CHAPTER- VI

INVENTORY SYSTEM WITH STOCK DEPENDENT DEMAND AND PARTIAL BACKLOGGING

Some inventory models were formulated in a static environment where the demand is assumed to be constant and steady over a finite planning horizon. In fact, the constant demand assumption is only valid during the maturity phase of time. However, for certain types of consumer goods (e.g., fruits, vegetables, donuts and other) of inventory, the demand rate may be influenced by the stock level. It has been noted by marketing researchers and practitioners that an increase in a product’s shelf space usually has a positive impact on the sales of that product and it is usually observed that a large pile of goods on shelf in a supermarket will lead the customer to buy more, this occurs because of its visibility, popularity or variety and then generate higher demand. In such a case, the demand rate is no longer a constant, but it depends on the stock level. This phenomenon is termed as ‘stock dependent consumption rate’. In general, ‘stock dependent consumption rate’ consists of two kinds. One is that the consumption rate is a function of order quantity (initial stock level) and the other is that the consumption rate is a function of inventory level at any instant of time.
The related analysis on such inventory system with stock-dependent consumption rate was studied by Levin et al. (1972) that “large piles of consumer goods displayed in a supermarket will lead customers to buy more. Yet, too many goods piled up in everyone’s way leave a negative impression on buyers and employees alike.” Silver and Peterson (1985) also noted that sales at the retail level tend to be proportional to the amount of inventory displayed. Gupta and Vrat (1986) assumed that the demand rate was a function of initial stock level. To quantify this, Baker and Urban (1988) established an economic order quantity model (or EOQ) for a power-form inventory-level dependent demand pattern (i.e., the demand rate at time t is $D(t)=\alpha[I(t)]^\beta$, where I(t) is the inventory level, $\alpha>0$, and $0<\beta<1$). Mandal and Phaujdar (1989) then incorporated deteriorating items with linearly stock-dependent demand. Vrat and Padmanaban (1990) developed an inventory model under a constant inflation rate for initial stock-dependent consumption rate. Datta and Pal (1990) modified the model of Baker and Urban’s (1988) by assuming that the stock-dependent demand rate was down to a given level of inventory, beyond which it is a constant. They presented an EOQ model in which the demand rate was dependent on the instantaneous stock amount displayed until a given level of inventory L is achieved. After this given inventory level, the demand rate becomes constant (i.e., $D(t)=\alpha[I(t)]^\beta$ if $I(t)>L$ and $D(t)=\alpha L^\beta$ if $0 \leq I(t) \leq L$). Goswami and Chaudhuri (1991) discussed different types of inventory models with linear trend in demand. Bar-Lev et al. (1994) developed an extension of the inventory-level-dependent demand-type EOQ model with random yield. Hariga (1995) studied the effects of inflation and time value of money on an inventory model with time-dependent demand rate and shortages. Bhunia, A.K. and Maiti, M. (1998) gave deterministic models of

In this chapter, an inventory model with depletion rate dependent holding cost has been developed. The demand rate is a power function of the on-hand inventory down to a certain stock level, at which the demand rate becomes a constant. We prove that the optimal replenishment policy not only exist but also is unique. Furthermore, we provide simple solution procedures for finding the maximum total profit per unit time. Numerical examples have also been given to illustrate the model.
2. NOTATIONS AND ASSUMPTIONS
2.1. Notations

To develop the mathematical model of inventory replenishment schedule, the notation adopted in this chapter is as below:

- \( A \) the replenishment cost per order.
- \( c \) the purchasing cost per unit.
- \( s \) the selling price per unit, where \( s > c \).
- \( Q \) the ordering quantity per cycle.
- \( I \) the maximum inventory level per cycle.
- \( C_1(t) \) the holding cost per unit per unit time.
- \( C_2 \) the backorder cost per unit per unit time.
- \( C_3 \) the opportunity cost (i.e., goodwill cost) per unit.
- \( t_1 \) the time at which the inventory level reaches \( S_0 \), where \( S_0 \) is given.
- \( t_2 \) the time at which the inventory level reaches zero.
- \( t_3 \) the length of period during which shortages are allowed.
- \( T \) the length of the inventory cycle, hence \( T = t_2 + t_3 \).
- \( I_1(t) \) the level of positive inventory at time \( t \), where \( 0 \leq t \leq t_1 \).
\( I_2(t) \) the level of positive inventory at time \( t \), where \( t_1 \leq t \leq t_2 \).

