CHAPTER – 1

INTRODUCTION AND PRELIMINARIES
1.1. GENERAL INTRODUCTION

Semiring theory stands with a foot in each of two mathematical domains. On one hand, semirings are abstract mathematical structures and their study is part of abstract algebra - arising ab initio from the work of Dedekind, Macaulay, krull, and others on the theory of ideals of a commutative ring and then through the more general work of Vandiver - and the tools used to study them are primarily the tools of abstract algebra. On the other, the modern interest in semirings arises primarily from fields of applied mathematics such as optimization theory, the theory of discrete-event dynamical systems, automata theory, and formal language theory, as well as from the allied areas of theoretical computer science and theoretical physics, and the questions being asked are, for the most part, motivated by applications.

The concept of semiring was first introduced by Vandiver in 1934. However the developments of the theory in semirings and ordered semirings have been taking place since 1950.

The theory of rings and the theory of semigroups have considerable impact on the developments of the theory of semirings and ordered semirings. The works of eminent people like S. Bourne [4, 5],
P.J. Allen [1, 2], K. Iseki [17, 18], M.P. Grillet [10], H.E. Stone[40], H.J. Weinert [42] and Sidney S. Mitchell and Kyungpook Porntip Sinutoke [23] are to be mentioned in the theory of semirings using ring theory techniques. P.H. Karvellas [20], M. Satyanarayana [29], H.J. Weinert [42], Andre Batbedat [3], John Zeleznikow [43], J. Hanumanthachari and K. Venuraju [12] have contributed to theory of semirings and ordered semirings using semigroup techniques.

During the last three decades, there is considerable impact of semigroup theory and semiring theory on the development of ordered semirings both in theory and applications, which are akin to ordered rings and ordered semirings. In this direction the works of H. J. Weinert [42], M. Satyanarayana [27, 29, 30, 31, 32, 33], J. Hanumanthachari, K. Venuraju and H.J. Weinert [13], M. Satyanarayana, J. Hanumanthachari and D. Umamaheswara Reddy [34, 35], K.P. Shum and C.S. Hoo [38], K.P. Shum and C.Y. Hung [39], Kehayopulu [21], G. Therrin, H.J. Shyr, Gerand Lallment and U. Zimmermann are to be worth mentioning.

The study of rings, which are special semirings shows that the multiplicative structures are quite independent though their additive
structures are abelian groups. However in semirings it is possible to derive the additive structures from their special multiplicative structures and vice versa. The developments of semirings and ordered semirings in this direction require semigroup techniques. It is well known that if the multiplicative structure of an ordered semiring is a rectangular band then its additive structure is a band. P.H. Karvellas [20] has proved that if the multiplicative structure of a semiring is a regular semigroup and if the additive structure is an inverse semigroup, then the additive structure is commutative.

Using the techniques of (ordered) semigroups, M. Satyanarayana [33] examines in his paper whether the multiplicative structure of semirings determines the order structure as well as the additive structure of the semirings. In other paper, M.Satyanarayana [32] deals with the problem determine when the multiplicative semigroup is o-Archimedean if its additive semigroup is o-Archimedean and conversely in totally ordered semirings. The aim of the author in this thesis is to contribute some new results in the theory of “Semirings and Ordered Semirings”.
1.2. A BRIEF SURVEY OF THE RESULTS ON SEMIRINGS AND ORDERED SEMIRINGS

Homomorphism theorems for semirings have been discussed by S.Bourne [5] and P.J.Allen [1]. M.P.Grillet [9] has considered subdivision rings of semirings and in [10] he has studied the structure of semirings in which additive semigroup is completely simple. The theory of ideals and quasi-ideals for semirings has been studied by K.Iseki [17, 18]. Radicals and semiradicals in semirings have been considered by S.Bourne [4], K.Iseki and Y.Miyanaga [19] and Bourne and Zassen-haus [6]. Henriksen [15] has generalized Jacobson’s theorem where as Allen [2] has extended the Hilbertbasis theorem to semirings. H.J.Weinert [42] has shown that each semiring is isomorphic to a subsemiring of certain semi-near-ring of partial transformations.

M.Satyanarayana [30] has studied some multiplicative conditions under which the additive structure of a semiring is a band (or) commutative, which again prompted K.Venuraju and J. Hanumanthachari [12] to study the question when the additive structure of semirings; in the case of semiring with identity element is band (or) commutative (or) a semilattice. Sidney S. Mitchell and
Kyungpook Porntip Sinutoke [23] studied the structure of positive rational domains and semifields.

