CHAPTER VI

STOCHASTIC MODEL TO DETERMINE THE OPTIMAL MANPOWER RESERVE AT FOUR NODES IN SERIES

6.1 Introduction

There are many industries and organizations where the skilled personnel are to be recruited and they must be given prior training before employment. In human resource planning the training and induction of the right type of personnel is a pressing problem. Recently there has been considerable interest in developing Stochastic models for describing optimal policies for item in inventory whose utility does not remain the same with the passage of time. Significant work has been done for describing optimal ordering policies for units with fixed life time. In this Chapter, we consider the optimal solution for the manpower to be kept as reserve inventory at four different nodes in series. This is based on the inventory model for finding the optimal size of reserve inventory at three successive stations in series.

6.2 Assumptions

(i) The manpower system comprises of four nodes namely the recruitment node, the training node, probation period employment node and permanent employment node.

(ii) The reserve of manpower inventory is at three points namely in between recruitment and training, training and probation period employment and similarly between probation period employment to permanent employment.

(iii) The delay in finding suitable candidates after training for employment is very costly.

* This work was already presented in the National Conference on Mathematical Modeling 2014, Department of Mathematics, Annamalai University, Annamalainagar, during 21-22 February 2014 and published in the International Journal of Ultra Scientist of Physical Sciences, (2013), Vol. 25(1) A, pp.139-155.
6.3 Notations

(i) The cost of excess candidates at the reserve points 1, 2 and 3 are denoted as \( h_1 \), \( h_2 \) and \( h_3 \) called holding costs.

(ii) \( d_1 \), \( d_2 \) and \( d_3 \) denote the cost arising due to the shortage of candidates or manpower at the three intermediate reserve points \( R_1, R_2 \) and \( R_3 \).

(iii) The breakdown occurs at the recruitment or entry point for a random duration \( \tau \), and has the probability density function is \( g(\tau) \) and cumulative density function is \( G(\tau) \).

(iv) \( r_1 \) denotes the constant rate of training of personnel at the node \( R_1 \).

(v) \( r_2 \) denotes the constant rate of probation period employment of personnel at the node \( R_2 \).

(vi) \( r_3 \) denotes the constant rate of permanent employment of personnel at the node \( R_3 \).

(vii) \( \mu_1, \mu_2, \mu_3 \) the mean interarrival time of breakdowns of system \( R_1, R_2 \) and \( R_3 \) respectively.

6.4 Manpower Reserve at Four Nodes in Series

We consider a manpower system in which the first node is the point of recruitment. This second node is the training of the selected before induction or employment. It may be noted that if there is any delay or breakdown at the first node namely the recruitment point then the training will be delayed and ultimately the shortage of trained personnel occurs. This in turn affects the running of the industry or organization. Similarly if there is any breakdown at the training node there will be again shortage of trained persons for employment. Therefore a reserve inventory of persons is to be maintained at three places namely in between first and second nodes, second and
third nodes and third and forth nodes. The following configuration explains the conceptualized model.

![Diagram](S_1 -> R_1 -> S_2 -> R_2 -> S_3 -> R_3 -> S_4)

S_1, S_2, S_3 and S_4 denote the four sections or systems devoted to different activities

S_1 = Recruitment division

S_2 = Training division

S_3 = Probation induction section

S_4 = Permanent induction section

R_1 = Reserve of persons at the first stage

R_2 = Reserve of persons at the second stage

R_3 = Reserve of persons at the third stage

The reserves are in terms of man hours.

In this model it is assumed that if the reserve of manpower is in excess, then there is a cost of excess and similarly in shortage, it involves the so called shortage cost. Here the problem is to determine the optimal size of reserve at the three different points namely between recruitment and training between training and probation period employment, and finally between probation period employment to permanent employment.