\( I_3(t) \) the level of negative inventory at time \( t \), where \( t_2 \leq t \leq T \).

\( \Pi(t_1,t_3) \) the total profit per unit time with two-component demand rate.

\( P(t_2,t_3) \) the total profit per unit time with constant demand rate.

2.2. ASSUMPTIONS

In addition, the following assumptions are imposed:

1. Replenishment rate is infinite, and lead time is zero.

2. The time horizon of the inventory system is infinite.

3. The demand rate is dependent on the on-hand inventory down to a level \( S_0 \), where \( S_0 \) is given and fixed, beyond which it is assumed to be a constant, that is, when the on-hand inventory level is \( I(t) \), the demand rate \( R(I(t)) \) of the item is considered to be of the form

\[
R(I(t)) = \begin{cases} 
\alpha[I(t)]^\beta, & I(t) \geq S_0 \\
D, & 0 \leq I(t) < S_0 
\end{cases}
\]

where \( \alpha > 0 \) and \( 0 < \beta < 1 \) are termed as scale and shape parameters respectively, \( D(>0) \) is a constant such that \( D = \alpha S_0^\beta \).

4. Shortages are allowed and the demand rate \( R(I(t)) \) is given by:
\[ R(I(t)) = D, \quad I(t) < 0: \]

We adopt the concept used in Abad, where some of the unsatisfied demand is backlogged, and the fraction of shortages backordered is \( \frac{1}{1 + \delta x} \), where \( x \) is the waiting time up to the next replenishment and \( \delta \) is a positive constant.

\[ \text{Fig 1: Graphical representation of the inventory system} \]

### 3. MATHEMATICAL FORMULATION

In the present model, the parameter \( S_0 \) is exogenous. Depending on the constant \( S_0 \) and the maximum inventory level \( I \), the inventory problem here has two situations: (i) \( I \geq S_0 \) and (ii) \( I < S_0 \).

#### 3.1. Inventory problem with \( I \geq S_0 \)

Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, we consider the following time intervals
separately, \([0,t_1],[t_1,t_2]\) and \([t_2,T]\). During the interval \([0,t_1]\), the inventory is depleted due to the effect of demand dependent on the on-hand inventory level and reaches the level \(S_0\) at time \(t = t_1\). Hence, the inventory level is governed by the following differential equation:

\[
\frac{dI_1(t)}{dt} = -\alpha [I_1(t)]^\beta, \quad 0 < t < t_1
\]

with the boundary condition \(I_1(t_1) = S_0\). Solving the differential equation, we get the inventory level as:

\[
I_1(t) = \left[ S_0^{1-\beta} + \alpha (1 - \beta)(t_1 - t) \right]^{\frac{1}{1-\beta}}, \quad 0 \leq t \leq t_1
\]

After the time \(t = t_1\), the demand rate becomes a constant \(D\), and the inventory level falls to zero at time \(t = t_2\). During the interval \([t_1,t_2]\), the inventory is depleted due to the effect of demand. Hence, the inventory level is governed by the following differential equation:

\[
\frac{dI_2(t)}{dt} = -D, \quad t_1 \leq t \leq t_2
\]

with the boundary condition \(I_2(t_2) = 0\). Solving the differential equation, we obtain the inventory level as

\[
I_2(t) = D(t_2 - t), \quad t_1 \leq t \leq t_2
\]

Due to the continuity of \(I_1(t)\) and \(I_2(t)\) at point \(t = t_1\), it follows that

\[
S_0 = D(t_2 - t).
\]
which implies

\[ t_2 = t_1 + \frac{S_0}{D}. \]

Thus, \( t_2 \) is a function of \( t_1 \). Furthermore, at time \( t_2 \), shortage occurs and the inventory level starts dropping below 0. During \([t_2, T]\), the inventory level only depends on demand, and a fraction \( \frac{1}{1+\delta(T-t)} \) of the demand is backlogged, where \( t \in [t_2, T] \). The inventory level is governed by the following differential equation:

\[
\frac{dI_3(t)}{dt} = -\frac{D}{1+\delta(T-t)}, \quad t_2 < t < T,
\]

with the boundary condition \( I_3(t_2) = 0 \). Solving the differential equation, we obtain the inventory level as

\[
I_3(t) = -\frac{D}{\delta} \{ \ln[1+\delta(T-t_2)] - \ln[1+\delta(T-t)] \}, \quad t_2 \leq t \leq T. \]

