FUZZY BELIEF, FUZZY PLAUSIBILITY AND RELATED CONCEPTS

2.1. INTRODUCTION

Normally, imprecision and indeterminacy are supposed to be statistical, random characteristics. Both these are taken into account by the methods of probability theory. Probability is an objective characteristic; the conclusions of probability theory can be tested experimentally.

The motivation for the development of fuzzy statistics is its philosophical and conceptual relation to subjective probability. Considering subjectively, probability represents the 'degree of belief' that a given person has in a given event on the basis of given evidence. James Bernoulli (1913), in his article 'Ars Conjectandi,' defines probability as a degree of confidence in a proposition of whose truth we cannot ascertain. His "degree of confidence" is identified with the probability of an event, which depends on the knowledge that the individual has in hand. The representation of a degree of belief by a probability function was reconsidered by Shafer in 1976 and Shafer introduced the concept of belief functions to quantify one's degree of belief in the truth of a proposition. [17]. Associated with every belief function is another function called its plausibility function. Shafer, in his article "Belief functions and Possibility Measures", describes the theory of belief functions and studies the rules for consonant plausibility functions.

In Shafer's work, belief functions are defined for non-fuzzy propositions that belong to a Boolean algebra of propositions defined on a referential Q.
Smets, in his work [18], extended the definition of belief functions to fuzzy propositions defined on Q and proved that such definition obeys to the definition of belief functions.

In this chapter, we introduce the concept of fuzzy belief and fuzzy plausibility. i.e. we fuzzify the three functions - belief function, plausibility function and basic probability assignment function by assigning values in the set of fuzzy numbers. All the three functions are defined for propositions that belong to a Boolean algebra of propositions defined on a referential Q and the properties of these functions are studied. This work is based on the work done by Dr. Philippe Smets [18], and Dr. Wang. Z and Dr. George. J. Klir [21].

2.2 **PRELIMINARIES**

In this section, we just have a look into the basic definitions and properties of belief functions and plausibility functions. [14].

2.2.1 **Defn.**

Let Q be a referential space and \( \mathcal{P} \) be a Boolean algebra of propositions 'A' defined on Q. A function \( \text{Bel} : \mathcal{P} \rightarrow [0,1] \) is a belief function if

1. \( \text{Bel} (\emptyset) = 0 \)
2. \( \text{Bel} (Q) = 1 \)

\&

3. \( \forall n \geq 1, \ A_1, A_2, \ldots, A_n \in \mathcal{P}, \text{Bel} (\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \text{Bel} (A_i) - \sum_{i<j}^{n} \text{Bel} (A_i \cap A_j) - \ldots - (-1)^n \text{Bel} (\bigcap_{i=1}^{n} A_i) \)
2.2.2. Defn:

Associated with each belief function ‘Bel’ there is a plausibility function ‘Pl’ defined by

\[ \text{Pl} : \mathcal{P} \rightarrow [0,1] \]

such that

1) \( \text{Pl} (\phi) = 0 \)

2) \( \text{Pl} (Q) = 1 \)

&

3) \( \forall n \geq 1, A_1, A_2, \ldots, A_n \in \mathcal{P} \)

\[ \text{Pl} \left( \bigcap_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} \text{Pl} \left( A_i \right) - \sum_{i>j}^{n} \text{Pl} \left( A_i \cup A_j \right) - \cdots - (-1)^n \text{Pl} \left( \bigcup_{i=1}^{n} A_i \right) \]

Belief functions and plausibility functions can be characterised by a function

\[ m : \mathcal{P} \rightarrow [0,1] \]

such that

1) \( m (\phi) = 0 \)

2) \( \sum_{A \in \mathcal{P}} m(A) = 1 \)

This function ‘m’ is called a basic probability assignment.