### 6.5 Results

In this model it can be seen that the total expected cost due to excess of manpower inventory and also shortage is
\[ E(c) = h_1 R_1 + h_2 R_2 + h_3 R_3 + \frac{d_1}{\mu_1} \int_{r_1}^{\infty} \left( \tau - \frac{R_1}{r_1} \right) g(\tau) \, d\tau \]

\[ + \frac{d_2}{\mu_2} \int_{r_1}^{r_2} \left( \tau - \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} \right) \right) g(\tau) \, d\tau + \frac{d_3}{\mu_3} \int_{r_1}^{r_2} \left( \tau - \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} \right) \right) g(\tau) \, d\tau \]

\[ \ldots (6.1) \]

It may be observed that the first three terms of eqn. (6.1) represent the cost of excess of reserve of manpower, the fourth term is the cost of training section due to shortage of candidates, the fifth and sixth term is the cost due to the failure production due to non availability of manpower.

To optimal values of \( R_1, R_2 \) and \( R_3 \) can be obtained by solving the equations.

\[ \frac{\partial E(c)}{\partial R_1} = 0 \quad \ldots (6.2) \]

\[ \frac{\partial E(c)}{\partial R_2} = 0 \quad \ldots (6.3) \]

\[ \frac{\partial E(c)}{\partial R_3} = 0 \quad \ldots (6.4) \]

On differentiating partially the above equations with respect to \( R_1, R_2 \) and \( R_3 \) and equating to zero we obtain,

\[ h_1 - \frac{d_1}{\mu_1 r_1} \left[ 1 - G \left( \frac{R_1}{r_1} \right) \right] - \frac{d_2}{\mu_1 r_1} \left[ 1 - G \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} \right) \right] - \frac{d_3}{\mu_1 r_1} \left[ 1 - G \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} \right) \right] = 0 \]

\[ \ldots (6.5) \]

\[ h_2 - \frac{d_2}{\mu_2 r_2} \left[ 1 - G \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} \right) \right] - \frac{d_3}{\mu_2 r_2} \left[ 1 - G \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} \right) \right] = 0 \]

\[ \ldots (6.6) \]

and

\[ h_3 - \frac{d_3}{\mu_3 r_3} \left[ 1 - G \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} \right) \right] = 0 \]

\[ \ldots (6.7) \]
It may be observed that Leibnitz rule for the differentiation of an integral namely

\[
\int_{a(x)}^{b(x)} f(x, t) \, dt = b'(x) f \left[ x, b(x) \right] - a'(x) f \left[ x, a(x) \right] + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x, t) \, dx
\]

Differentiating eqn. (6.1) with respect to \( R_1, R_2 \) and \( R_3 \) respectively. Solving equations (6.5), (6.6) and (6.7), the optimal values of \( R_1, R_2 \) and \( R_3 \) are obtained.

### 6.6 Model I

Consider the case when \( G(.) \) has uniform distribution over \([0, a]\). The down time density of the recruitment is a constant and is independent of time, that is, the down time distribution is proportional to time.

The holding cost will arise when \( a < \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} + \frac{R_3}{\mu_3} \right) \).

When \( a > \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} + \frac{R_3}{\mu_3} \right) \), the equations (6.5), (6.6) and (6.7) present the following,

\[
h_1 = \frac{d_1}{\mu_1 r_1} \left[ a - \left( \frac{R_1}{\mu_1} \right) \right] - \frac{d_2}{\mu_2 r_2} \left[ a - \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} \right) \right] - \frac{d_3}{\mu_3 r_3} \left[ a - \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} + \frac{R_3}{\mu_3} \right) \right] = 0
\]

\[
h_2 = \frac{d_2}{\mu_2 r_2} \left[ a - \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} \right) \right] - \frac{d_3}{\mu_3 r_3} \left[ a - \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} + \frac{R_3}{\mu_3} \right) \right] = 0
\]

\[
h_3 = \frac{d_3}{\mu_3 r_3} \left[ a - \left( \frac{R_1}{\mu_1} + \frac{R_2}{\mu_2} + \frac{R_3}{\mu_3} \right) \right] = 0
\]