Hence

\[
I_1(t) = \left[ S_0^{1-\beta} + \alpha(1-\beta)(t_1-t) \right]^{1-\beta}, \quad 0 \leq t \leq t_1.
\]

\[
I(t) = \begin{cases} 
I_1(t) = \left[ S_0^{1-\beta} + \alpha(1-\beta)(t_1-t) \right]^{1-\beta}, & 0 \leq t \leq t_1, \\
I_2(t) = D(t_2-t), & t_1 \leq t \leq t_2, \\
I_3(t) = -\frac{D}{\delta} \{ \ln[1+\delta(T-t_2)] - \ln[1+\delta(T-t)] \}, & t_2 \leq t \leq T.
\end{cases}
\]

Therefore, the ordering quantity over the replenishment cycle can be determined as

\[
Q = I_1(0) - I_3(T) = \left[ S_0^{1-\beta} + \alpha(1-\beta)t_1 \right]^{1-\beta} + \frac{D\ln(1+\delta t_3)}{\delta}. \quad (2)
\]
and the maximum inventory level per cycle is

\[ I = I_1(0) = \left[ S_0^{1-\beta} + \alpha(1 - \beta)t_1 \right]^{1-\beta}. \]

Based on Eqs. (1) and (2), the total profit per cycle consists of the following elements:

1. Ordering cost per cycle = \( A \),

2. We assumed that holding cost per unit per unit time when demand depends upon stock level

\[ C_1(t) = \begin{cases} 
    a + \frac{b}{I_1}, & t \in [0, t_1]. \\
    a, & t \in [t_1, t_2].
\end{cases} \]

Where, \( a \) and \( b \) are positive constant with \( aD > b \).

Holding cost per cycle

\[ = \frac{a}{\alpha(2 - \beta)} \left[ S_0^{2-\beta} - \left( S_0^{1-\beta} + \alpha(1 - \beta)t_1 \right)^{2-\beta} \right] + \frac{aS_0^2}{2D} - \frac{bt_1S_0^{1-\beta}}{\alpha} - \frac{b(1 - \beta)t_1^2}{2}, \]

3. Backorder cost per cycle = \( \frac{C_2D}{\delta^2} [\delta t_3 - \ln(1 + \delta t_3)] \),

4. Opportunity cost due to lost sales per cycle = \( \frac{C_3D}{\delta} [\delta t_3 - \ln(1 + \delta t_3)] \),

5. Purchase cost per cycle = \( cQ = c \left[ S_0^{1-\beta} + \alpha(1 - \beta)t_1 \right]^{1-\beta} + \frac{cD}{\delta} \ln(1 + \delta t_3), \)
6. Sales revenue per cycle: 
\[ sQ = s \left( S_0^{1-\beta} + \alpha (1-\beta)t_1 \right)^{\frac{1}{1-\beta}} + \frac{sD}{\delta} \ln(1+\delta t_3) . \]

Therefore, the total profit per unit time of our model is obtained as follows:

\[ \Pi(t_1, t_3) = \frac{1}{(t_2 + t_3)} \left[ (s-c)[S_0^{1-\beta} + \alpha (1-\beta)t_1]^{\frac{1}{1-\beta}} - A + \alpha (\beta-2) \frac{a}{\alpha} [S_0^{1-\beta} + \alpha (1-\beta)t_1]^{\frac{2-\beta}{1-\beta}} \right. \]

\[ \left. - aS_0^{2-\beta} \left( \frac{1}{\alpha (\beta-2)} + \frac{S_0^{\beta}}{2D} + \frac{b}{\alpha} S_0^{1-\beta} t_1 + \frac{b}{2} (1-\beta)t_1^2 + D(s-c)t_3 \right) \right] \]

\[ - \frac{D[C_2 + \delta(s-c+C_3)]}{\delta^2} \left[ \delta t_3 - \ln(1+\delta t_3) \right] . \]

