In this chapter we give sufficient number of examples to make our investigations in the earlier chapters meaningful. Recall that we have characterized completely those W-T extensions which are compact or linkage compact. We did this both in terms of the trace systems of the extensions as well as in terms of the nearnesses generating the extensions. These may be thought to be necessary and sufficient conditions for W-T extensions to be compact or linkage compact. These conditions would be meaningless if all W-T extensions are compact or linkage compact. Thus the question naturally arose whether all W-T extensions are compact or linkage compact. We list below the examples which answer the above question including some others which have already been raised or to be raised.

1. Example of a LO-nearness which is proximal but fails to be Wallman (4.1.1).
2. Example of a LO-nearness which is contigual but fails to be proximal as well as Wallman (4.1.3).

3. Example of cluster generated LO-nearness which fails to be contigual as well as Wallman (4.1.4).

4. Example of proximal Wallman nearness (4.1.5).

5. Example of contigual Wallman nearness which fails to be proximal (4.1.8).

6. Example of a Wallman nearness which fails to be contigual (4.1.9).

The following diagram explains the position of the examples.

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<table>
<thead>
<tr>
<th></th>
<th>Contigual</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Cluster</td>
<td></td>
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<tr>
<td>generated</td>
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<tr>
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<tr>
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<td></td>
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<tr>
<td>2</td>
<td></td>
<td>Proximal</td>
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<tr>
<td>1</td>
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<tr>
<td>Wallman</td>
<td>Cont.Wallman</td>
<td>Prox.Wallman</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>
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In addition to the above examples we also present two more examples:

7. Example of a covered proximity $\pi$ such that there exists contigual LO-nearness in the proximity class of $\pi$ which fails to be Wallman (4.1.2).

8. Example of a covered proximity $\pi$ such that the nearness generated by all the $\pi$-clans is different from the nearness generated by all the Wallman $\pi$-clans (4.1.7).

Before we begin to give examples we recall the facts:

If $X$ is a $T_1$-space then

(i). $\omega \rightarrow E_\omega$ is a bijection from $\mathcal{N}(X)$ to $\mathcal{E}(X)$ (Theorem 2.1.23).

(ii). Restriction of the map $\omega \rightarrow E_\omega$ to $\mathcal{N}_W(X)$ is a bijection from $\mathcal{N}_W(X)$ to $\mathcal{E}_W(X)$ (Remark 2.3.6).

(iii). $\omega$ is contigual (proximal) if and only if $E_\omega$ is compactification (linkage compactification) (Corollary 2.1.24).

In view of above facts Examples as listed above will either be given in terms of extensions or in terms of nearnesses.

First we give an example of a linkage compact principal $T_1$-extension which is not a $W$-$T$ extension.
Example 4.1.1. Let $X$ be a compact $T_1$-space such that it has two infinite components. Then the relation $\pi$ defined by

$$\pi = \{ (E, F) : \overline{E} \cap \overline{F} \neq \emptyset \text{ or } E \text{ and } F \text{ are both infinite } \}$$

is a LO-proximity compatible with the space $X$ and $\pi$ is not covered (see Remark 2.2.8).

Let

$$A^*(\pi) = \{ A \subseteq \mathcal{P}(X) : A \text{ is contained in a } \pi\text{-clan } \}.$$ 

Here let us simply write $A^*$ instead of writing $A^*(\pi)$ frequently.

Obviously $A^*$ is the largest nearness in the proximity class of $\pi$. Moreover $A^*$ is proximal and is a cluster generated LO-nearness compatible with the space $X$. Hence $E_{A^*}$ is a linkage compact principal $T_1$-extension of $X$ (see Corollary 2.1.24).

We claim that $E_{A^*}$ is not a W-T extension. If it is, then $A^*$ is a Wallman nearness (Remark 2.3.6). Thus

$$A^* = A_{W}(\pi_{A^*}) = A_{W}(\pi) \text{ for } A^* \text{ belongs to the proximity class of } \pi \text{ and hence } A^* \text{ is a Wallman nearness in the proximity class of } \pi.$$ 

Thus it follows from Theorem 2.2.10, That $\pi$ is covered - a contradiction to the fact that $\pi$ is not covered.
Note that if \( \pi \) is a LO-proximity on a \( T_1 \)-space such that \( \pi \) is not covered then every nearness in the proximity class of \( \pi \) is not Wallman (Theorem 2.2.10). In particular \( \mathcal{V}^*(\pi) \) is not a Wallman nearness which is of course contigual. In fact \( \mathcal{V}^*(\pi) \) is proximal.

Thus it would be interesting to find an example of a contigual LO-nearness in the proximity class of a covered proximity which fails to be Wallman. We are led to the following example.

**Example 4.1.2.** Let \( X \) be a noncompact \( T_1 \)-space such that there are three distinct n.p. ultraclosed filters \( \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \) on \( X \). Note that such spaces do exist (Example 1.3.9). Let

\[
\mathcal{V} = \{ b(\mathcal{U}) : \mathcal{U} \in \Psi(X) - \{ \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \} \}
\]

\[
\cup \{ b(\mathcal{U}_1 \cup \mathcal{U}_2), b(\mathcal{U}_2 \cup \mathcal{U}_3), b(\mathcal{U}_3 \cup \mathcal{U}_1) \}.
\]

Clearly \( \mathcal{V} \) generates a nearness \( \mathcal{V}_\mathcal{V} \) compatible with the space \( X \). Since in view of Theorem 3.1.1(iii) and (vi), each member of \( \mathcal{V} \) is a c-grill and also each member of \( \mathcal{V} \) is maximal (with respect to the set inclusion) it follows that \( \mathcal{V}_\mathcal{V} \) is a LO-nearness on the space \( X \) which is cluster generated and \( X^{\mathcal{V}_\mathcal{V}} = \mathcal{V} \) (Proposition 1.4.5 and Theorem 1.4.8 (iv)).
Now we show that $\pi_\mathcal{V}$ is a covered proximity. Let $(A, B) \in \pi_\mathcal{V}$ then $\{A, B\} \in \mathcal{V}$. Since $\mathcal{V}$ is cluster generated, there exists $\mathcal{F}_0 \in \mathcal{V}$ such that $\{A, B\} \subseteq \mathcal{F}_0$. Since $\mathcal{F}_0 \in \Gamma(X) \cap \Gamma(\pi_\mathcal{V})$, $\pi_\mathcal{V}$ is a covered proximity. Thus $\mathcal{V}$ is in the proximity class of a covered proximity.