On collecting the coefficients of \( R_1, R_2 \) and \( R_3 \) these three equations will be reduced to the three simultaneous, linear equations in \( R_1, R_2 \) and \( R_3 \) as follows. Consider the eqn. (6.9) and it can be written as,
Consider the eqn. (1.11) and it can be written as,

\[
\begin{align*}
&\hat{R}_1 = \frac{a_1 (d_1 - h_1 r_1 \mu_1 + h_2 r_2 \mu_1)}{d_1} \\
&\hat{R}_2 = \frac{a_2 (d_2 h_1 r_1 - d_1 h_2 r_2 - d_2 h_2 r_2 + d_1 h_3 r_3) \mu_2}{d_1 d_2} \\
&\hat{R}_3 = \frac{a_3 (-d_3 h_2 r_2 + d_2 h_3 r_3 + d_3 h_3 r_3) \mu_3}{d_2 d_3}
\end{align*}
\]

\[\text{(6.15)}\]

\[\text{(6.16)}\]

\[\text{(6.17)}\]

6.6.1 Numerical Example

The Variation in $\hat{R}_1$ consequent to the changes in $h_1$, keeping $r_1=10$, $r_2=20$, $d_1=30$, $h_2=45$, $\mu_1=1.0$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.1.
The Variation in $\hat{R}_1$ consequent to the changes in $h_1$

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_1$</td>
<td>415</td>
<td>365</td>
<td>315</td>
<td>265</td>
<td>215</td>
<td>165</td>
<td>115</td>
</tr>
</tbody>
</table>

Figure 6.1: The Variation in $\hat{R}_1$ consequent to the changes in $h_1$

The Variation in $\hat{R}_1$ consequent to the changes in $d_1$, keeping $r_1=10$, $r_2=20$, $h_1=10$, $h_2=45$, $\mu_1=1.0$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.2.

Table 6.2: The Variation in $\hat{R}_1$ consequent to the changes in $d_1$

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
<th>75</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_1$</td>
<td>255</td>
<td>233</td>
<td>215</td>
<td>199</td>
<td>186</td>
<td>175</td>
<td>165</td>
</tr>
</tbody>
</table>

Figure 6.2: The Variation in $\hat{R}_1$ consequent to the changes in $d_1$
The Variation in $\ddot{R}_1$ consequent to the changes in $\mu_1$, keeping $r_1=10$, $r_2=20$, $h_1=10$, $h_2=45$, $d_1=50$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.3.

**Table 3: The Variation in $\ddot{R}_1$ consequent to the changes in $\mu_1$**

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{R}_1$</td>
<td>255</td>
<td>303</td>
<td>351</td>
<td>399</td>
<td>447</td>
<td>495</td>
<td>543</td>
</tr>
</tbody>
</table>

**Figure 6.3: The Variation in $\ddot{R}_1$ consequent to the changes in $\mu_1$**

The Variation in $\ddot{R}_2$ consequent to the changes in $h_2$, keeping $r_1=10$, $r_2=20$, $r_3=15$, $d_1=50$, $d_2=40$, $h_1=10$, $h_3=80$, $\mu_2=1.0$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.4.

**Table 6.4: The Variation in $\ddot{R}_2$ consequent to the changes in $h_2$**

<table>
<thead>
<tr>
<th>$h_2$</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{R}_2$</td>
<td>420</td>
<td>393</td>
<td>366</td>
<td>339</td>
<td>312</td>
<td>285</td>
<td>258</td>
</tr>
</tbody>
</table>
Figure 6.4: The Variation in $\hat{R}_2$ consequent to the changes in $h_2$

The Variation in $\hat{R}_2$ consequent to the changes in $d_2$, keeping $r_1=10$, $r_2=20$, $r_3=15$, $d_1=50$, $h_1=10$, $h_2=20$, $h_3=80$, $\mu_2=1.0$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.5.

Table 6.5: The Variation in $\hat{R}_2$ consequent to the changes in $d_2$

<table>
<thead>
<tr>
<th>$d_2$</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_2$</td>
<td>420</td>
<td>353</td>
<td>300</td>
<td>256</td>
<td>220</td>
<td>189</td>
<td>163</td>
</tr>
</tbody>
</table>

Figure 6.5: The Variation in $\hat{R}_2$ consequent to the changes in $d_2$
The Variation in $\hat{R}_2$ consequent to the changes in $\mu_2$, keeping $r_1=10$, $r_2=20$, $r_3=15$, $d_1=50$, $d_2=40$, $h_1=10$, $h_2=45$, $h_3=125$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.6.