(3)

To maximize the total profit per unit time, taking the first derivative of \( \Pi(t_1, t_3) \) with respect to \( t_1 \) and \( t_3 \), respectively, we obtain

\[ \frac{\partial \Pi(t_1, t_3)}{\partial t_1} = -\frac{1}{(t_1 + t_3 + S_0^D)} \left[ \Pi(t_1, t_3) - [S_0^{1-\beta} + \alpha (1-\beta)t_1]^{\frac{1}{1-\beta}} (\alpha(s-c) \]

\[ - a[S_0^{1-\beta} + \alpha (1-\beta)t_1] + \frac{b}{\alpha} [S_0^{1-\beta} + \alpha (1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}} \right] . \]

(4)

and
\[
\frac{\partial \Pi(t_1,t_3)}{\partial t_3} = -\frac{1}{t_1 + t_3 + \frac{S_0}{D}} \left[ \Pi(t_1,t_3) - D(s-c) + \frac{D[C_2 + \delta(s-c+C_3)]t_3}{1 + \delta t_3} \right].
\]

(5)

The optimal solution of \( \Pi(t_1,t_3) \) must satisfy the equations \( \frac{\partial \Pi(t_1,t_3)}{\partial t_1} = 0 \) and \( \frac{\partial \Pi(t_1,t_3)}{\partial t_3} = 0 \), simultaneously, which implies

\[
\Pi(t_1,t_3) = a\alpha(1-\beta)[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}} \left[ \frac{(s-c)}{a(1-\beta)} - \frac{S_0^{1-\beta}}{a(1-\beta)} + \frac{b}{a(1-\beta)\beta^2} \left[ S_0^{1-\beta} + \alpha(1-\beta)t_1 \right]^{\frac{1-2\beta}{1-\beta}} - t_1 \right],
\]

(6)

and

\[
\Pi(t_1,t_3) = D(s-c) + \frac{D[C_2 + \delta(s-c+C_3)]t_3}{1 + \delta t_3}
\]

(7)

respectively. Because both the left hand sides in Eqs. (6) and (7) are the same, hence the right hand sides in these equations are equal, that is,

\[
\frac{D[C_2 + \delta(s-c+C_3)]t_3}{1 + \delta t_3} = D(s-c) - a\alpha(1-\beta)[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}} \left[ \frac{(s-c)}{a(1-\beta)} - \frac{S_0^{1-\beta}}{a(1-\beta)} + \frac{b}{a(1-\beta)\beta^2} \left[ S_0^{1-\beta} + \alpha(1-\beta)t_1 \right]^{\frac{1-2\beta}{1-\beta}} - t_1 \right].
\]
\[ \frac{S_0^{1-\beta}}{\alpha(1-\beta)} + \frac{b}{a(1-\beta)\alpha^2} \left[ S_0^{1-\beta} + \alpha(1-\beta) t_1 \right]^{1-\beta} - t_1 \] .

(8)

On the other hand, we substitute \( \Pi(t_1, t_3) \) in (3) into Eq. (7) and obtain

\[ \frac{D[C_2 + \delta(s-c+C_3)](t_1 + t_3 + \frac{S_0}{D})}{1 + \delta t_3} = A - (s-c)[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{1-\beta} \]

\[ -\frac{a}{\alpha(\beta-2)} [S_0^{1-\beta} + \alpha(1-\beta)t_1]^{2-\beta} + a\beta_0^{2-\beta} \left( \frac{1}{\alpha(\beta-2)} + \frac{S_0^\beta}{2D} \right) - \frac{b}{\alpha} S_0^{1-\beta} t_1 - \frac{b}{2} (1-\beta)t_1^2 \]

\[ + \frac{D[C_2 + \delta(s-c+C_3)]}{\delta^2} [\delta t_3 - \ln(1+\delta t_3)] + D(s-c)(t_1 + \frac{S_0}{D}) . \]

(9)

Now, we want to find the value of \( (t_1, t_3) \) which satisfies Eqs. (8) and (9), simultaneously. For convenience, we first let \( K(t_1) \) denote the right hand side of Eq. (8), that is,

\[ K(t_1) = D(s-c) - a\alpha(1-\beta)[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\beta} \left[ \frac{(s-c)}{a(1-\beta)} - \frac{S_0^{1-\beta}}{\alpha(1-\beta)} + \right] \]

\[ \frac{b}{a(1-\beta)\alpha^2} \left[ S_0^{1-\beta} + \alpha(1-\beta)t_1 \right]^{1-\beta} - t_1 \] , \hspace{1cm} t_1 \geq 0 .

