-2}$$

(6.29)

Otherwise, possible for finding the higher amplitude soliton profile from the nonlinear wave equation could be some how difficult. Now our interest is to describe dynamical behaviour of soliton nature with the modification of nonisothermality in plasma. Following the modified Boltzmannian relation as [Das and Sen(1991), Das et al.(1998)]

$$n_{el} = \mu \exp \left[ \frac{\phi}{\mu + \nu\beta} \right] - b_l \left[ \frac{\phi}{\mu + \nu\beta} \right]^\frac{3}{2}$$

(6.30)

and

$$n_{eh} = \nu \exp \left[ \frac{\beta\phi}{\mu + \nu\beta} \right] - b_h \left[ \frac{\beta\phi}{\mu + \nu\beta} \right]^\frac{3}{2}$$

(6.31)

along with the basic Eqs.(6.1) - (6.5), the reductive perturbation technique derives the modified Z-K (mZ-K) equation in the following form

$$\frac{\partial\phi^{(1)}}{\partial \tau} + C \left( \phi^{(1)} \right)^{\frac{1}{2}} \frac{\partial\phi^{(1)}}{\partial \zeta} + B \frac{\partial^3\phi^{(1)}}{\partial\zeta^3} + \frac{\partial}{\partial \zeta} \left[ \frac{\partial^2\phi^{(1)}}{\partial\zeta^2} + \frac{\partial^2\phi^{(1)}}{\partial\eta^2} \right] = 0$$

(6.32)

The equation, by using the algorithm (6.17), but in different form \( \left( \phi^{(1)} \right)^{\frac{1}{2}} \), derives the soliton profile as:

$$\phi^{\frac{1}{2}}(\xi, \eta, \zeta, \tau) = \frac{30}{C} \frac{(l^2 + m^2 + Bn^2)}{sech^2[n\zeta + l\xi + m\eta - 16n(l^2 + m^2 + Bn^2)\tau]}$$

(6.33)
For weak nonlinearity, the electron density is derived [Das and Sen(1991)] as:

\[ n_{el} = \mu \exp \left[ \frac{\phi}{\mu + \nu \beta} \right] - \epsilon \frac{1}{2} b_1 \left[ \frac{\phi}{\mu + \nu \beta} \right]^{\frac{3}{2}} \]  \hspace{1cm} (6.34)

and

\[ n_{ch} = \nu \exp \left[ \frac{\beta \phi}{\mu + \nu \beta} \right] - \epsilon \frac{1}{2} b_2 \left[ \frac{\beta \phi}{\mu + \nu \beta} \right]^{\frac{3}{2}} \]  \hspace{1cm} (6.35)

and correspondingly the modified Z-K equation has been derived as:

\[ \frac{\partial \phi^{(1)}}{\partial \tau} + \left[ A \phi^{(1)} + C(\phi^{(1)})^{\frac{1}{2}} \right] \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial}{\partial \zeta} \left[ \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right] = 0 \]  \hspace{1cm} (6.36)

The above form of equation with its soliton solution has been discussed in different configuration by different method elsewhere [Das and Sen(1991)]. We solved it by using the algorithm (6.17) for \( \left( \phi^{(1)} \right)^{\frac{1}{2}} \) and gives rise to a soliton profile as:

\[ \phi^{\frac{1}{2}}(\xi, \eta, \zeta, \tau) = -\frac{2 C}{5 A} + \frac{\epsilon \sqrt{8} C}{5 A} \text{sech} \left( \frac{16 C^2 n \tau}{75 A} + n \zeta + a \zeta \sqrt{\frac{2 C^2}{75 A} - m^2 - B n^2 + m \eta} \right) \]  \hspace{1cm} (6.37)

where \( \epsilon = \pm 1 \) and \( a = \pm 1 \) respectively, giving 4 families of solutions.

We further modify the Boltzmannian relations with further higher order nonisothermality in the following forms

\[ n_{el} = \mu \exp \left( \frac{\phi}{\mu + \nu \beta} \right) - \epsilon \frac{1}{2} b_1 \left( \frac{\phi}{\mu + \nu \beta} \right)^{\frac{3}{2}} - \epsilon^{-\frac{1}{2}} b_2 \left( \frac{\phi}{\mu + \nu \beta} \right)^{\frac{3}{2}} \]  \hspace{1cm} (6.38)

\[ n_{ch} = \nu \exp \left( \frac{\beta \phi}{\mu + \nu \beta} \right) - \epsilon \frac{1}{2} b_1 \left( \frac{\beta \phi}{\mu + \nu \beta} \right)^{\frac{3}{2}} - \epsilon^{-\frac{1}{2}} b_2 \left( \frac{\beta \phi}{\mu + \nu \beta} \right)^{\frac{3}{2}} \]  \hspace{1cm} (6.39)

a further mZ-K (FmZ-K) equation has been derived as:

\[ \frac{\partial \phi^{(1)}}{\partial \tau} + \left[ C(\phi^{(1)})^{\frac{1}{2}} + A \phi^{(1)} + K(\phi^{(1)})^{\frac{1}{2}} \right] \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial}{\partial \zeta} \left[ \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right] = 0 \]  \hspace{1cm} (6.40)
Eq. (6.40) admits a couple of soliton solutions according to the algorithm (6.17), and the first of which is:

\[
\phi^\frac{1}{2}(\xi, \eta, \zeta, \tau) = \frac{7A}{48K} \left[ -2 + 20^\frac{1}{2} \text{sech}^\frac{3}{2} \left( \frac{49A^3n\tau}{1152K^2} + n\zeta + \frac{\sqrt{49A^3}}{6144K^2} - m^2 - Bn^2 + m\eta \right) \right]
\]

(6.41)

with the constraint on the plasma parameters as: \( C = \frac{35A^2}{96K} \), while other is

\[
\phi^\frac{1}{2}(\xi, \eta, \zeta, \tau) = \left[ \frac{70(l^2 + m^2 + Bn^2)}{9K} \right]^\frac{1}{2} \text{sech}^\frac{3}{2} \left[ n\zeta + l\xi + m\eta - \frac{16n(l^2 + m^2 + Bn^2)}{9} \tau \right]
\]

(6.42)

with the constraints on the plasma parameters as \( A = C = 0 \).

In this way, the nonisothermality effect continue to the generalized form as:

\[
\frac{\partial \phi^{(1)}}{\partial \tau} + \sum_{r=1}^{n} A_r \left( \frac{\partial \phi^{(1)}}{\partial \zeta} \right)^{\frac{1}{2}} \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + \partial \left[ \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \phi^{(1)}}{\partial \eta^2} \right] = 0
\]

(6.43)

correspondingly, following the hyperbolic-method, the soliton solution evaluates the profile as a \( \text{sech}(-\cdot) \) function with a different nature of soliton propagation as:

\[
\phi^\frac{1}{2}(\xi, \eta, \zeta, \tau) =
\]

\[
\left[ \frac{(p + 4)}{(p + 1)(p + 3)} \frac{A_{p-1}}{A_p} \pm \frac{(p + 2)(p + 4)(\frac{1}{2} - Mn)}{4A_p} \right]^{\frac{1}{p}} \text{sech}^{\frac{2}{p}} \left[ \frac{l\xi + m\eta + n\zeta - U\tau}{\delta} \right]
\]

(6.44)

where

\[
\delta = \left[ \frac{8n(Bn^2 + m^2 + p^2)}{p^2(\frac{1}{2} - Mn)} \right]^{\frac{1}{2}} \text{with } M = \sum_{i=1}^{p} \frac{2(i+1)A_i^\lambda A_i}{i+4} \text{ and } \lambda = -\frac{p+4}{(p+1)(p+3)} \frac{A_{p-1}}{A_p}
\]

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6.3 Results and discussions

The present study derives various form of Zakharov-Kuznetsov equation in an isothermal magnetized plasma with different types of ions. The hyperbolic transformation-typed method yields the coexistences of various solitary wave propagation showing the interaction of multiple charges with the solitons in magnetized plasma. The analysis of the Z-K wave equation with higher order nonlinearity revealed the spiky and explosive solitary waves along with double layers. The different features of compressive and rarefactive solitary waves are shown to be functions of plasma parameters. It is also shown that it, at higher order nonlinearity, derives other forms of solitary waves. The theoretical observations on the formation of spikes in solitons, collapsed or explosive solitons and double layers have a closed relation to those observed by the satellites in heliosphere as well as in surrounding astrophysical problems. We will not be far off if we believe in the possible interlinking of the soliton formation with the wave phenomena in solar flares e.g. solar radio burst or solar radio emission produced by the interaction of solar wind with shock structure, radiation because of the depression produced in plasma density. Moreover, in contrast to the earlier studies, the nonlinear wave equation with different form of nonlinearity has been solved by using the hyperbolic-method. The main aim is to develop a hyperbolic-method to exhibit the coexisting of different nonlinear acoustic waves. The Z-K equation has been derived too in nonisothermal plasma with the high amplitude pulses and have shown convincingly the use of the method for finding spiky or explosive solitary waves as well as double layers. The totality of the investigations discusses the effect of strong as well as weak, nonlinearity on the existences of different acoustic modes in magnetized plasma. The proposed technique yields successfully the soliton features and does not depend on the order of nonlinearity appearing in the dynamical system. Finally, we conclude that the theoretical attempt shows a large class of solitons through the Z-K equation and the observations can relate the explanation on the existence of dip and hump solitons, spikes in soliton, double layers, collapsed or explosive solitons, shock-like structures of the solitary waves.
References

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