**Table 6.6: The Variation in $\hat{R}_2$ consequent to the changes in $\mu_2$**

<table>
<thead>
<tr>
<th>$\mu_2$</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_2$</td>
<td>251</td>
<td>302</td>
<td>352</td>
<td>402</td>
<td>452</td>
<td>503</td>
<td>553</td>
</tr>
</tbody>
</table>

![Figure 6.6: The Variation in $\hat{R}_2$ consequent to the changes in $\mu_2$](image)

The Variation in $\hat{R}_3$ consequent to the changes in $h_3$, keeping $r_2=20$, $r_3=15$, $d_2=40$, $d_3=20$, $h_2=25$, $\mu_3=1.0$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.6.

**Table 6.7: The Variation in $\hat{R}_3$ consequent to the changes in $h_3$**

<table>
<thead>
<tr>
<th>$h_3$</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_3$</td>
<td>281</td>
<td>534</td>
<td>788</td>
<td>1041</td>
<td>1294</td>
<td>1547</td>
<td>1800</td>
</tr>
</tbody>
</table>
The Variation in $\hat{R}_3$ consequent to the changes in $d_3$, keeping $r_2=20$, $r_3=15$, $d_2=40$, $h_2=20$, $h_3=80$, $\mu_3=1.0$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.8.

<table>
<thead>
<tr>
<th>$d_3$</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_3$</td>
<td>1800</td>
<td>1125</td>
<td>900</td>
<td>788</td>
<td>720</td>
<td>675</td>
<td>643</td>
</tr>
</tbody>
</table>

Figure 6.7: The Variation in $\hat{R}_3$ consequent to the changes in $h_3$

Figure 6.8: The Variation in $\hat{R}_3$ consequent to the changes in $d_3$
The Variation in $\bar{R}_3$ consequent to the changes in $\mu_1$, keeping $r_2=20$, $r_3=15$, $d_2=40$, $d_3=60$, $h_2=45$, $h_3=75$ and $a=1.5$ all are fixed, the simulated values are shown in table 6.9.

**Table 6.9: The Variation in $\bar{R}_3$ consequent to the changes in $\mu_3$**

<table>
<thead>
<tr>
<th>$\mu_3$</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{R}_3$</td>
<td>548</td>
<td>658</td>
<td>768</td>
<td>878</td>
<td>987</td>
<td>1097</td>
<td>1207</td>
</tr>
</tbody>
</table>

**Figure 6.9: The Variation in $\bar{R}_3$ consequent to the changes in $\mu_3$**

On the basis of the model I discussed the following conclusions are drawn using the numerical illustration.

(i) As the cost of manpower reserve between stages $S_1$, $S_2$, $S_3$ and $S_4$ namely $h_1$ increases a small reserve $R_1$ is suggested as an optimal one. This is indicated in table 6.1 and figure 6.1.

(ii) If the cost of shortages at the first node namely $d_1$ increases, it is suggested that a larger manpower at $R_1$ is desirable as indicated in table 6.2 and figure 6.2.

(iii) The interarrival time between breakdown $\mu_1$ increase then a increased in a value of $R_1$ is suggested as indicated in table 6.3 and figure 6.3.
(iv) If the cost of manpower reserve at node 2 is higher namely as $h_2$ increases then a decrease in the value of $R_2$ is suggested as indicated in table 6.4 and figure 6.4.

(v) If $d_2$, the cost of shortages at the second node increases a larger reserve namely $R_2$ is desirable as indicated in table 6.5 and figure 6.5.

(vi) The interarrival time between breakdown $\mu_2$ increase then a increased in a value of $R_2$ is suggested as indicated in table 6.6 and figure 6.6.

(vii) If the cost of manpower reserve at node 3 is higher namely as $h_3$ increases then a increase in the value of $R_3$ is suggested as indicated in table 6.7 and figure 6.7.