(10)
It notes that $K(t_1)$ is a continuous function in $t_1 \in [0, \infty)$. Then Eq. (8) becomes

$$K(t_1) = \frac{D[C_2 + \delta(s - c + C_3)]t_3}{1 + \delta t_3}$$

(11)

which implies,

$$t_3 = \frac{K(t_1)}{D[C_2 + \delta(s - c + C_3)] - \delta K(t_1)}$$

(12)

Thus, $t_3$ is a function of $t_1$, and further we have

$$\frac{dt_3}{dt_1} = \frac{D[C_2 + \delta(s - c + C_3)]}{\{D[C_2 + \delta(s - c + C_3)] - \delta K(t_1)\}^2} \frac{dK(t_1)}{dt_1}.$$  

(13)

Furthermore, motivated by Eq. (9), we get

$$G(t_1) = A - (s - c)[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{1}{1-\beta}} - \frac{a}{\alpha(\beta-2)}[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{2-\beta}{1-\beta}} +$$

$$+aS_0^{2-\beta}\left(\frac{1}{\alpha(\beta-2)} + \frac{S_0^\beta}{2D}\right) - \frac{b}{\alpha}S_0^{1-\beta}t_1 - \frac{b}{2}(1-\beta)t_1^2 + D(s-c)(t_1 + \frac{S_0}{D})$$

$$+\frac{D[C_2 + \delta(s - c + C_3)]}{\delta^2}[\delta t_3 - \ln(1 + \delta t_3)] - \frac{D[C_2 + \delta(s - c + C_3)](t_1 + t_3 + \frac{S_0}{D})t_3}{1 + \delta t_3}$$
where $t_3$ is given as in Eq. (12). Taking the derivative of $G(t_1)$ with respect to $t_1$ and by using the relations shown in Eqs. (8), (10) and (13), we obtain

$$\frac{dG(t_1)}{dt_1} = -\frac{D[C_2 + \delta(s - c + C_3)](t_1 + t_3 + \frac{S_0}{D})}{(1 + \delta t_3)^2} \frac{dt_3}{dt_1}$$

$$= -(t_1 + t_3 + \frac{S_0}{D}) \frac{dK(t_1)}{dt_1}.$$  

(15)

In order to prove the existence and uniqueness of the optimal solution $t^*_1$ which satisfies equation $G(t^*_1) = 0$, we have to investigate the property of function $K(t_1)$. Taking the derivative of $K(t_1)$ with respect to $t_1$, we have

$$\frac{dK(t_1)}{dt_1} = -\beta[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{2\beta-1}{1-\beta}} \left[ \alpha(s-c) + [S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}} \right]$$

$$- b - a\alpha[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}} - b(1 - 2\beta) + a\alpha(1-\beta)[S_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}}$$

(16)
Because $K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} > 0$, since $(aD > b)$ and it can be shown that \[
\lim_{t_1 \to \infty} K(t_1) = \infty, \text{ thus, we have the following result.}
\]

**Lemma 1:** Let $K(t_1)$ be defined as in Eq. (8), we have, If $aD > b$, then $K(t_1)$ is a strictly increasing function in $t_1 \in [0, \infty)$, and the minimum value of

$K(t_1)$ is $K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} > 0$.

**Proof:** See Appendix A.

Now, let us consider the following two sub cases: (i)

(i) \[
K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \geq \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}
\]

(ii) \[
K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} < \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}
\]

For convenience, we let $F(t_3) = \frac{D[C_2 + \delta(s - c + C_3)t_3]}{1 + \delta t_3}, t_3 \geq 0$

(17)

**3.1.1 Case 1:** When $K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \geq \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}$, Eqs. (17),

becomes \[
F(t_3) < \frac{D[C_2 + \delta(s - c + C_3)t_3]}{1 + \delta t_3} \leq aS_0 - \frac{bS_0^{1-\beta}}{\alpha} = K(0), \text{ for } t_3 \in [0, \infty).
\]