2.2.3. Defn:

Every set \( A \in \mathcal{P} \) for which \( m(A) > 0 \) is called a focal element of \( m \); and the pair \((F,m)\) where \( F \) and \( m \) denote a set of focal elements and the associated basic assignment, respectively, is called a body of evidence.
Given a basic assignment \( 'm' \), a belief function and a plausibility function are uniquely determined for all sets \( A \in \mathcal{P} \) by the formulas

\[
\text{Bel} (A) = \sum_{B \subseteq A} m(B) \\
\text{Pl} (A) = \sum_{B / A \cap B \neq \emptyset} m(B)
\]

The relationship between \( m(A) \), \( \text{Bel} (A) \) and \( \text{Pl} (A) \) can be expressed as follows:

\( m(A) \) represents the degree of belief that the element in question belongs exactly to set \( A \); while \( \text{Bel} (A) \) represents the total evidence or belief that the element belongs to \( A \) as well as to the various special subsets of \( A \). The plausibility function represents not only the total evidence or belief that the element in question belongs to set \( A \) or to any of its subsets, but also the additional evidence or belief associated with sets that overlap with \( A \). Hence,

\[
\text{Pl} (A) \geq \text{Bel} (A) \quad \text{for all } A \in \mathcal{P}
\]

Belief functions and plausibility functions satisfy the following properties [14].

1) \( \text{Bel} (A) \leq \text{Bel} (\overline{A}) \) for all \( A,B \in \mathcal{P} \) with \( A \subseteq B \)

2) \( \text{Bel} (A) + \text{Bel} (\overline{A}) \leq 1, \forall A \in \mathcal{P} \)

3) \( \text{Pl} (A) + \text{Pl} (\overline{A}) \geq 1, \forall A \in \mathcal{P} \)

4) \( \text{Bel} (A) \leq \text{Pl} (A), \forall A \in \mathcal{P} \)
2.3. **INTRODUCTION TO FUZZY BELIEF**

In this section, we introduce the concept of fuzzy belief. Fuzzy belief functions are defined for propositions belonging to a Boolean algebra $\mathcal{P}$ defined on a referential $Q$ and the properties of fuzzy belief functions are studied.

Considering the applications of belief functions, we find that fuzzy belief functions serve as a better tool in representing epistemic probabilities. The application of fuzzy belief functions in medical diagnosis is dealt, in detail, in the last chapter.

2.3.1. **Defn:**

Let $Q$ be a referential space and $\mathcal{P}$ be a Boolean algebra of propositions ‘$A$’ defined on $Q$. A function

$$\text{FB} : \mathcal{P} \to [0, 1]$$

is a fuzzy belief function, if

1. $\text{FB} (\emptyset) = 0$ (fuzzy number zero)
2. $\text{FB} (Q) = 1$ (fuzzy number one)

\&

3. \( \forall n \geq 1, \ A_1, A_2, \ldots, A_n \in \mathcal{P}, \)

$$\text{FB} \left( \bigcup_{i=1}^{n} A_i \right) \geq \sum_{i=1}^{n} \text{FB} (A_i) - \sum_{i<j}^{n} \text{FB} (A_i \cap A_j) - \cdots - (-1)^n \text{FB} \left( \bigcap_{i=1}^{n} A_i \right)$$

For each $A \in \mathcal{P}$, $\text{FB} (A)$ can be considered as the degree of fuzzy belief allocated to the occurrence of the event $A$ and is a fuzzy set in the unit interval $[0, 1]$. ie, $\text{FB}(A)$ corresponds to a fuzzy number.
2.3.2 Note

We restrict to fuzzy belief functions being represented by triangular fuzzy numbers.

A fuzzy belief function can also be defined in terms of a fuzzy basic probability assignment.

2.3.3 Defn:

Let \( Q \) be a referential space and \( \mathcal{P} \) be a Boolean algebra of a propositions ‘A’ defined on \( Q \). A fuzzy basic probability assignment is a function

\[
m: \mathcal{P} \rightarrow I
\]

such that

1) \( m(\phi) = 0 \)

&

2) \( \sum_{E \in \mathcal{P}} m(E) = 1 \)

2.3.4. Defn:

If ‘\( m \)’ is a fuzzy basic probability assignment on \( \mathcal{P} \), then the fuzzy belief function.

\[
FB_m : \mathcal{P} \rightarrow I
\]

induced by ‘\( m \)’, is defined by

\[
FB_m (E) = \sum_{F \subseteq E} m(F) \text{ for any } E \in \mathcal{P}.
\]
Next, we state the following lemma, which is necessary for proving the
forthcoming theorems.