We now show that $\mathcal{V}$ is contigual. Choose a grill $\mathcal{F}_0$ on $X$ such that each finite subfamily of $\mathcal{F}_0$ is contained in a member of $\mathcal{V}$. We claim that $\mathcal{F}_0 \subseteq \mathcal{F}$ for all $\mathcal{F} \in \mathcal{V}$. If $\mathcal{F}_0 \neq b(\mathcal{U}_1 \cup \mathcal{U}_2), \mathcal{F}_0 \neq b(\mathcal{U}_2 \cup \mathcal{U}_3), \mathcal{F}_0 \neq b(\mathcal{U}_3 \cup \mathcal{U}_1)$, then we can choose $A_1 \in \mathcal{F}_0 - b(\mathcal{U}_1 \cup \mathcal{U}_2)$, $A_2 \in \mathcal{F}_0 - b(\mathcal{U}_2 \cup \mathcal{U}_3), A_3 \in \mathcal{F}_0 - b(\mathcal{U}_3 \cup \mathcal{U}_1)$.

Let $\{B_1, B_2, \ldots, B_n\}$ be any arbitrary finite subfamily of $\mathcal{F}_0$. Then $\{A_1, A_2, A_3, B_1, B_2, \ldots, B_n\}$ is contained in some member of $\mathcal{V}$. Since none of $b(\mathcal{U}_1 \cup \mathcal{U}_2), b(\mathcal{U}_2 \cup \mathcal{U}_3)$ and $b(\mathcal{U}_3 \cup \mathcal{U}_1)$ contains $\{A_1, A_2, A_3, B_1, B_2, \ldots, B_n\}$ it follows that there exists an ultraclosed filter $\mathcal{U} \neq \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ such that $\{A_1, A_2, A_3, B_1, B_2, \ldots, B_n\} \subseteq b(\mathcal{U})$. Hence $\bar{B}_1 \cap \bar{B}_2 \cap \ldots \cap \bar{B}_n \neq \emptyset$ . Thus $\mathcal{F}$ has the F.I.P.

By Proposition 3.1.3 there exists an ultraclosed filter $\mathcal{U}_0$ such that $\mathcal{F}_0 \subseteq b(\mathcal{U}_0)$. Clearly $\mathcal{U}_0 \neq \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ and
hence $b(\mathcal{U}_0) \in \gamma$. Thus $x^{\gamma} = \gamma$ is a finitely determined collection of grills on $X$ and hence by Theorem 1.4.15 $\mathcal{U}_x$ is contigual.

Next we show that $\mathcal{U}_x$ is not a Wallman nearness. Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ be three n.p. ultrafilters such that $\mathcal{U}_1 \subseteq \mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{U}_2, \mathcal{U}_3 \subseteq \mathcal{U}_3$.

Set

$$\mathcal{Q}^* = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3.$$ 

By Theorem 3.1.1 (v), $\mathcal{U}_K \subseteq b(\mathcal{U}_K)$ for all $K = 1, 2, 3$.

Note that $\mathcal{Q}^* \in \Gamma^w(X)$ and each two-element subfamily of $\mathcal{Q}^*$ is contained in one of $b(\mathcal{U}_1 \cup \mathcal{U}_2), b(\mathcal{U}_2 \cup \mathcal{U}_3)$ and $b(\mathcal{U}_3 \cup \mathcal{U}_1)$ but in view of Theorem 3.1.1 (vi) & (vii), $\mathcal{Q}^*$ is not contained in any element of $\gamma$.

Thus $\gamma$ fails to be a binary collection with respect to $\Gamma^w(X)$ and hence by Corollary 2.3.5 $\mathcal{U}_x$ is not Wallman.

Remark 4.1.3. The nearness $\mathcal{U}^\gamma$ of the above example is cluster generated LO-nearness on the space $X$. Note that $\gamma$ is the set of all $\mathcal{U}_x$-clusters. It has already been proved that $\mathcal{U}_x$ is contigual. Since $\gamma$ is not a binary collection with respect to $\Gamma^w(X)$, in particular $\gamma$ is not a binary collection and hence $\mathcal{U}_x$ is not proximal (Theorem 1.4.15).

Thus $\mathcal{U}_x$ is also an example of the fact that there exists
a contigu. LO-nearness on a $T_1$-space which not only fails
to be Wallman but also fails to be proximal.

Now we shall give an example of a well known extension
which fails to be Wallman-type as well as compactification.

**Example 4.1.4.** Let $\mathbb{R}$ be the set of reals with usual topo-
logy and $\mathbb{Q}$ be the subspace of $\mathbb{R}$ consisting to the rationals.
Then $E = (i, \mathbb{R})$ is a $T_1$-extension of $\mathbb{Q}$ where $i : \mathbb{Q} \to \mathbb{R}$
is the inclusion map.

Note that for any set $X \subset \mathbb{R}$,

$$\overline{X} \subset \bigcap \{ \overline{B} : B \subset \mathbb{Q}, \overline{B} \supset X \}$$

where $X$ is the
closure of $X$ in $\mathbb{R}$.

Also let $x \notin \overline{X}$ then there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \cap X = \emptyset.$$ 

So $X \subset \mathbb{R} - (x - \epsilon, x + \epsilon) = (-\infty, x - \epsilon] \cup [x + \epsilon, \infty)$. 

Let $A = \mathbb{Q} \cap (-\infty, x - \epsilon] \cup \mathbb{Q} \cap [x + \epsilon, \infty)$. 

Then $\overline{A} = (-\infty, x - \epsilon] \cup [x + \epsilon, \infty)$. 

Clearly $A \subset \mathbb{Q}$ such that $\overline{A} \supset X$ but $x \notin \overline{A}$ and
hence $\overline{X} \supset \bigcap \{ \overline{B} : B \subset \mathbb{Q}, \overline{B} \supset X \}$.

Thus $\overline{X} = \bigcap \{ \overline{B} : B \subset \mathbb{Q}, \overline{B} \supset X \}$.
This shows that \( E \) is also a principal extension of \( \mathcal{A} \) which is not compact. In fact \( E \) is equivalent to the principal \( T_1 \)-extension \( \mathcal{E}_\omega \) where \( \omega \) is given by

\[
\omega = \{ A \subseteq \mathcal{P}(\mathcal{A}) : \cap A \neq \emptyset \}.
\]

Clearly \( \omega \) is a cluster generated LO-nearness on \( \mathcal{A} \). In fact \( \omega = \omega_{\mathcal{E}} \) and hence by Theorem 2.1.23, \( E_\omega = E \).