By lemma, we know that $K(t_1)$ is strictly increasing function in $t_1 \in [0, \infty)$, hence for any
given \( t_1 \in [0, \infty) \), there does not exist a value \( t_3 \in [0, \infty) \) such that \( K(t_1) = F(t_3) \), i.e. for any given \( t_1 \in (0, \infty) \), we cannot find a value \( t_3 \) which satisfies Eqs. (8). However, for this situation, from Eqs. (4), (7), (9) and (17), we have

\[
\frac{\partial \Pi(t_1, t_3)}{\partial t_1} = \frac{F(t_3) - K(t_1)}{(t_1 + t_3 + \frac{S_0}{D})} < \frac{F(t_3) - K(0)}{(t_1 + t_3 + \frac{S_0}{D})} < 0, \text{ for any } t_1 \in (0, \infty) \text{ and } t_3 \in (0, \infty).
\]

Therefore, if \( K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \geq \frac{D[C_2 + \delta(s - c + C_3)]}{\delta} \), the maximum value of \( \Pi(t_1, t_3) \) occurs at the boundary point \( t_1^* = 0 \). In special circumstance that \( t_1^* = 0 \), the optimal values of \( t_2 \) (denoted by \( t_2^* \)) can be obtained by \( t_2 = t_1 + \frac{S_0}{D} \) and is \( t_2^* = \frac{S_0}{D} \).

Then the total profit per unit time in Eq. (3) becomes,

\[
\Pi(t_3) \equiv \Pi(0, t_3) = D(s - c) - \frac{1}{(t_3 + \frac{S_0}{D})} \left[ A + \frac{aS_0^2}{2D} + \frac{D[C_2 + \delta(s - c + C_3)]}{\delta^2} \left[ \delta t_3 - \ln(1 + \delta t_3) \right] \right]
\]

(18)

The necessary condition to find the optimal solution of \( \Pi(t_3) \) is \( \frac{d \Pi(t_3)}{dt_3} = 0 \), which implies

\[
A + \frac{aS_0^2}{2D} + \frac{D[C_2 + \delta(s - c + C_3)]}{\delta^2} \left[ \delta t_3 - \ln(1 + \delta t_3) \right] - \frac{D[C_2 + \delta(s - c + C_3)]t_3 + \frac{S_0}{D}}{1 + \delta t_3} = 0.
\]
(19) Let $Z(t_3) = A + \frac{aS_0^2}{2D} + \frac{D[C_2 + \delta(s - c + C_3)]}{\delta^2}\left[t_3 - \ln(1 + \delta t_3)\right]$ -

$D[C_2 + \delta(s - c + C_3)]t_3(t_3 + \frac{S_0}{D})$ \hspace{1cm} $1 + \delta t_3$

The derivative of $Z(t_3)$ with respect to $t_3$ is

$$\frac{dZ(t_3)}{dt_3} = -\frac{D[C_2 + \delta(s - c + C_3)](t_3 + \frac{S_0}{D})}{(1 + \delta t_3)^2} < 0,$$

thus, $Z(t_3)$ is strictly decreasing function in $t_3 \in [0, \infty)$. Furthermore we have

$Z(0) = A + \frac{aS_0^2}{2D} > 0$ and $\lim_{t_3 \to \infty} Z(t_3) < 0$. By using the intermediate value theorem, there exists a unique solution $t_3^* \in (0, \infty)$ such that $Z(t_3^*) = 0$, that is, $t_3^*$ is a unique value which satisfies Eq. (19). Summarize the above arguments, we obtain the following theorem

**Theorem 1:** For $aS_0 - \frac{bS_0}{\alpha} \geq \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}$, the optimal value of

$(t_1, t_2, t_3)$ is given by $t_1^* = 0, t_2^* = \frac{S_0}{D}$ and $t_3^*$ is the value which satisfies Eq. (19).

When, $t_1^* = 0$, the inventory problem becomes the regular EOQ with constant demand rate and partial backordering. Once the optimal value $(t_1^*, t_3^*) = (0, t_3^*)$ is obtained by Eqs. (18) and (19), the maximum total profit per unit
time $\prod(t^*_3) = D(s-c) + \frac{D[C_2 + \delta(s-c+C_3)]t^*_3}{1+\delta t^*_3}$  \hspace{1cm} (20)

follows. The maximum inventory level per cycle is $I^* = S_0$.