(viii) If $d_3$, the cost of shortages at the third node increases a larger reserve namely $R_3$ is desirable as indicated in table 6.8 and figure 6.8.

(ix) The interarrival time between breakdown $\mu_3$ increase then a increased in a value of $R_3$ is suggested as indicated in table 6.9 and figure 6.9.

(x) If $\mu_1, \mu_2, \mu_3$ the parameters of the interarrival times between the breakdowns of $S_1, S_2, S_3$ and $S_4$ increases then the interarrival times will be shorter since $E(\tau)=1/\mu_1$, $E(\tau)=1/\mu_2$, $E(\tau)=1/\mu_3$ decreases. Hence a larger reserve at $R_1, R_2$ and $R_3$ is suggested as indicated in table 6.3, table 6.6 and table 6.9 and the figure 6.3, figure 6.6 and figure 6.9.

6.7 Model II

In the above model an additional assumption that the down time of $R_1$ is a random variable which satisfies the so called Setting the Clock Back to Zero (SCBZ) property is introduced. This property has been due to Raja Rao and Talwalker (1990). Under this property the p.d.f. of the down time $\tau$ which is namely $g(\tau)$ is such that
\( g(\tau) = \begin{cases} 
\theta_1 e^{-\theta_1 \tau} & \text{if } \tau \leq \tau_0 \\
\theta_2 e^{-\theta_2 \tau} e^{\tau_0 (\theta_2 - \theta_1)} & \text{if } \tau > \tau_0 
\end{cases} \)

When \( \tau_0 \) is called the truncation point.

It can be shown that \( \int_0^\infty g(\tau) d\tau = 1 \), and \( \tau_0 \) is called the truncation point which itself is a random variable that follows exponential distribution with parameter \( \lambda \). Now the cost function under this assumption is given as,

\[
E(c) = h_1 R_1 + h_2 R_2 + h_3 R_3 \\
+ \frac{d}{\mu_1} \left[ \int_{R_1/r_1}^{\tau_0} \left( \tau - \frac{R_1}{r_1} \right) \theta_1 e^{-\theta_1 \tau} d\tau p[\tau \leq \tau_0] + e^{\tau_0 (\theta_2 - \theta_1)} \int_{\tau_0}^\infty \theta_1 e^{-\theta_1 \tau} d\tau p[\tau > \tau_0] \right] \\
+ \frac{d}{\mu_2} \left[ \int_{R_2/r_2}^{\tau_0} \left( \tau - \frac{R_2}{r_2} \right) \theta_2 e^{-\theta_2 \tau} d\tau p[\tau \leq \tau_0] + e^{\tau_0 (\theta_2 - \theta_1)} \int_{\tau_0}^\infty \theta_2 e^{-\theta_2 \tau} d\tau p[\tau > \tau_0] \right] \\
+ \frac{d}{\mu_3} \left[ e^{\tau_0 (\theta_2 - \theta_1)} \int_{\tau_0}^\infty \left( \tau - \frac{R_3}{r_3} \right) \theta_3 e^{-\theta_3 \tau} d\tau p[\tau > \tau_0] \right]
\]

\[
E(c) = h_1 R_1 + h_2 R_2 + h_3 R_3 \\
+ \frac{d}{\mu_1} \left[ \theta_1 \int_{R_1/r_1}^{\tau_0} \left( \tau - \frac{R_1}{r_1} \right) e^{-\theta_1 \tau} e^{-\lambda \tau} d\tau + e^{\tau_0 (\theta_2 - \theta_1)} \theta_1 \int_{\tau_0}^\infty \left( \tau - \frac{R_1}{r_1} \right) e^{-\theta_1 \tau} d\tau \right] \\
+ \frac{d}{\mu_2} \left[ \theta_2 \int_{R_2/r_2}^{\tau_0} \left( \tau - \frac{R_2}{r_2} \right) e^{-\theta_2 \tau} e^{-\lambda \tau} d\tau + e^{\tau_0 (\theta_2 - \theta_1)} \theta_2 \int_{\tau_0}^\infty \left( \tau - \frac{R_2}{r_2} \right) e^{-\theta_2 \tau} d\tau \right]
\]
\[
\frac{\partial E(c)}{\partial R_1} = h_1 + \frac{d_1}{\mu_1} \left\{ 0 - \frac{1}{r_1} (0) + \theta_1 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_1 \tau} e^{-\lambda \tau_0} d\tau \right\} \\
+ \frac{d_3}{\mu_3} \left\{ 0 - \frac{1}{r_1} (0) + \theta_1 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_1 \tau} e^{-\lambda \tau_0} d\tau \right\} \\
+ e^{\tau_0 (\theta_2 - \theta_1)} \frac{d_2}{\mu_2} \left[ \theta_2 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_2 \tau} [1 - e^{-\lambda \tau_0}] d\tau \right] \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\]