Set

\[
\mathcal{B} = \{ \{ r, \infty \} \cap \mathcal{A} : r \in \mathbb{R} \}.
\]

Clearly \( \mathcal{B} \) is a base for a filter \( \mathcal{K} \) on \( \mathcal{A} \). Since \( \cap \mathcal{B} = \emptyset \), \( \mathcal{B} \notin \omega \) and hence \( \mathcal{K} \notin \omega \). Thus in view of Theorem 3.1.5 (iv), \( \omega \) is not a Wallman nearness, consequently by Remark 2.3.6, \( E_\omega \) is not a W-T extension.

Now we give an example of a Wallman nearness which is proximal.

**Example 4.1.5.** Let \( X \) be an infinite set with the cofinite topology. Then \( X \) is a \( T_1 \)-compact space. So by Theorem 3.1.6, the elementary nearness

\[
\mathcal{V}_\omega = \{ A \subseteq \mathcal{P}(X) : \cap A \neq \emptyset \}
\]

is a Wallman nearness compatible with the space \( X \) such that \( X^{\mathcal{V}_\omega} = \{ \emptyset \times X \} \). We shall show that \( \mathcal{V}_\omega \) is proximal. Let us first prove that \( X \) is linkage compact. For this let
be a linked grill on X. Then there are two cases to be considered.

**Case 1.** \( G \in \Gamma^W(X) \). Since \( \Psi(X) = \{ U_x : x \in X \} \)
there exists an \( x \in X \) such that \( U_x \subseteq G \). Let \( A \in G \).
Since \( G \) is linked and \( \{ x \} \in G \) we have
\( \{ x \} \neq \emptyset \) and hence \( x \in A \), consequently \( A \subseteq G_x \).
Thus \( G \subseteq G_x \).

**Case 2.** \( G \in \Gamma(X) - \Gamma^W(X) \). Then \( G \) does not contain
any finite set. If possible let \( G \) contains a finite set \( A \).
Then there exists an \( a \in A \) such that \( \{ a \} \in G \) and hence the
ultraclosed filter \( \mathcal{U}_a \subseteq G \) which implies \( G \in \Gamma^W(X) \)
a contradiction.

So \( A \subseteq G \) implies \( A \) is infinite and hence \( A = X \). Thus
\( G \subseteq G_x \) for all \( x \in X \).

Thus in any case \( G \subseteq G_x \) for some \( x \in X \). Therefore \( X \)
is linkage compact.

To prove \( \mathcal{U} \) is proximal, let \( G \in \Gamma(X) \) be such that
every two-element subfamily of \( G \) is contained in some
element \( G_x \) of \( X^\omega \) and hence \( G \) is a linked grill on \( X \)
and since \( X \) is linkage compact, there exists an \( x \in X \) such
that \( G \subseteq G_x \in X^\omega \). Thus \( X^\omega \) is a binary collection
of grills on \( X \). So in view of Theorem 1.4.15, \( \mathcal{U} \) is proximal.
Remark 4.1.6. Given a LO-proximity π it is natural to ask whether the nearness generated by all the π-clans is different from the nearness generated by all the Wallman π-clans.

We now wish to give an example of a covered proximity π such that the set of all Wallman π-clans and the set of all π-clans generate different nearnesses.

Before we give such an example, it is interesting to note that if π is a LO-proximity on a $T_1$-space such that π is not covered (for the existence of such a proximity see Remark 2.2.8) then there exists $(A,B) \in π$ such that no Wallman π-clan contain both A, B but there exists a maximal π-clan $\mathcal{G}$ such that $\mathcal{G} \supset \{A,B\}$. Hence $\mathcal{V}_W(\pi) \neq \mathcal{V}(\pi)$.

Thus for a LO-proximity which is not covered, it is always true that the Wallman π-clans and the π-clans do generate different nearnesses. It is worth mentioning here that $\mathcal{V}_W(\pi)$ always belongs to the proximity class of π but $\mathcal{V}_W(\pi)$ fails to belong to the proximity class of π. Thus it would be much more interesting to find a covered proximity π such that the set of all π-clans and the set of all Wallman π-clans generate different nearnesses.

Now we are led to the following example.
Example 4.1.7. Let $X$ be the compact $T_1$-space of Example 7.2 of [11]. We know that $X$ is not linkage compact. Let us define $\pi$ by

$$\pi = \{ (A, B) : A, B \subseteq X, \overline{A} \cap \overline{B} \neq \emptyset \}.$$ 

Now for all $x \in X$, $\mathcal{U}_x \subseteq \mathcal{F}_x$. Also since every ultraclosed filter in this space is of the form $\mathcal{U}_x$ for some $x \in X$ (Proposition 1.3.6), it follows that $\{ \mathcal{F}_x : x \in X \}$ is the set of all maximal Wallman $\pi$-clans.

Clearly $\pi$ is a LO-proximity compatible with the space $X$. Also if $A, B \subseteq X$, $(A, B) \in \pi$ then there exists an $x \in X$ such that $x \in \overline{A} \cap \overline{B}$ and hence $\{ A, B \} \subseteq \mathcal{F}_x$. Thus $\pi$ is a covered proximity on the space $X$. Since $X$ is not linkage compact there exists a linked grill $\mathcal{F}_o$ on $X$ such that $\mathcal{F}_o \notin \mathcal{F}_x$ for all $x \in X$ and $\mathcal{F}_o \notin \mathcal{U}_W(\pi)$. Since $\mathcal{F}_o$ is linked, $\mathcal{F}_o$ is a $\pi$-clan and hence $\mathcal{F}_o \in \mathcal{U}^*(\pi)$.

Thus $\mathcal{U}_W(\pi) \neq \mathcal{U}^*(\pi)$.

It should be noted that $\mathcal{U}_W(\pi)$, since $\pi$ is covered proximity, belongs to the proximity class of $\pi$ (see Theorem 2.2.10).

This example shows that there exists a covered proximity $\pi$ such that the Wallman $\pi$-clans and the $\pi$-clans generate different nearnesses both belonging to the proximity class of $\pi$.
Remark 4.1.8. Consider the Wallman nearness $\mathcal{V}_w(\pi)$ of the above example. It is obviously the elementary nearness on the space $X$. Since $X$ is $T_1$-compact, $\mathcal{V}_w(\pi)$ is contigual (Theorem 3.1.6). Since each two-element subfamily of $\mathcal{F}_o$ belongs to $\mathcal{V}_w(\pi)$ and $\mathcal{F}_o \not\subseteq \mathcal{V}_w(\pi)$ it follows from Theorem 2.3.8 (b) that $\mathcal{V}_w(\pi)$ is not proximal. Thus there exists a contigual Wallman nearness which fails to be proximal.