3.1.2. Case 2: Where $K(0) = aS_0 - \frac{bs_0^{1-\beta}}{\alpha} < \frac{D[C_2 + \delta(s-c+C_3)]}{\delta}$, from lemma 1, $K(t_1)$ is a strictly increasing function of $t_1 \in [0, \infty)$, thus we can find a unique value $\hat{t}_1 \in (0, \infty)$ such that $K(\hat{t}_1) = \frac{D[C_2 + \delta(s-c+C_3)]}{\delta}$. Furthermore, for any given $t_1 \geq \hat{t}_1$, we have

$$K(t_1) \geq K(\hat{t}_1)$$

$$= \frac{D[C_2 + \delta(s-c+C_3)]}{\delta} > \frac{D[C_2 + \delta(s-c+C_3)]}{\delta} \left[1 - \frac{1}{1+\delta t_3}\right] = F(t_3).$$

It implies that we cannot find a $t_3 \in [0, \infty)$ such that Eq. (8) holds. Therefore, the optimal solution of $t_1$ which satisfies Eq. (8) will occur in the interval $(0, \hat{t}_1)$. On the other hand, from the definition of $F(t_3)$ in Eq. (17), it can be shown that $F(t_3)$ is a continuous and strictly increasing function in $t_3 \in [0, \infty)$. Besides, we have

$$F(0)=0, \text{ and } \lim_{t_3 \to \infty} F(t_3) = \frac{D[C_2 + \delta(s-c+C_3)]}{\delta} = K(\hat{t}_1) > K(t_1), \text{ for any } t_1 \in [0, \hat{t}_1).$$

Thus, for any given $t_1 \in (0, \hat{t}_1)$, there exists a unique value $t_3 \in (0, \infty)$ such that $F(t_3) = K(t_1)$. Consequently, for any given $t_1 \in [0, \hat{t}_1)$,
when \( K(0) = aS_0 - \frac{bS_0^{1-\beta}}{\alpha} < \frac{D[C_2 + \delta(s-c+C_3)]}{\delta} \), we can find a unique value \( t_3 \in (0, \infty) \) satisfying Eq. (8). Therefore, optimal value \( t_1^* \in [0, \hat{t}_1) \) is obtained, the optimal solutions of \( t_2, t_3 \) and \( T \) are as follows \( t_2^* = t_1^* + \frac{S_0}{D} \)

(21)

\[
t_3^* = \frac{K(t_1^*)}{D[C_2 + \delta(s-c+C_3)] - \delta K(t_1^*)}
\]

(22)

\[
T^* = t_2^* + t_3^*
\]

(23)

Now, we want to prove the existence and uniqueness of \( t_1^* \) in \((0, \hat{t}_1)\). By using

\[
(t_1 + t_3 + \frac{S_0}{D}) > 0 \text{ and } \frac{dK(t_1)}{dt_1} > 0 \text{ for } t_1 \in (0, \hat{t}_1), \text{ from Eq. (15), we obtain } \frac{dG(t_1)}{dt_1} < 0.
\]

Therefore, \( G(t_1) \) is a strictly decreasing function in \( t_1 \in [0, \hat{t}_1) \). Furthermore, from Eq. (12), we have \( t_3 \to \infty \) as \( t_1 \to \hat{t}_1^- \), and \( \lim_{t_1 \to \hat{t}_1^-} G(t_1) = -\infty < 0 \) and

\[
G(0) = A + \left( aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \right) \left( \frac{1}{\delta} - \frac{S_0}{2D} \right) + \frac{bS_0^{2-\beta}}{2D\alpha}
\]
\[
- \frac{D[C_2 + \delta(s - c + C_3)]}{\delta^2} \ln \frac{D[C_2 + \delta(s - c + C_3)]}{D[C_2 + \delta(s - c + C_3)] - \delta \left( aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \right)}
\]

(24)

Note that the value in the brace is well defined, becomes we have

\[
\frac{D[C_2 + \delta(s - c + C_3)]}{\delta} > K(0) = \left( aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \right).
\]

Then we have the following result.