\[
I_1 = h_1 + \frac{d_1}{\mu_1} \left[ \theta_1 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_1 \tau} e^{-\lambda \tau_0} d\tau \right] \\
= h_1 - \frac{d_1 \theta_1}{\mu_1 r_1} \left[ e^{-\lambda \tau_0} \int_{\tau_0}^{\infty} e^{-\theta_1 \tau} d\tau \right] \\
= h_1 - \frac{d_1 \theta_1}{\mu_1 r_1} e^{-\lambda \tau_0} \left[ \frac{e^{-\theta_1 \tau_0} - e^{-\theta_1 \tau_0}}{-\theta_1 \tau_0} \right] \\
= h_1 + \frac{d_1}{\mu_1 r_1} e^{-\lambda \tau_0} \left[ e^{-\theta_1 \tau_0} - e^{-\theta_1 \tau_0} \right]
\]
\[ I_1 = h_1 + \frac{d_1}{\mu_1 r_1} \left[ e^{-(\lambda + \theta_1)\tau_0} - e^{-\theta_1 \frac{r_1}{r_1} \lambda \tau_0} \right] \]

\[ I_2 = \frac{d_1}{\mu_1} e^{(\theta_2 - \theta_1)\tau_0} \theta_2 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_2 \tau} \left[ 1 - e^{-\lambda \tau} \right] d\tau \]

\[ = -\frac{d_1}{\mu_1 r_1} e^{(\theta_2 - \theta_1)\tau_0} \left[ 1 - e^{-\lambda \tau_0} \right] \int_{\tau_0}^{\infty} e^{-\theta_2 \tau} d\tau \]

\[ = -\frac{d_1}{\mu_1 r_1} e^{(\theta_2 - \theta_1)\tau_0} \left[ 1 - e^{-\lambda \tau_0} \right] \left[ e^{-\theta_2 \tau} \right]_{\tau_0}^{\infty} \]

\[ = \frac{d_1}{\mu_1 r_1} e^{(\theta_2 - \theta_1)\tau_0} \left[ 1 - e^{-\lambda \tau_0} \right] (-e^{-\theta_2 \tau_0}) \]

\[ = -\frac{d_1}{\mu_1 r_1} \left[ 1 - e^{-\lambda \tau_0} \right] (e^{-\theta_1 \tau_0}) \]

\[ I_1 + I_2 = h_1 + \frac{d_1}{\mu_1 r_1} \left[ -e^{-\theta_1 \frac{r_1}{r_1} \lambda \tau_0} - e^{-\theta_1 \tau_0} \right] \]

\[ = h_1 - \frac{d_1}{\mu_1 r_1} \left[ e^{-\theta_1 \frac{r_1}{r_1} \lambda \tau_0} + e^{-\theta_1 \tau_0} \right] \]

\[ I_3 = \frac{d_2}{\mu_2 r_1} \theta_1 \int_{r_1}^{\tau_0} \left( -\frac{1}{r_1} \right) e^{-\theta_1 \tau} e^{-\lambda \tau_0} d\tau \]

\[ = -\frac{d_2 \theta_1}{\mu_2 r_1} e^{-\lambda \tau_0} \left[ \int_{r_1}^{\tau_0} \frac{r_1}{r_1} e^{-\theta_1 \tau} d\tau \right] \]