2.3.5. **Lemma [21]**

If $E$ is a non-empty finite set, then

$$|F|$$

$$\sum (-1) = 0, \text{ where } |F| \text{ denotes the cardinality of } F.$$  

$F \subset E$

2.3.6. **Theorem**

If $FB_m : \mathcal{P} \to \mathcal{I}$ is a fuzzy belief function induced by the fuzzy
basic probability assignment ‘m’, then

1) $FB_m (\emptyset) = 0$

2) $FB_m (Q) = 1$

$\&$

3) $FB_m \left( \bigcup_{i=1}^{n} E_i \right) \geq \sum_{i \in I \subseteq \{1, 2, \ldots, n\} \setminus \{i \neq \phi\}} (-1)^{|I|+1} FB_m (\bigcap_{i \in I} E_i)$

Proof

$FB_m (\emptyset) = \sum_{F \subseteq \emptyset} m(F) = m(\emptyset) = 0$

Hence (1).

$FB_m (Q) = \sum_{F \subseteq Q} m(F)$

$= \sum_{F \in \mathcal{P}} m(F) = 1$

Hence (2).
Now, we show that (3) holds.

Consider

\[ \sum_{I \subset \{1, 2, \ldots, n\}} (-1)^{|I|} F_B m \left( \bigcap_{i \in I} E_i \right) \]

\[ = \sum_{I \subset \{1, 2, \ldots, n\}} (-1)^{|I|} \sum_{F \subset \bigcap_{i \in I} E_i} m(F) \]

Arbitrarily given a finite subclass \( \{ E_1, E_2, \ldots, E_n \} \) of \( \mathcal{P} \), set

\( I(F) = \{ i / 1 \leq i \leq n, \ F \subset E_i \} \) for any \( F \in \mathcal{P} \).

Then,

\[ \sum_{I \subset \{1, 2, \ldots, n\}} (-1)^{|I|} \sum_{F \subset \bigcap_{i \in I} E_i} m(F) \]

\[ = \sum_{I \subset \{1, 2, \ldots, n\}} (-1)^{|I|} \sum_{F \subset E_i, \ i \in I} m(F) \]

\[ \leq \sum_{F/F \subset E_i \text{ for some } i} m(F) \]

\[ \leq \sum_{F \subset \bigcup_{i=1}^n E_i} m(F) \quad \text{for some } i \]

\[ = F_B m \left( \bigcap_{i=1}^n E_i \right) \]

Hence (3). Hence, the theorem.
2.3.7. Lemma [21]

If $E$ is a finite set, $F \subseteq E$ and $F \neq E$, then

$$\sum_{G/F \subseteq G \subseteq E} |G| (-1) = 0$$

2.3.8. Lemma

Let $Q$ be finite, $\lambda$ and $\nu$ be fuzzy set functions defined on finite sets belonging to $\mathcal{P}$. Then, we have

$$\lambda(E) = \sum_{F \subseteq E} \nu(F) \text{ for any } E \in \mathcal{P} \rightarrow (A)$$

iff

$$\nu(E) = \sum_{F \subseteq E} (-1)^{|E-F|} \lambda(F) \text{ for any } E \in \mathcal{P} \rightarrow (B)$$

Proof

If $(A)$ is true, then

$$\sum_{F \subseteq E} (-1)^{|E-F|} \lambda(F) = (-1)^{|E|} \sum_{F \subseteq E} (-1)^{|F|} \lambda(F)$$

$$= (-1)^{|E|} \left[ \sum_{F \subseteq E} (-1)^{|F|} \sum_{G \subseteq F} \nu(G) \right] \quad [\text{by } (A)]$$

$$= (-1)^{|E|} \sum_{G \subseteq E} \left[ \nu(G) \left( \sum_{F \subseteq E} (-1)^{|F|} \right) \right]$$

$$= (-1)^{|E|} \nu(E) (-1) \quad \text{(by lemma 2.3.7)}$$

$$= \nu(E)$$

Conversely, if $(B)$ is true, then
\[
\sum_{F \subseteq E} \nu(F) = \sum_{F \subseteq E} \sum_{G \subseteq F} (-1)^{|F-G|} \lambda(G) \quad \text{[by (B)]}
\]
\[
= \sum_{G \subseteq E} \left[ (-1)^{|G|} \lambda(G) \sum_{F/G \subseteq E} (-1)^{|F|} \right]
\]
\[
= (-1)^{|E|} \left[ (-1)^{|E|} \right] = \lambda(E)
\]

Hence, the lemma.