Lemma 2: For \( \frac{D[C_2 + \delta(s - c + C_3)]}{\delta} > \left( aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \right) \) we have;

(a) If \( G(0) \leq 0 \), then the optimal value of \( t_1^* \) is \( t_1^* = 0 \).

(b) If \( G(0) > 0 \), then the solution \( t_1^* \in (0, \hat{t}_1) \) which satisfies Eq. (9) not only exist but also is unique.

Proof: See Appendix B.

Lemma 2(a) indicates that if \( \frac{D[C_2 + \delta(s - c + C_3)]}{\delta} > \left( aS_0 - \frac{bS_0^{1-\beta}}{\alpha} \right) \) and \( G(0) \leq 0 \) then the optimal time at which the inventory level reaches \( S_0 \) is \( t_1^* = 0 \). It implies the maximum inventory level in this system is \( I^* = S_0 \). The corresponding value of \( t_3 \) can be found from
Eq. (22) and is given by \( t_3^* = \frac{K(0)}{D[C_2 + \delta(s - c + C_3)] - \delta K(0)} \). However,

since \( \Pi(0, t_3^*) = \Pi(t_3^*) \), where \( t_3^* \) is the value of \( t_3 \) which satisfies Eq. (19). Thus, \( t_3^* \) is the optimal value of \( t_3 \) and the maximum total profit per unit time \( \Pi(t_3^*) \) is obtained in Eq. (20).

Lemma 2(b) reveals that if \( D[C_2 + \delta(s - c + C_3)] \delta > aS_0 - bS_0^{1-\beta} / \alpha \) and \( G(0) > 0 \), then \( t_4^* \in (0, \hat{t}_1) \) and is unique. Furthermore, the unique solution will be proved to be indeed a global maximum point by checking the second order optimality conditions, that is, we have the following main result.

**Theorem 2:** For \( D[C_2 + \delta(s - c + C_3)] \delta > aS_0 - bS_0^{1-\beta} / \alpha \), if \( G(0) > 0 \), then the point \((t_1^*, t_3^*)\) which satisfies Eqs. (8) and (9) simultaneously is the global maximum point of the total profit per unit time.

**Proof:** See Appendix C.

Once the optimal solution \((t_1^*, t_3^*)\) is obtained, we substitute \((t_1^*, t_3^*)\) into Eqs. (3), optimal ordering quantity per cycle, \( Q^* \), and the maximum total profit per unit time \( \Pi(t_1^*, t_3^*) \), are as

\[
Q^* = \left[ S_0^{1-\beta} + \alpha(1-\beta)t_1^* \right]^{1-\beta} + \frac{DIn(1+\delta t_3^*)}{\delta}
\]
\[
\Pi(t_4^*, t_3^*) = D(s-c) + \frac{D[C_2 + \delta(s-c+C_3)]t_3^*}{1+\delta t_3^*}
\]

(26)

3.2. Inventory problem with \( I < S_0 \)

When the stock level \( S_0 \) at which the demand rate changes from being inventory level dependent to a constant \( D \) is relatively high, an optimal inventory control policy would never order enough to rise \( I \) to \( S_0 \). Under this situation, that \( I < S_0 \), the demand rate is never a function of the inventory level and is always the constant \( D \). That is, what is being discussed in this section is the same as what was discussed in the previous section, but with a very large value of \( S_0 \).
5 CONCLUSIONS:
An order level inventory models deals with stock dependent demand and variable holding cost. Partially backlogged shortages with time dependent backlogging rate are permitted in this study. Backlogging rate is taken as waiting time for the next replenishment. In stock dependent demand phenomena, demand is initially high and decrease with stock. But holding cost contains two parts first is constant as fixed rent, insurance, tax, etc. and second is variable part which depends upon the marketing policy of warehouse holder, time for which inventory held, quantity in warehouses. Every warehouse holder wants to take more and more order of stocks to hold. Consequently, warehouse holder may set his marketing policy based on depletion of stock due to having limited storage space. This policy affects the holding cost. Using this fact, we have considered a different function of holding cost. We provide simple solution procedures for finding the maximum total profit per unit time. Numerical examples have also been given to illustrate the model.

The proposed model can be extended in several ways. For instance, we may consider the permissible delay in payments and the deterministic demand function to stochastic fluctuating demand patterns.