In the recent papers on ordered semirings, the works of M. Satyanarayana [31], J. Hanumanthachari, K. Venuraju and H.J. Weinert [13], M. Satyanarayana, J. Hanumanthachari and D. Umamaheswareddy [34, 35] are to be mentioned. M. Satyanarayana [31] has studied how far the properties of multiplicative structure are reflected in the additive structure and vice-versa. J. Hanumanthachari, K. Venuraju and H.J. Weinert [13] have studied weak partially ordered semirings (w.p.o.s.r) in which additive structures are idempotents. M. Satyanarayana, J. Hanumanthachari and D. Umamaheswara Reddy [34] have studied additive structure of totally ordered semirings \((S, +, \cdot)\) in which the multiplicative structures \((S, \cdot)\) are \(o\)-Archimedean and p.t.o. They proved that \((S, +)\) is a band in which the addition is one of the four types namely commutative minimum addition, commutative maximum addition, left zero addition and the right zero addition or \((S, +)\) is \(o\)-Archimedean and positively ordered in the strict sense in the case \((S, \cdot)\) is non-cancellative, \((S, +)\) is a nilsemigroup. They have also studied in [35], the multiplicative structure of totally ordered
semirings \((S, +, \cdot)\) in which \((S, +)\) is o-Archimedean and r.n.t.o. 
In [14], D.Uمامaheswara Reddy and J.Hanumanthachari have studied the properties of maximal and minimal elements in the additive and multiplicative structures of certain classes of totally ordered \((t.o.)\) semirings. Also the inter-relation between cancellation laws and zero-divisors in a totally ordered semiring \((t.o.s.r)\) is discussed. The main interest in all these properties is to make the additive structure either into non-negatively ordered (or) into non-positively ordered. The influence of the maximal and minimal elements and the concept of zero-divisor on the additive structure of certain classes of t.o. semirings are also discussed in this paper. The results obtained by the author on semirings and ordered semirings are presented in section 1.3.

For future study, the author is interested to study the applications of semirings and ordered semirings to computer science.

**Notations:**

\(E[+]\) is the set of all additive idempotents.

\(E[\cdot]\) is the set of all multiplicative idempotents.
1.3 SUMMARY OF THE RESULTS OBTAINED BY THE AUTHOR:

Chapter one deals with a general introduction and brief survey on the developments of semirings and ordered semirings which also includes the major results of the author which are going to be proved in the subsequent chapters.

In chapter two, we study the zeroid in a semiring. We also study the properties of zeroid semirings and ordered zeroid semirings. The main results of chapter 2 are:

1.3.1: Let \((S, +, \cdot)\) be a zero-square semiring with additive identity 0. If \((S, +)\) is a zeroid, then \(S^2 = \{0\}\).

1.3.2: Let \((S, +, \cdot)\) be a semiring. If \(x \in Z\), where \(Z\) is the zeroid of semiring, then every power of \(x\) is a zeroid.

1.3.3: Let \((S, +, \cdot)\) be a semiring and \(Z\) be a zeroid, then \(Z\) is a multiplicative ideal.
1.3.4: Let \((S, +, \cdot)\) be a semiring and \((S, +)\) be commutative. Then \((\mathbb{Z}, +)\) is a subsemigroup of \((S, +)\).

1.3.5: Let \((S, +, \cdot)\) be a semiring with IMP in which \((S, +)\) is a zeroid. If \((S, +)\) is cancellative, then \((S, \cdot)\) is a band.

Chapter three deals with \(C\)–semirings and ordered \(C\)–Semirings. It is proved that if \((S, +, \cdot)\) is a \(C\)–semiring and \((S, +)\) is left cancellative, then \((S, \cdot)\) is a band. We also discuss the structure of positive rational domains and the structure of ordered positive rational domains. The principal results of this chapter are:

1.3.6: Let \((S, +, \cdot)\) be a \(C\)–semiring and \((S, +)\) be left cancellative, then \((S, \cdot)\) is a band.

1.3.7: Let \((S, +, \cdot)\) be a \(C\)–semiring. If \((S, \cdot)\) is regular, then \((S, \cdot)\) is a band.
1.3.8: Let \((S, +, \cdot)\) be a \(C\)–semiring. Then \(S\) contains two elements \(a\) and \(b\) such that \(ab = a + b + ab\) if and only if \(ab = a + b = a = b\).

1.3.9: Let \((S, +, \cdot)\) be a t.o. \(C\)–semiring in which \((S, +)\) is p.t.o.(n.t.o.), then for every \(x, y\) in \(S\), \(x + y = x\) or \(y\).