\[ = -\frac{d_2 \theta_1}{\mu_2 r_1} e^{-\lambda \tau_0} \left[ e^{-\theta_1 \tau} \right]_{r_1}^{\tau_0} \]

\[ = -\frac{d_2 \theta_1}{\mu_2 r_1} e^{-\lambda \tau_0} \left[ e^{-\theta_1 \frac{r_1}{r_1} \lambda \tau_0} \right] \]

\[ = \frac{d_2}{\mu_2 r_1} e^{-\lambda \tau_0} \left[ e^{-\theta_1 \tau_0} - e^{-\theta_1 \frac{r_1}{r_1} \lambda \tau_0} \right] \]

\[ I_4 = \frac{d_2}{\mu_2} e^{(\theta_2 - \theta_1)\tau_0} \theta_2 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_2 \tau} \left[ 1 - e^{-\lambda \tau} \right] d\tau \]
\[ I_3 + I_4 = \frac{d_2}{\mu_2 r_1} e^{-\lambda_1 \tau_0} \left[ e^{-\theta_1 \tau_0} - e^{-\theta_1 \left(\frac{r_1}{r_2} + \frac{r_2}{r_3}\right)} \right] - \frac{d_2}{\mu_2 r_1} \left[ 1 - e^{-\lambda_1 \tau_0} \right] (e^{-\theta_1 \tau_0}) \]

\[ I_5 = \frac{d_3}{\mu_3} \left[ \frac{1}{r_1} \int_{\frac{r_1}{r_2 + \frac{r_2}{r_3}}}^{\tau_0} \left( -\frac{1}{r_1} \right) e^{-\theta_1 \tau} e^{-\lambda_1 \tau_0} d\tau \right] \]

\[ = \frac{d_2}{\mu_3} \left( -\frac{1}{r_1} \right) e^{-\lambda_1 \tau_0} \left[ e^{-\theta_1 \tau} \int_{\frac{r_1}{r_2 + \frac{r_2}{r_3}}}^{\tau_0} \frac{e^{-\theta_1 \tau}}{\frac{r_1}{r_2 + \frac{r_2}{r_3}}} d\tau \right] \]

\[ = \frac{d_3}{\mu_3} \left( -\frac{1}{r_1} \right) e^{-\lambda_1 \tau_0} \left[ e^{-\theta_1 \tau_0} - e^{-\theta_1 \left(\frac{r_1}{r_2 + \frac{r_2}{r_3}}\right)} \right] \]

\[ I_6 = \frac{d_3}{\mu_3} e^{(\theta_2 - \theta_1) \tau_0} \theta_2 \int_{\tau_0}^{\infty} \left( -\frac{1}{r_1} \right) e^{-\theta_2 \tau} \left[ 1 - e^{-\lambda_1 \tau_0} \right] d\tau \]

\[ = \frac{d_3}{\mu_3} e^{(\theta_2 - \theta_1) \tau_0} \theta_2 \left( -\frac{1}{r_1} \right) \left[ 1 - e^{-\lambda_1 \tau_0} \right] \int_{\tau_0}^{\infty} e^{-\theta_2 \tau} d\tau \]

\[ = \frac{d_3}{\mu_3} e^{(\theta_2 - \theta_1) \tau_0} \theta_2 \left( -\frac{1}{r_1} \right) \left[ e^{-\theta_2 \tau} \right]^{\infty}_{\tau_0} \]
\[
I_5 + I_6 = \frac{d_3}{\mu_3 r_1^2} \left[ e^{-(\lambda + \Theta_1) \tau_0} - e^{-\lambda \tau_0 - \Theta_1 \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} \right)} - e^{-\Theta_1 \tau_0} + e^{-(\lambda + \Theta_1) \tau_0} \right]
\]

... (6.20)

\[
\frac{\partial E(c)}{\partial R_3} = h_3 + \frac{d_3}{\mu_3 r_1^2} \left[ e^{-(\lambda + \Theta_1) \tau_0} - e^{-\lambda \tau_0 - \Theta_1 \left( \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} \right)} - e^{-\Theta_1 \tau_0} + e^{-(\lambda + \Theta_1) \tau_0} \right] = 0
\]