Examples of this chapter answer several questions those arose naturally out of our investigations in earlier chapters. It seems to us that most important are the following:

Given a covered proximity $\pi$ on a $T_1$-space, do the set of all $\pi$-clans and the set of all Wallman $\pi$-clans generate the same nearness?

Does there exist a noncompact Wallman-type extension of a $T_1$-space?

We have already answered the former in the negative (Example 4.1.7).

To conclude in what follows we answer the latter in the affirmative. In fact we give an example of a Wallman nearness which fails to be contigual and hence it will follow that noncompact Wallman-type extensions exist (Remark 2.3.6 and Corollary 2.1.24).
Example 4.1.9. Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{Z}$ be the set of all finite subsets of $\mathbb{N}$. Now we define for each $n \in \mathbb{N}$,

$$X_n = \{ (m, n) : m \in \mathbb{N} \} ,$$

$$Z_n = \{ I \in \mathbb{Z} : n \text{ is the first element of } I \} .$$

Note that $X_m \cap X_n = \emptyset$ and $Z_m \cap Z_n = \emptyset$ for all $m, n, m \neq n$.

Also $Z = \bigcup Z_n : n \in \mathbb{N}$.

Next define

$$X = Y \cup Z \cup \{ \infty \} ,$$

where $Y = \bigcup X_n : n \in \mathbb{N}$. It is understood that $Y \cap Z = \emptyset$ and $\infty \notin Y \cup Z$.

Let us define $c : \wp(X) \rightarrow \wp(X)$ by

$$c(B) = B \cup \{ I \in Z : |B \cap X_i| > \aleph_0 \text{ for some } i \in I \} .$$

$$\bigcup \{ \infty : \exists \text{ an infinite subset } M_1 \text{ of } \mathbb{N}$$

such that $B \cap X_i \neq \emptyset \forall i \in M_1$

or $\exists$ an infinite subset $M_2$ of $\mathbb{N}$

such that $|B \cap Z_i| > \aleph_0 \forall i \in M_2$.

Let us first verify the following results:

4.1.9.A. For each $A \subseteq Y$, $\infty \notin c(A)$ implies that $\infty \notin c(c(A))$. 
Proof: Since $A \subseteq Y$ and $\omega \notin c(A)$ it follows that $A$ intersects at most finitely many $X_i$. Let $n$ be the largest positive integer such that $A \cap X_n \neq \emptyset$. Thus if $I \in c(A)$ then there exists $s \in I$ such that $s \leq n$ and hence $I \notin Z_{n+k}$ for all $k = 1, 2, \ldots$, consequently $c(A) \cap Z_{n+k} = \emptyset$ for all $k = 1, 2, \ldots$. Also $A \cap X_i = \emptyset$ if and only if $c(A) \cap X_i = \emptyset$. In view of these facts we conclude that $\omega \notin c(c(A))$.

4.1.9.B. For each $A, B \subseteq X$, $c(\emptyset) = \emptyset$, $c(A) \supseteq A$ and $c(A) \supseteq c(B)$ whenever $A \supseteq B$. Hence $c(A \cup B) \supseteq c(A) \cup c(B)$ for all $A, B \subseteq X$. Also $c(A) = A$ for all finite $A \subseteq X$.

Proof: The results follow immediately from the definition of $c$.

4.1.9.C. For each $A, B \subseteq X$, $c(A \cup B) \subseteq c(A) \cup c(B)$.

Proof: Let $x \in X$ and $x \notin c(A) \cup c(B)$. To complete the proof we show that $x \notin c(A \cup B)$. Let us consider the following three cases.

Case 1. $x \in Y$. In this case we know that (from the definition of $c$) for every $D \subseteq X$, $x \in D$ if and only if $x \in c(D)$. Thus $x \notin A \cup B$ as $x \notin c(A) \cup c(B)$ and hence $x \notin c(A \cup B)$.
Case II. \( x \in Z \), i.e., \( x = I \) for some \( I \in Z \). Since \( I \notin c(A) \cup c(B) \) it follows that

\[
I \notin A \text{ and } |A \cap X_i| < \xi_1 \text{ for all } i \in I
\]

and

\[
I \notin B \text{ and } |B \cap X_i| < \xi_1 \text{ for all } i \in I.
\]

Thus \( I \notin A \cup B \) and \( |(A \cup B) \cap X_i| < \xi_1 \) for all \( i \in I \) and hence \( I \notin c(A \cup B) \).

Case III. \( x = \infty \). Since \( \infty \notin c(A) \cup c(B) \) it follows that

\[
\infty \notin A \text{ and } A \cap X_i \neq \emptyset \text{ for at most finitely many } X_i \text{'s and } |A \cap Z_i| > \xi_1 \text{ for at most finitely many } Z_i \text{'s.}
\]

and

\[
\infty \notin B \text{ and } B \cap X_i \neq \emptyset \text{ for at most finitely many } X_i \text{'s and } |B \cap Z_i| > \xi_1 \text{ for at most finitely many } Z_i \text{'s.}
\]

Thus \( \infty \notin A \cup B \) and \( (A \cup B) \cap X_i \neq \emptyset \) for at most finitely many \( X_i \)’s and \( |(A \cup B) \cap Z_i| > \xi_1 \) for at most finitely many \( Z_i \)’s and hence \( \infty \notin c(A \cup B) \).

This completes the proof.

In view of the above results we have
4.1.9.D. For all $A, B \subseteq X$, $c(A \cup B) = c(A) \cup c(B)$.

4.1.9.E. If $A \subseteq X$ then $c(c(A)) = c(A)$.

Proof: First observe that if $B \subseteq Z$ then $c(B) \subseteq B \cup \{\infty\}$ and hence either $c(B) = B$ or $c(B) = B \cup \{\infty\}$. Thus in both the cases $c(c(B)) = c(B)$. If $B$ is a subset of $Y$ then

$$c(B) = B \cup \{I \in Z : |B \cap X_i| > \xi_0 \text{ for some } i \in I\}$$

or

$$\cup \{\infty : \exists M \subseteq \mathbb{N}, |M| > \xi_0 \text{ such that } B \cap X_i \neq \emptyset \forall i \in M\}.$$

Two possibilities may occur. Namely $\infty \in c(B)$ or $\infty \notin c(B)$.