1.3.10: If \(|S| > 1\) in a totally ordered PRD in which \((S, +)\) is cancellative semigroup, then one of the following is true.

(i) \((S, +)\) is positively ordered in strict sense

(ii) \((S, +)\) is negatively ordered in strict sense

1.3.11: Let \((S, +, \cdot)\) be a t.o. PRD and \(x \not\in x + S\) and \(x \not\in S + x\) for every \(x \in S\). Then \((S, +)\) is positively ordered in the strict sense or negatively ordered in the strict sense.

In chapter four, we discuss properties of semirings and ordered semirings with IMP and regular semirings. The main results of this chapter are:
1.3.12: Let \((S, +, \circ)\) be a semiring with IMP. If \((S, \circ)\) is regular and also contain the multiplicative identity, then \((S, +)\) is periodic.

1.3.13: Let \((S, +, \circ)\) be a t.o.s.r. If an element \(a\) in \(S\) satisfies \(a^2 = na\), where \(n\) is an even positive integer and \(a^2 = a + a + a^2\), then \(a^2\) is an additive idempotent.

1.3.14: Let \((S, +, \circ)\) be a semiring. If \(S\) satisfies \(ab = a + b + ab\) and \((S, +)\) is left cancellative, then \((S, +)\) is regular.

1.3.15: Let \((S, +, \circ)\) be semiring in which \((S, \circ)\) is regular and p.t.o. Then the following are true.

(i) \(a \geq x\) for some \(x \in S\) which arises in the regularity of \(a\).

(ii) \(a = a^3\) if \((S, \circ)\) is an inverse semigroup.

1.3.16: Let \((S, +, \circ)\) be a t.o.s.r, \((S, \circ)\) be p.t.o. and regular, then the following are true.

(i) \((S, \circ)\) is a band.

(ii) \((S, +)\) is non - negatively ordered if \((S, \circ)\) is commutative.
In Chapter five, some special classes of semirings and ordered semirings are studied.

1.3.17: Let \((S, +, \cdot)\) be a semiring in which \((S, +)\) is cancellative and \((S, \cdot)\) is a band, then \((S, +)\) is commutative.

1.3.18: Let \((S, +, \cdot)\) be a semiring and \(E[+] = \emptyset\). If for every \(a, b \in S\), \(a < b\) implies \(b = a + c\) for some \(c \in S\) and if \((S, +)\) is left cancellative, then \((S, \cdot)\) is cancellative.

1.3.19: Let \((S, +, \cdot)\) be a semiring with multiplicative identity which is an additive idempotent, then \((S, +)\) is a band.

1.3.20: Every Boolean semiring \(S\) in which \((S, +)\) is cancellative has the following properties.

(i) \(S = \{a, 2a\} \cup \{b, 2b\} \cup \ldots \text{ for all } a, b \ldots \in S\).

(ii) \(a = a + ab + ba\) and \(b = b + ba + ab\).
1.3.21: Let $(S, +, \cdot)$ be a semiring and satisfy $ab = a + b + ab$, for all $a, b$ in $S$ and $(S, +)$ be right cancellative. If $(S, \cdot)$ is commutative, then $(S, +)$ is commutative.

1.3.22: Let $(S, +, \cdot)$ be a t.o.s.r. If $(S, \cdot)$ is p.t.o., and $S$ contains multiplicative identity $1$, then $(S, +)$ is non-negatively ordered.

1.3.23: Let $(S, +, \cdot)$ be a t.o.s.r. and $E[+] = \emptyset$. Then $(S, +)$ is either strictly non-negatively ordered or strictly non-positively ordered.

1.3.24: Let $(S, +, \cdot)$ be a t.o.s.r. such that for every $a \in S$, $a^2 \geq na$, for some positive integer $n$. If $(S, +)$ is a band, then $(S, \cdot)$ is p.t.o., if one of the following conditions is satisfied.

(i) $(S, \cdot)$ is O-Archimedean

(ii) $(S, \cdot)$ is left cancellative without idempotents

(iii) $(S, \cdot)$ is cancellative

1.3.25: Let $(S, +, \cdot, \leq)$ be a t.o.s.r. in which $(S, +)$ is o-Archimedean. Then $(S, +)$ is either non-negatively ordered or non-positively ordered.
1.3.26: Let \((S, +, \cdot)\) be a t.o. zero sum semiring. If \((S, +)\) is non-negatively (non-positively) ordered, then zero is the maximum (minimum) element.

Some of the results of chapter 4 and chapter 5 are published in “Southeast Asian Bulletin of Mathematics”, Vol.35 (2011), pp.149-156.