Using the above lemma, we prove the following theorem.

2.3.9. Theorem

Let \( Q \) be finite. If a fuzzy set function \( \mu : \mathcal{P} \rightarrow \mathcal{I} \) satisfies the conditions

1) \( \mu(\phi) = 0 \)

2) \( \mu(Q) = 1 \)

&

3) \( \mu\left( \bigcup_{i=1}^{n} E_i \right) \geq \sum_{I \subseteq \{1, 2, \ldots, n\}} (-1)^{|I|+1} \mu\left( \bigcap_{i \in I} E_i \right) \),

where \( \{E_1, E_2, \ldots, E_n\} \) is any finite subclass of \( \mathcal{P} \), then the fuzzy set function \( 'm' \) determined by

\[
m(E) = \sum_{F \subseteq E} (-1)^{|E-F|} \mu(F) \quad \text{for any } E \in \mathcal{P}
\]

is a fuzzy basic probability assignment, and \( '\mu' \) is just the fuzzy belief function induced by \( 'm' \).

ie, \( \mu(E) = FB_m(E) = \sum_{F \subseteq E} m(F) \).
Proof

First, we show that the set function ‘m’ determined by

\[ m(E) = \sum_{F \subseteq E} (-1)^{|E-F|} \mu(F) \]

satisfies the conditions of a fuzzy basic probability assignment.

\[ m(\emptyset) = \sum_{F \subseteq \emptyset} (-1)^{|\emptyset-F|} \mu(F) = \mu(\emptyset) = 0 \]

Next, we show that \( \sum_{E \in \mathcal{P}} m(E) = 1 \)

\[
\sum_{E \in \mathcal{P}} m(E) = \sum_{E \in \mathcal{Q}} \left[ \sum_{F \subseteq E} (-1)^{|E-F|} \mu(F) \right] \\
= \sum_{F \subseteq Q} (-1)^{|F|} \mu(F) \sum_{E/F \subseteq E \in \mathcal{Q}} (-1)^{|E|} \\
= (-1)^{|Q|} \mu(Q) (-1) \quad \text{(By lemma 2.3.7)} \\
= \mu(Q) = 1
\]

Hence, ‘m’ is a fuzzy basic probability assignment on \( \mathcal{P} \).

Now, we show that ‘\( \mu \)’ is the fuzzy belief function induced by ‘m’

Consider \( \sum_{F \subseteq E} m(F) \)

\[
= \sum_{F \subseteq E} \left[ \sum_{G \subseteq F} (-1)^{|F-G|} \mu(G) \right] \\
= \sum_{G \subseteq E} (-1)^{|G|} \mu(G) \sum_{F/G \subseteq F \subseteq E} (-1)^{|F|} \mu(E) \\
= (-1)^{|E|} \mu(E) (-1)
\]
= \mu(E)

Hence, the theorem.

Now, we consider some properties of fuzzy belief functions.

2.3.10. Property

Let \( A, B \in \mathcal{P} \) with \( A \subseteq B \). Then, \( FB(A) \leq FB(B) \).

Proof

Let \( C = B - A \)

Then, \( A \cup C = B \) and \( A \cap C = \emptyset \)

Consider \( FB(A \cup C) \).

By defn 2.3.1,

\[
FB(A \cup C) \geq FB(A) + FB(C) - FB(A \cap C)
\]

ie, \( FB(B) \geq FB(A) + FB(C) - FB(\emptyset) \)

\( \Rightarrow FB(B) \geq FB(A) + FB(C) \quad (\therefore FB(\emptyset) = 0) \)

\( \Rightarrow FB(B) \geq FB(A) \quad (\therefore FB(C) \geq 0) \)

So, whenever \( A \subseteq B \), \( FB(A) \leq FB(B) \).
2.3.11. Property

\[ FB(A) + FB(\overline{A}) \leq 1, \quad \forall A \in \mathcal{P} \]

Proof

By defn. 2.3.1.

\[ FB(A \cup \overline{A}) \geq FB(A) + FB(\overline{A}) - FB(A \cap \overline{A}) \]

ie, \[ FB(Q) \geq FB(A) + FB(\overline{A}) - FB(\phi) \]

ie, \[ 1 \geq FB(A) + FB(\overline{A}) \]

ie, \[ FB(A) + FB(\overline{A}) \leq 1. \]

Hence, the result.