Case I. Let $\infty \notin c(B)$. Then $c(B) = B \cup D$ where $D$ is given by

$$D = \{I \in Z : |B \cap X_i| > \xi_0 \text{ for some } i \in I\}.$$

Since $B \subseteq Y$ and $\infty \notin c(B)$, from 4.1.9.A it follows that $\infty \notin c(c(B)) = c(B) \cup c(D)$ and hence in particular $\infty \notin c(D)$, consequently $c(D) = D$ for $D \subseteq Z$. Thus $c(c(B)) = c(B) \cup D = B \cup D \cup D = B \cup D = c(B)$.

Case II. $\infty \in c(B)$. Then $c(B) = B \cup E \cup \{\infty\}$ where

$$E = \{I \in Z : |B \cap X_i| > \xi_0 \text{ for some } i \in I\}.$$
Thus \( c(c(B)) = c(B) \cup c(E) \cup c(\{\infty\}) \)
\[ = c(B) \cup E \cup \{\infty\} \]
\[ = c(B), \text{ for } E \cup \{\infty\} \subset c(B). \]

So in this case also \( c(c(B)) = c(B) \).

Thus for any subset \( A \subset X \) we have
\[
A = (A \cap Y) \cup (A \cap Z) \cup (A \cap \{\infty\})
\]
and in view of the above results we have
\[
c(c(A)) = c(c(A \cap Y)) \cup c(c(A \cap Z)) \cup c(c(A \cap \{\infty\}))
\[ = c(A \cap Y) \cup c(A \cap Z) \cup c(A \cap \{\infty\}) \]
\[ = c(A). \]

From the above results we have

4.1.9.F. \( (X,c) \) is a \( T_1 \)-topological space.

4.1.9.G. Let \( A \subset Y \). Then \( A \) is closed in \( (X,c) \) if and only if \( A \) is finite.

Proof: Since \( (X,c) \) is a \( T_1 \)-space it follows that for every finite subset \( A \) of \( X \), \( c(A) = A \).

Conversely suppose that \( A \subset Y \) and \( c(A) = A \). Then in particular \( \infty \notin c(A) \) and hence there are finitely many \( i_1, i_2, \ldots, i_m \in \mathbb{N} \) such that \( A \cap X_i = \emptyset \) for all
\[
i \in \mathbb{N} - \{i_1, i_2, \ldots, i_m\}\text{. Again since } c(A) = A \text{ and } A \cap Z = \emptyset, I \notin c(A) \text{ for all } I \in Z. \text{ Thus } |A \cap X_n| < s_{i_0} \text{ for all } n \in \mathbb{N} \text{ and hence we have}
\]
\[
|A| = |(A \cap X_{i_1}) \cup (A \cap X_{i_2}) \cup \ldots \cup (A \cap X_{i_m})|
\]
\[
= |A \cap X_{i_1}| + |A \cap X_{i_2}| + \ldots + |A \cap X_{i_m}|
\]
\[
< s_{i_0}.
\]

This completes the proof.

4.1.9.H. \textbf{Let } A \subseteq Z. \textbf{ Then } A \textbf{ is closed if and only if}
\[
|A \cap Z_n| > s_{i_0} \text{ for at most finitely many } Z_n.
\]
\textbf{In particular for every } n \in \mathbb{N}, \textbf{ each subset of}
\[
Z_1 \cup Z_2 \cup \ldots \cup Z_n \textbf{ is closed.}
\]

\textbf{Proof:} It is an immediate consequence of the definition of \( c \).

4.1.9.I. \textbf{For every } n \in \mathbb{N}, \textbf{ } c(X_n) \cap Z_{n+k} = \emptyset \textbf{ for all } k = 1,2,3,\ldots.

\textbf{Proof:} Note that if } I \in c(X_n) \cap Z \text{ then } n \in I \text{ and hence}
\[
I \notin Z_{n+k} \text{ for all } k = 1,2,3,\ldots. \text{ Thus } c(X_n) \cap Z_{n+k} = \emptyset \text{ for all } k = 1,2,3,\ldots.
4.1.9.J. Let \( m \) be a positive integer. If \( I \in \mathbb{Z} \) such that \( m \) is the largest element of \( I \) then \( I \notin c(X_{m+k}) \) for all \( k = 1, 2, \ldots \).

**Proof**: This also follows from the fact that if \( J \in c(X_{m+k}) \cap \mathbb{Z} \) then \( m + k \in J \).

4.1.9.K. \((X, c)\) is not compact.

**Proof**: Clearly \( \cap c(X_1) \cap c(X_2) \cap \ldots \) and also if \( I \in \mathbb{Z} \) and \( m \) is the largest element of \( I \) then by the above result \( I \notin c(X_{m+k}) \) for all \( k = 1, 2, 3, \ldots \) and hence \( I \notin c(X_1) \cap c(X_2) \cap \ldots \) for all \( I \in \mathbb{Z} \). Thus \( c(X_1) \cap c(X_2) \cap \ldots = \emptyset \). But if \( i_1, i_2, \ldots, i_n \in \mathbb{N} \) then by the definition of \( c \) it follows that

\[
\bigcap_{i_1, i_2, \ldots, i_n} c(X_{i_1}) \cap c(X_{i_2}) \cap \ldots \cap c(X_{i_n}).
\]

Thus \( \bigcap_{n \in \mathbb{N}} \{ n \} \) is a family of closed sets with the F.I.P. which has empty intersection. Thus \((X, c)\) is not compact.

4.1.9.L. There exist non-principal ultraclosed filters on \((X, c)\) and each non-principal ultraclosed filter necessarily contains \( Z \cup \{ \infty \} \) and does not contain \( Y \).
Proof: Since \((X,c)\) is a \(T_1\)-noncompact space, there exist non-principal ultraclosed filters on \((X,c)\) (see Corollary 1.3.5).

Let \(\mathcal{U}\) be such a non-principal ultraclosed filter on \((X,c)\). Since \(Y \cup (Z \cup \{\infty\}) = X\) and \(Z \cup \{\infty\}\) is closed it follows that \(Y\) is open. Since \(\mathcal{U}\) is ultraclosed filter, either \(Y \in \mathcal{U}\) or \(Z \cup \{\infty\} \in \mathcal{U}\) (Theorem 1.3.2(iv)). We claim that \(Y \notin \mathcal{U}\). If not, then there is a closed set \(A \in \mathcal{U}\) such that \(A \subseteq Y\), consequently in view of 4.1.9.G, \(A\) is finite - a contradiction for \(\mathcal{U}\) is non-principal. Hence \(Z \cup \{\infty\} \in \mathcal{U}\).