... (6.23)

Solving eqn. (6.21), eqn. (6.22) and eqn. (6.23) we get the eqn. (6.24), eqn. (6.25) and eqn. (6.26) after simplification.

\[
\hat{R}_1 = \frac{\theta_1}{d_{r_1}} \left[ \frac{-\Theta_1 \left( \frac{R_2 + R_3 - \tau_0}{r_2 + r_3} \right)}{d_3 r_2 \mu_1 \mu_2 e^{-\frac{R_2}{r_2} + \frac{R_3}{r_3}}} \left( \frac{R_3 \theta_1}{d_2 r_1 \mu_1 + e^{-\frac{R_2}{r_2} - \frac{R_3}{r_3}}} \left( \frac{R_2 \theta_1}{d_1 r_2 \mu_2} \right) \mu_3 \right) \right] \]

... (6.24)
Given the values of the parameters and the costs the expressions for \( R_2 \), \( R_3 \) and \( R_4 \) can be evaluated numerically using simulation.

6.7.1 Numerical Illustration

Using computer routines by fixing \( r_1=30, r_2=40, r_3=50, R_2=30, R_3=40, \mu_1=1, \mu_2=2, \mu_3=3, d_1=200, d_2=300, d_3=400, \theta_1=1, \tau_0=1 \) and \( \lambda=0.5 \), we find the optimal values of \( R_1 \) as shown in Table 6.10.

**Table 6.10: The Variation in \( \tilde{R}_1 \) consequent to the changes in \( h_1 \)**

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{R}_1 )</td>
<td>27</td>
<td>24</td>
<td>22</td>
<td>21</td>
<td>19</td>
<td>18</td>
<td>16</td>
</tr>
</tbody>
</table>

**Figure 6.10: The Variation in \( \tilde{R}_1 \) consequent to the changes in \( h_1 \)**
Using computer routines by fixing \( r_1=30, r_2=40, r_3=50, R_2=415, R_3=420, \mu_1=1, \mu_2=2, \mu_3=3, h_2=1, d_2=300, d_3=400, \theta_1=1, \tau_0=1 \) and \( \lambda=0.5 \), we find the optimal values of \( R_1 \) as shown in table 6.11.

**Table 6.11: The Variation in \( \tilde{R}_1 \) consequent to the changes in \( h_1 \)**

<table>
<thead>
<tr>
<th>( d_1 )</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
<th>160</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{R}_1 )</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
</tr>
</tbody>
</table>

![Graph showing the variation in \( \tilde{R}_1 \) consequent to the changes in \( d_1 \)](image)

**Figure 6.11: The Variation in \( \tilde{R}_1 \) consequent to the changes in \( d_1 \)**

Using computer routines by fixing \( r_1=10, r_2=20, r_3=30, R_2=100, R_3=120, \mu_2=3, \mu_3=3, h_1=1, d_1=500, d_2=600, d_3=700, d_3=400, \theta_1=1.5, \tau_0=2.5 \) and \( \lambda=0.5 \), we find the optimal values of \( R_1 \) as shown in table 6.12.

**Table 6.12: The Variation in \( \tilde{R}_1 \) consequent to the changes in \( \mu_1 \)**

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{R}_1 )</td>
<td>21</td>
<td>16</td>
<td>14</td>
<td>12</td>
<td>10</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>
Figure 6.12: The Variation in $\hat{R}_1$ consequent to the changes in $\mu_1$

6.8 Conclusion

On the basis of the model II discussed the following conclusions are drawn using the numerical illustration.

(i) As the holding cost $h_1$ increases a small reserve $R_1$ is decreases, is suggested as an optimal one. This is indicated in table 6.10 and figure 6.10.

(ii) If the cost of shortages at the first stage $d_1$ increases, larger manpower at $R_1$ is desirable as indicated in table 6.11 and figure 6.11.

(iii) The interarrival time between breakdown $\mu_1$ increases then a decreases in a value of $R_1$ is suggested as indicated in table 6.12 and figure 6.12.