2.4. FUZZY PLAUSIBILITY FUNCTIONS

Shafer proposed the plausibility of an event ‘A’ as a function which expresses the extent to which one fails to doubt in ‘A’.

ie, \[ Pl(A) = 1 - Dou(A) \]

In other words, \( Pl(A) \) measures the amount of belief that might be assigned to ‘A’.

Shafer proposed

\[ Pl(A) = 1 - Dou(A) = 1 - Bel(\overline{A}). \]

Thus, in the crisp case, the axioms for belief functions could evidently be based on plausibility functions.
Smets, in his work ([19]), describes the method for computing the a-posteriori degrees of belief and plausibility that a patient presenting a particular symptom belongs to a fuzzy diagnostic group.

In this section, we define the notion of the degree of fuzzy plausibility. The plausibility functions defined for propositions belonging to a Boolean algebra defined on a referential space are fuzzified by assigning values in the set of fuzzy numbers (c.f. defn. 1.4.1). We also study the properties of fuzzy plausibility functions and then find that the axioms for fuzzy belief functions do not depend on fuzzy plausibility functions. An application of fuzzy plausibility function in medical diagnosis is given in the last chapter.

2.4.1. Defn.

Let \( Q \) be a referential space and \( \mathcal{P} \) be a Boolean algebra of propositions 'A' defined on \( Q \). A function

\[
FPI : \mathcal{P} \rightarrow [0,1]
\]

is a fuzzy plausibility function, if

1) \( FPI (\emptyset) = 0 \)

2) \( FPI (Q) = 1 \)

&

3) For each event \( A \in \mathcal{P} \), \( FPI (A) \) represents the degree of fuzzy plausibility allocated to 'A' and is a fuzzy set in the unit interval [0,1].

ie, \( FPI (A) \) is a fuzzy number.
2.4.2. **Note**

We restrict to fuzzy plausibility functions being represented by triangular fuzzy numbers.

A fuzzy plausibility function can also be defined in terms of a fuzzy basic probability assignment ‘m’ (defined by defn.2.3.3) as follows:

2.4.3. **Defn.**

If ‘m’ is a fuzzy basic probability assignment on $\mathcal{P}$, then the fuzzy set function

$$F_{\text{PI}}_m : \mathcal{P} \rightarrow I^i$$

determined by

$$F_{\text{PI}}_m (E) = \sum_{F \cap E \neq \phi} m(F) \text{ for any } E \in \mathcal{P}$$

is called a fuzzy plausibility function induced from ‘m’

2.4.4. **Theorem**

If $F_{\text{PI}}_m : \mathcal{P} \rightarrow I^i$ is a fuzzy plausibility function induced from the fuzzy basic probability assignment ‘m’, then

1. $F_{\text{PI}}_m (\emptyset) = 0$
2. $F_{\text{PI}}_m (\mathcal{P}) = 1$

&

3. $F_{\text{PI}}_m (\bigcap_{i=1}^n E_i) \leq \sum_{\{I \in \mathcal{P} : I \neq \emptyset \}} (-1)^{|I|^{+i}} F_{\text{PI}}_m (\bigcup_{i \in I} E_i)$
where \( \{ E_1, E_2, \ldots, E_n \} \) is any finite subclass of \( \mathcal{P} \).

**Proof**

\[
FP_{m} (\phi) = \sum_{F \cap \phi \neq \phi} m(F)
= m(\phi) = 0
\]

Hence (1)

\[
FP_{m} (Q) = \sum_{F \cap Q \neq \phi} m(F)
= \sum_{F \in \mathcal{P}} m(F) = 1
\]

Hence (2).

Now, we show that (3) holds.