4.1.9.M. If \(\mathcal{U}\) is a n.p. ultraclosed filter and \(k \in \mathbb{N}\) such that \(Z_n \notin \mathcal{U}\) for all \(n = k, k+1, \ldots\) then there exists a closed set \(B \in \mathcal{U}\) such that \(B \subseteq Z\) and \(B\) intersects each of \(Z_k, Z_{k+1}, Z_{k+2}, \ldots\) in at most finitely many points.

Proof: Since \(\mathcal{U} \neq \mathcal{U}_\infty\), there exists a closed set \(B_1 \in \mathcal{U}\) such that \(B_1 \subseteq Z\). Clearly \(B_1\) intersects at most finitely many \(Z_i\) each at infinitely many points. Let \(m\) be the largest natural number such that \(|B_1 \cap Z_m| > k_0\). Since each \(Z_i\) is closed and \(Z_i \notin \mathcal{U}\) for all \(i > k\) it follows that \(\bigcup \{Z_i : k < i \leq m\} \notin \mathcal{U}\).
Note that

\[ B_1 = B \cup \left( B_1 \cap \bigcup \{ Z_i^k : k \leq i \leq m \} \right) \]

where \[ B = B_1 - \bigcup \{ Z_i^k : k \leq i \leq m \} \].

Since both of \( B \) and \( B_1 \cap \left( \bigcap \{ Z_i^k : k \leq i \leq m \} \right) \) are closed (see 4.1.9.H) and \( B_1 \cap \left( \bigcap \{ Z_i^k : k \leq i \leq m \} \right) \notin \mathcal{U}, \ B \in \mathcal{U}. \)

Obviously \( B \subset Z \) and \( |B \cap Z_n| < \aleph_0 \) for all \( n = k, k+1, k+2, \ldots \) for \( |B_1 \cap Z_{m+j}| < \aleph_0 \) for all \( j = 1, 2, \ldots \) and \( B \cap Z_n = \emptyset \) for all \( n = k, k+1, k+2, \ldots, m \).

4.1.9.N. For each \( n \in \mathbb{N} \),

(I). there are infinitely many n.p. ultrafilters on \( X \) containing \( X_n \)

and

(II). there exists n.p. ultraclosed filter on \( (X, c) \) containing \( Z_n \).

Proof: (I) follows from Proposition 1.2.9, for \( X_n \) is an infinite subset of \( X \).

Note that \( Z_n \) is an infinite closed discrete subspace of \( (X, c) \) and hence \( Z_n \) is not compact, consequently in view of Proposition 1.3.8, it follows that there exists a n.p. ultraclosed filter on \( (X, c) \) containing \( Z_n \). Thus (II) holds.
Henceforth, \( U_1^n, U_2^n \) will denote two distinct n.p. ultrafilters on \( X \) containing \( X_n \) for all \( n \in \mathbb{N} \) and \( U_n \) will denote a n.p. ultraclosed filter on \((X, c)\) containing \( Z_n \). We shall simply write the space \( X \) instead of writing the space \((X, c)\) and \( \bar{A} \) will be used to denote \( c(A) \) for each \( A \subset X \).

**Main construction:** At first let us define

\[
\begin{align*}
\Phi_n^* &= \bigcup \{ U_2^k : k \in \mathbb{N} - \{n-1\} \} \text{ for all } n = 2, 3, \ldots \\
\Phi_n &= b(U_n) \cup U_1^{n-1} \cup \Phi_n^* \text{ for all } n = 2, 3, \ldots.
\end{align*}
\]

Set

\[
\gamma = \{ \Phi_x : x \in X \} \cup \{ \Phi_n : n = 2, 3, \ldots \} \cup \{ b(U) : U \text{ n.p. ultraclosed filter on } (X, c), U \neq U_n \text{ for all } n = 2, 3, \ldots \}.
\]

Since \( \Phi_x \) is a c-grill for each \( x \in X \) and \( b(U) \) is a c-grill for each ultraclosed filter \( U \) (see Theorem 3.1.1(iii)), in order to conclude that each element of \( \gamma \) is a c-grill it is sufficient to check that \( U_i^k \) is a c-grill for all \( i = 1, 2 \) and \( k = 1, 2, 3, \ldots \).

Note that for each \( A \subset X \),

\[
\bar{A} = (\bar{A} \cap \gamma) \cup (\bar{A} \cap (Z \cup \{ \omega \}))
\]
Thus if $\bar{A} \in \mathcal{U}_i^k$ then either $\bar{A} \cap Y \in \mathcal{U}_i^k$ or
$\bar{A} \cap (Z \cup \{\infty^2\}) \in \mathcal{U}_i^k$. Since $X_k \in \mathcal{U}_i^k$ and

$x_k \cap (Z \cup \{\infty^2\}) = \emptyset$, $Z \cup \{\infty^2\} \notin \mathcal{U}_i^k$ and hence

$\bar{A} \cap Y \in \mathcal{U}_i^k$. It is immediate from the definition of

c that $\bar{A} \cap Y = A \cap Y$. Thus we conclude that for each

$A \subseteq X$, $\bar{A} \in \mathcal{U}_i^k$ implies that $A \cap Y \in \mathcal{U}_i^k$ and hence

$A \in \mathcal{U}_i^k$. Hence each $\mathcal{U}_i^k$ is a c-grill on $(X, c)$.

Now we claim that no two elements of $\mathcal{Y}$ are comparable

with respect to the set inclusion.

Let $m, n \in \mathbb{N}$ such that $m \neq n$.

Note that $Z_m$ is a closed set in $X$ such that

$Z_m \in b(\mathcal{U}_m) \subseteq \mathcal{F}_m$ but $Z_m \notin \mathcal{F}_n$ for $Z_m \notin b(\mathcal{U}_n)$ and

$Z_m \notin \mathcal{U}_1^{n-1} \cup \{ \mathcal{U}_2^k : k \in \mathbb{N} - \{n-1\} \}$. Thus $\mathcal{F}_m \neq \mathcal{F}_n$.

We know that $b(\mathcal{U}) \notin b(\mathcal{V})$ whenever $\mathcal{U}, \mathcal{V}$ are two
distinct ultraclosed filters (see Theorem 3.1.1(vi)).

In view of these facts, to show that no two elements

of $\mathcal{Y}$ are comparable it is sufficient to check that for

all ultraclosed filters $\mathcal{U} \neq \mathcal{U}_2, \mathcal{U}_3, \ldots$, $b(\mathcal{U}) \neq \mathcal{F}_n$

for all $n = 2, 3, 4, \ldots$. Choose a closed set $A \in \mathcal{U} - \mathcal{U}_n$. 

Set \( A \cap (Z \cup \{\infty\}) = B \). Since \( Z \cup \{\infty\} \in \mathcal{U} \) (see Result 4.1.9.L), \( B \in \mathcal{U} - \mathcal{U}_{\infty} \). Note that \( B \in \mathcal{U}_{k} \) for all \( k = 1, 2, \ldots \) and \( i = 1, 2 \). Thus \( B \in \mathcal{U}_{1}^{n-1} \cup \mathcal{U}_{n}^{*} \). Also \( B \in b(\mathcal{U}) - b(\mathcal{U}_{n}) \) for \( B \) is a closed set in \( X \). Thus \( B \in b(\mathcal{U}) - \mathcal{U}_{n} \).

Finally note that \( \{x\} \in \mathcal{G}_{x} \) for all \( x \in X \). Thus in view of the above facts and Proposition 1.4.5, Theorem 1.4.8(iv) it follows that

\[
\mathcal{U} = \{ A \subseteq \mathcal{P}(X) : A \subseteq \mathcal{G} \text{ for some } \mathcal{G} \in \mathcal{U} \}
\]

is a cluster generated LO-nearness compatible with the space \( X \) such that \( X^{\mathcal{U}} = \mathcal{U} \).

Now we prove that \( \mathcal{U} \) is a Wallman nearness.

Note that \( \mathcal{G}_{x} = b(\mathcal{U}_{x}) \) for all \( x \in X \) and \( \mathcal{U} \subseteq b(\mathcal{U}) \) for all \( \mathcal{U} \in \Psi(X) \) (see Theorem 3.1.1(i) & (iv)). Thus each member of \( \mathcal{U} \) (\( = X^{\mathcal{U}} \)) contains an ultraclosed filter. Thus in view of Corollary 2.3.5, to show that \( \mathcal{U} \) is a Wallman nearness it suffices to check that \( \mathcal{U} \) is a binary collection with respect to \( \Gamma^{W}(X) \).

Let \( \mathcal{G}_{0} \in \Gamma^{W}(X) \) such that each two-element subfamily of \( \mathcal{G}_{0} \) is contained in an element of \( \mathcal{U} \). Since \( \mathcal{G}_{0} \in \Gamma^{W}(X) \), there exists \( \mathcal{U}_{0} \in \Psi(X) \) such that \( \mathcal{U}_{0} \subseteq \mathcal{G}_{0} \).
Now let us consider the following possibilities:

**Case 1.** $\mathcal{U}_0 = \mathcal{U}_x$ for some $x \in X$.

Then $\{x\} \in \mathcal{G}_0$. Since each two-element subfamily of $\mathcal{G}_0$ is contained in an element of $\mathcal{U}$, $\{A, \{x\}\} \in \mathcal{U}$ for all $A \in \mathcal{G}_0$ and hence $x \in \mathcal{U}(A) = c(A) = \overline{A}$ for all $A \in \mathcal{G}_0$. Thus $\mathcal{G}_0 \subseteq \mathcal{U}_x$.

**Case 2.** $\mathcal{U}_0 = \mathcal{U}_n$ for some $n = 2, 3, 4, \ldots$.

In this case we claim that $\mathcal{G}_0 \subseteq \mathcal{U}_n$. If not, there exists $A \in \mathcal{G}_0 - \mathcal{U}_n$. Clearly $A \notin b(\mathcal{U}_n)$ and hence there exists a closed set $F \in \mathcal{U}_n$ such that $\overline{A} \cap F = \emptyset$ (Theorem 1.3.2(ii)). Note that $\{A, Z \cap F\} \subseteq \mathcal{G}_0$.

Since $Z \notin b(\mathcal{U}_m)$, $m \neq n$, $\overline{A} \cap F = \emptyset$ and $A \notin \mathcal{G}_n$ it follows that $\{A, Z \cap F\}$ is not contained in any member of $\mathcal{U}$, even though it is a two-element subfamily of $\mathcal{G}_0$ - a contradiction to the fact that each two-element subfamily of $\mathcal{G}_0$ is contained in some element of $\mathcal{U}$.

**Case 3.** $\mathcal{U}_0$ is a n.p. ultraclosed filter on $X$ such that $\mathcal{U}_0 \neq \mathcal{U}_n$ for all $n = 2, 3, 4, \ldots$ and there exists an $m = 1, 2, 3, \ldots$ such that $Z_0 \in \mathcal{U}_0$.

In this case we claim $\mathcal{G}_0 \subseteq b(\mathcal{U}_0)$. If not, then there exists $A \in \mathcal{G}_0 - b(\mathcal{U}_0)$. We can choose a closed set $F_1 \in \mathcal{U}_0$ such that $\overline{A} \cap F_1 = \emptyset$. 
If $m = 1$, i.e., when $z_1 \in \mathcal{U}_0$ then $\{ A, Z_1 \cap F_1 \} \not\subseteq \mathcal{F}_0$. Since $Z_1 \cap F_1 \notin \mathcal{F}_n$ for all $n > 2$ and $\overline{A} \cap F_1 = \emptyset$, it follows that the two-element subfamily $\{ A, Z_1 \cap F_1 \}$ of $\mathcal{F}_0$ is not contained in any element of $\mathcal{Y}$ — a contradiction.

If $m > 2$ then, since $\mathcal{U}_0 \neq \mathcal{U}_m$, we can choose $F_m \in \mathcal{U}_0 - \mathcal{U}_m$. Note that $Z_m \cap F_1 \cap F_m \in \mathcal{U}_0$ but $Z_m \cap F_1 \cap F_m \notin b(\mathcal{U}_n)$ for all $n = 2, 3, 4, \ldots$ and $\overline{A} \cap F_1 = \emptyset$ and hence it follows that $\{ A, Z_n \cap F_1 \cap F_m \}$, even though a two-element subfamily of $\mathcal{F}_0$, is not contained in any element of $\mathcal{Y}$ — a contradiction.

Case 4. $\mathcal{U}_0$ is n.p. ultraclosed filter on $X$ such that $Z_n \notin \mathcal{U}_0$ for all $n = 1, 2, 3, \ldots$.