Consider

\[
\sum_{I \subseteq \{1, 2, \ldots, n\}} (-1)^{\mid I \mid + 1} \sum_{F \cap (\bigcup_{i \in I} E_i) \neq \phi} m(F)
\]

Arbitrarily given a finite subclass \( \{ E_1, E_2, \ldots, E_n \} \) of \( \mathcal{P} \), set

\( I(F) = \{ i \mid 1 \leq i \leq n, \; F \cap E_i \neq \phi \} \) for any \( F \in \mathcal{P} \).

Then,

\[
\sum_{I \subseteq \{1, 2, \ldots, n\}} (-1)^{\mid I \mid + 1} \sum_{F \cap (\bigcup_{i \in I} E_i) \neq \phi} m(F)
\]
\[
\begin{align*}
&= \sum_{I \in \mathcal{I}(F)} (-1)^{|I|+1} \sum_{F \cap E \neq \emptyset} m(F) \\
&\geq \sum_{I \in \mathcal{I}(F)} (-1)^{|I|} \sum_{F \cap (\bigcap_{i \in I} E_i) \neq \emptyset} m(F) \\
&= \sum_{F \cap (\bigcap_{i=1}^n E_i) \neq \emptyset} m(F) = \text{FPI}_m (\bigcap_{i=1}^n E_i)
\end{align*}
\]

Hence (3).

Hence, the theorem.

Now, we consider some properties of fuzzy plausibility functions.

2.4.5. Property

\[\text{FPI}(A) + \text{FPI}(\overline{A}) \geq 1, \forall A \in \mathcal{P}\]

Proof

By defn 2.4.1.

\[\text{FPI}(A \cap \overline{A}) \leq \text{FPI}(A) + \text{FPI}(\overline{A}) - \text{FPI}(A \cup \overline{A})\]

ie, \(\text{FPI}(\emptyset) \leq \text{FPI}(A) + \text{FPI}(\overline{A}) - \text{FPI}(Q)\)

ie, \(\text{FPI}(A) + \text{FPI}(\overline{A}) - \text{FPI}(Q) \geq \text{FPI}(\emptyset) \rightarrow (A)\)
Let $\alpha [\text{FPI} (A)] = [a_1, a_2]$ and $\alpha [\text{FPI} (\overline{A})] = [b_1, b_2]$, for any $\alpha \in (0, 1]$.

We have

$$\text{FPI} (\emptyset) = \emptyset = [0, 0] \text{ and } \text{FPI} (Q) = 1 = [1, 1]$$

Then, (A) gives

$$[a_1, a_2] + [b_1, b_2] - [1, 1] \geq [0, 0]$$

ie $[a_1, a_2] + [b_1 - 1, b_2 - 1] \geq [0, 0]$ \hspace{1cm} (by result 1.4.1)

ie, $[a_1 + b_1 - 1, a_2 + b_2 - 1] \geq [0, 0]$

which implies

$$a_1 + b_1 - 1 \geq 0, a_2 + b_2 - 1 \geq 0$$

$$\therefore a_1 + b_1 \geq 1, a_2 + b_2 \geq 1$$

which implies

$$[a_1, a_2] + [b_1, b_2] \geq [1, 1]$$

ie, $\text{FPI} (A) + \text{FPI} (\overline{A}) \geq 1$

2.4.6. Property

It is not necessary that $\text{FB} (A) \leq \text{FPI} (A), \forall A \in \mathcal{P}$.

Proof

We have

$$\text{FB} (A) + \text{FB} (\overline{A}) \leq 1 \hspace{1cm} \text{(by property 2.3.11)}$$
The above property 2.4.5. gives

$$\text{FPI}(A) + \text{FPI}(\overline{A}) \geq 1.$$  

Hence, we have

$$\text{FB}(A) + \text{FB}(\overline{A}) \leq 1 \leq \text{FPI}(A) + \text{FPI}(\overline{A})$$

Let, for any $\alpha \in (0,1]$,

$$\alpha [\text{FB}(A)] = [a_1, a_2], \quad \alpha [\text{FB}(\overline{A})] = [b_1, b_2],$$

$$\alpha [\text{FPI}(A)] = [c_1, c_2] \quad \& \quad \alpha [\text{FPI}(\overline{A})] = [d_1, d_2].$$