In this case also we claim $\mathcal{F}_0 \subseteq b(\mathcal{U}_0)$. If not, then there exists $A \in \mathcal{F}_0 - b(\mathcal{U}_0)$ and consequently one can find a closed set $F_1$ in $\mathcal{U}_0$ such that $\overline{A} \cap F_1 = \emptyset$.

Since $Z_n \notin \mathcal{U}_0$ for all $n \geq 1$, in view of Result 4.1.9.M, there is a closed set $F_2 \subseteq Z$ such that $F_2 \in \mathcal{U}_0$ and $|F_2 \cap Z_n| < \mathcal{F}_0$ for all $n \geq 1$. Since $F_1 \cap F_2 \subseteq Z$ and $|F_1 \cap F_2 \cap Z_n| < \mathcal{F}_n$ for all $n \geq 2$, $\{ A, F_1 \cap F_2 \} \not\subseteq \mathcal{F}_n$ for all $n \geq 2$ and since $\overline{A} \cap F_1 \cap F_2 = \emptyset$,

$\{ A, F_1 \cap F_2 \} \not\subseteq b(\mathcal{U})$ for all ultraclosed filters $\mathcal{U}$ on $X$. 
Hence \( \mathfrak{A} \), \( F_1 \cap F_2 \neq \emptyset \), even though being a two-element subfamily of \( \mathfrak{F}_0 \), is not contained in any element of \( \mathfrak{Y} \) - a contradiction.

This completes the proof of the fact that \( \mathfrak{Y} \) is a Wallman nearness.

Finally we prove that \( \mathfrak{Y} \) is not contigual.

Define

\[
\mathfrak{F}^* = \mathcal{U}_2^1 \cup \mathcal{U}_2^2 \cup \mathcal{U}_2^3 \cup \ldots
\]

Let \( \{ A_1, A_2, \ldots, A_k \} \) be a finite subfamily of \( \mathfrak{F}^* \).

Then for each \( r = 1, 2, 3, \ldots, k \) there is \( i_r \) such that

\[ A_r \in \mathcal{U}_2^{i_r} \]

which implies that \( |A_r \cap X_1| > \mathfrak{F}_0 \) for all \( r = 1, 2, \ldots, k \) and hence \( i_1, i_2, \ldots, i_k \in c(A_1) \cap c(A_2) \cap \ldots \cap c(A_k) \). Thus \( \mathfrak{F}^* = \{ c(A) : A \in \mathfrak{F}^* \} \) has the F.I.P.

Then by Proposition 3.1.3, there are ultraclosed filters \( \mathcal{U} \) on \( X \) such that \( \mathfrak{F}^* \subseteq b(\mathcal{U}) \).

Since \( \bigcap \{ c(X_n) : n \in \mathbb{N} \} = \emptyset \) (see the proof of 4.1.9.K) and \( c(X_1) \cap Z_n = \emptyset \) for all \( n = 2, 3, 4, \ldots \) (see 4.1.9.I), it follows that if \( \mathcal{U} \) is an ultraclosed filter such that \( \mathfrak{F}^* \subseteq b(\mathcal{U}) \) then \( \mathcal{U} \) must be nonprincipal and \( Z_n \notin \mathcal{U} \) for all \( n = 2, 3, 4, \ldots \). For any such \( \mathcal{U} \) one can choose (by 4.1.9.M) a closed set \( F \subset Z \) such that \( F \in \mathcal{U} \) and \( |F \cap Z_n| < \mathfrak{F}_0 \) for all \( n > 2 \). Obviously \( F \notin \mathcal{U}_n \) for
all \( n \geq 2 \). For each such \( \mathcal{U} \) we define \( \mathcal{F}_n = (\mathbb{Z} \cup \{\infty\}) - \mathcal{F} \). Clearly \( \mathcal{F}_n \in \mathcal{U}_n \) for all \( n = 2, 3, 4, \ldots \).

Now define

\[
\mathcal{A}^* = \{ \mathcal{F}_n : \mathcal{F}^* \subseteq b(\mathcal{U}), \mathcal{U} \in \psi(x) \}.
\]

Next consider the family \( \mathcal{F}^* \cup \mathcal{A}^* \). First we shall show that \( \mathcal{F}^* \nsubseteq \mathcal{F}_n \) for all \( n = 2, 3, 4, \ldots \).

Choose \( A \in \mathcal{U}_{n-1}^n - \mathcal{U}_{1}^{n-1} \). Then \( A \cap X_{n-1} \in \mathcal{U}_2^{n-1} \) and \( c(A \cap X_{n-1}) \cap Z_n = \emptyset \) for all \( n \geq 2 \) and \( A \cap X_{n-1} \notin \mathcal{U}_2^k \) for all \( k \neq n-1 \). Thus we have found \( B \in \mathcal{U}_2^{n-1} \) such that \( B \notin b(\mathcal{U}_{n-1}) \cup \mathcal{U}_1^{n-1} \cup \mathcal{F}_n \) for all \( n = 2, 3, 4, \ldots \) and clearly \( B \in \mathcal{F}^* \).

From the definition of \( \mathcal{A}^* \) it follows that \( \mathcal{A}^* \subseteq b(\mathcal{U}) \) for all ultraclosed filters \( \mathcal{U} \) such that \( \mathcal{F}^* \subseteq b(\mathcal{U}) \).

Thus \( \mathcal{F}^* \cup \mathcal{A}^* \) is not contained in any element of \( \mathcal{V} \).

To complete the proof of the fact that \( \mathcal{V} \) is not contiguity one needs to check only that each finite subfamily of \( \mathcal{F}^* \cup \mathcal{A}^* \) is contained in an element of \( \mathcal{V} \).

Let \( \mathcal{A} \) be a finite subfamily of \( \mathcal{F}^* \cup \mathcal{A}^* \). Then \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \) where \( \mathcal{A}_1 \subseteq \mathcal{F}^* \) and \( \mathcal{A}_2 \subseteq \mathcal{A}^* \).
Let $A_1 = \{ A_1, A_2, \ldots, A_k \}$. Then there are $i_1, i_2, \ldots, i_k$ such that $A_s \in \mathcal{U}_{ij}$ for all $s = 1, 2, \ldots, k$. Choose a positive integer $m$ such that $m-1 > \max \frac{i}{2} i_1, i_2, \ldots, i_k$. Then $A_1 \subset C^*_m$. Since $A_2 \subset b(\mathcal{U}_n)$ for all $n = 2, 3, 4, \ldots$ it follows that $A \subset C^*_m$.

Thus we have proved that $\omega$ is a Wallman nearness which fails to be contigual.