So, the inequality $\text{FB}(A) + \text{FB}(\overline{A}) \leq \text{FPI}(A) + \text{FPI}(\overline{A})$

implies

$$[a_1, a_2] + [b_1, b_2] \leq [c_1, c_2] + [d_1, d_2]$$

$$\Rightarrow a_1 + b_1 \leq c_1 + d_1; \quad a_2 + b_2 \leq c_2 + d_2$$  \hspace{1cm} \text{(by result 1.4.1)}

$$\Rightarrow a_1 \leq c_1 + d_1; \quad a_2 \leq c_2 + d_2$$  \hspace{1cm} \text{(since $b_1, b_2 \geq 0$)}

$$\not\Rightarrow a_1 \leq c_1 \quad \& \quad a_2 \leq c_2.$$  

Hence, it is not necessary that $[a_1, a_2] \leq [c_1, c_2]$.

ie, it is not necessary that $\text{FB}(A) \leq \text{FPI}(A), \ \forall A \in \mathcal{P}$.

2.4.7. Property

It is not necessary that $\text{FPI}(A) \leq \text{FPI}(B), \ \forall A, B \in \mathcal{P}$ with $A \subseteq B$.  

46
Proof

Let $C = B - A$

Then $A \cup C = B$ and $A \cap C = \emptyset$

For any $\alpha \in (0, 1]$, let

$$\alpha [FPI (A)] = [a_1, a_2], \quad \alpha [FPI (B)] = [b_1, b_2] \text{ and}$$

$$\alpha [FPI (C)] = [c_1, c_2].$$

By Defn. 2.4.1.

$$FPI (A \cap C) \leq FPI (A) + FPI (C) - FPI (A \cup C)$$

i.e., $FPI (\emptyset) \leq FPI (A) + FPI (C) - FPI (B)$

which implies

$$[0, 0] \leq [a_1, a_2] + [c_1, c_2] - [b_1, b_2]$$

i.e., $[0, 0] \leq [a_1 + c_1 - b_2, a_2 + c_2 - b_1]$

$$\therefore a_1 + c_1 - b_2 \geq 0, \quad a_2 + c_2 - b_1 \geq 0$$

i.e., $b_2 \leq a_1 + c_1, \quad b_1 \leq a_2 + c_2$

Since $a_1 + c_1 \leq a_2 + c_2$ and $b_1 \leq b_2$, we get

$$b_1 \leq b_2 \leq a_1 + c_1 \leq a_2 + c_2.$$

This relation doesn't imply $a_1 \leq b_1$ and $a_2 \leq b_2.$
Hence, it is not necessary that \([a_1, a_2] \leq [b_1, b_2]\).

ie, it is not necessary that \(\text{FPI}(A) \leq \text{FPI}(B)\), \(\forall A, B \in \mathcal{P}\) with \(A \subseteq B\).

For example,

let \(A = [0.4, 0.7]\) and \(B = [0.2, 0.8]\)

Clearly, \(A \subseteq B\).

If we take \(\alpha[\text{FPI}(A)] = A\) and \(\alpha[\text{FPI}(B)] = B, \forall \alpha \in (0, 1]\),

then by result 1.41.6,

\(\alpha[\text{FPI}(A)] \nsubseteq \alpha[\text{FPI}(B)]\)

and hence, by the same result, \(\text{FPI}(A) \nsubseteq \text{FPI}(B)\).

2.4.8 Theorem

Let \(\text{FB}_m\) and \(\text{FPI}_m\) be the fuzzy belief function and the fuzzy plausibility function respectively, induced from a fuzzy basic probability assignment \(m\). If \(\text{FB}_m\) coincides with \(\text{FPI}_m\), then \(m\) focuses only on singletons.

Proof

Let \(E \in \mathcal{P}\) which is not a singleton of \(\mathcal{P}\) such that \(m(E) > 0\)

Then, for any \(x \in E\),

\(\text{FB}_m(\{x\}) = m\{x\}\)

\(< m(\{x\}) + m(E)\)

\(\leq \sum_{F \cap \{x\} \neq \phi} m(F) = \text{FPI}_m(\{x\})\)

This contradicts the coincidence of \(\text{FB}_m\) and \(\text{FPI}_m\).

Hence, \(E\) should be singleton.

Hence, the theorem.
CHAPTER - III

FUZZY MEASURES