2.1 INTRODUCTION

The study of dispersion of a solute in flowing fluids has several applications in Industries, Chemical engineering, Biomedical engineering, environmental sciences, physiological fluid dynamics and various other branches of science. The study of dispersion facilitates to understand the transport of nutrients in blood and various artificial devices (Middleman, 1972; Lightfoot, 1974; Cooney, 1976; Jayaraman et al. 1981). The indicator dilution technique is often employed in physiological system to measure the cardiac output. In this method a quantity of dye is introduced into the blood stream to measure its concentration at some downstream point along the blood. Several intravenous medicines have therapeutic value at low concentration, but at high concentration they are toxic. Hence, the dispersion theory is of great value to know the rate of dispersion of drugs.

The concept of longitudinal dispersion was introduced by Taylor (1953). His work was based on the experimental observations of Griffiths (1911) conducted to measure the viscosity of water at very low speeds of flow. Taylor studied the dispersion of a bolus of a passive tracer in a circular pipe and showed that if a solute is introduced into a fluid (solvent) flowing in the pipe, the lateral molecular diffusion and variation of velocity over the cross section would spread the solute diffusively with effective molecular diffusivity $D_{\text{eff}}$ given by $D_{\text{eff}} = \frac{a^2 w_m^2}{48 D_m}$, where $D_m$ is the molecular diffusivity,
$w_m$ is the average axial velocity and ‘a’ is the radius of the pipe. Aris (1956) showed that the effective molecular diffusivity as $D_{\text{eff}} = D_m + \frac{a^2 w_m^2}{48 D_m}$ by including the effect of axial molecular diffusion. The experimental study of Bailey and Gogarty (1962) on the dispersion of a solute in a fluid flow revealed that the dispersion coefficients obtained from the results of experiments conducted at various flow times with a fixed velocity increased with time. They also obtained a numerical solution valid for all times.

Evans and Kenney (1965) obtained the dispersion coefficient of a tracer gas injected into a flowing stream of a second gas, under the condition that the tracer gas could also be exchanged by diffusion with a stagnant gaseous zone held in a porous solid. Ananthakrishnan et al. (1965) obtained a numerical solution for the complete convective diffusion equation considering both radial and molecular diffusion and hence the solution was valid for a large range of non-dimensional time ($t = \frac{D_m t}{a^2}$) varying from 0.1 to 30 and Peclet number ($Pe = \frac{a w_m}{D_m}$) in the range from 1 to 23,000. The Taylor – Aris theory was found to be applicable for $t > 0.8$. Gill and Ananthakrishnan (1966) studied the effect of inlet boundary conditions on the transient approach to the asymptotic Taylor – Axis theory which was validated with the experimental work of Reejhsingani et al. (1966) on dispersion in horizontal pipes. Neglecting axial molecular diffusion, Lighthill (1966) presented a
theory complimentary to the Taylor’s theory which takes into account the initial concentration distribution and is valid for small values of time less than \( \frac{0.1a^2}{D_m} \). All the above models are valid only for large or small times.

Gill (1967) generalized the work of Taylor by giving a series expansion about the mean concentration to describe the local concentration. Thus, in this approach, the expression obtained for the effective molecular diffusivity (dispersion coefficient) reduced to Taylor’s results in the limit for large Peclet numbers and to Aris’s results for small Peclet numbers where axial molecular dispersion is significant. This theory was extended by Gill and Ananthakrishnan (1967) by including the effect of finite slug inputs on the dispersion process.

Gill and Sankarasubramanian (1970) in their subsequent paper showed that the method of series solution mentioned above provides an exact solution to the unsteady convective diffusion problem for laminar flow in a circular tube provided that the coefficients in dispersion model are obtained as a suitable function of time ‘t’. This model was widely known as generalized dispersion model. By considering two terms in the series solution and neglecting higher order terms, Gill and Sankarasubramanian (1970) showed that the mean concentration profile of the solute was symmetric about a point moving with the mean velocity of the fluid for all values of time. Using a different analysis, in a series of papers Yu (1976, 1979, 1981) showed that the
mean concentration profile was not symmetric at small times after the injection of the solute into the fluid. It was observed that although the complete dispersion equation of Gill and Sankarasubramanian’s generalized dispersion model is exact, the truncated two terms from it with time-dependent coefficient is exact only at large values of time. For small values of Peclet number, the two term approximation was found to be reasonably satisfactory over all values of time. The problem of dispersion of solutes in non-Newtonian fluids is also investigated by a few authors. The review on the dispersion phenomena in powerlaw fluids flowing through circular pipes is given in Agarwal (1994) and Agarwal and Jayaraman (1994).

Based on Taylor - Aris fundamental theory of dispersion, Sharp (1993) investigated the shear- augmented dispersion in non-Newtonian fluids (Casson, Bingham plastic and power law fluids) flowing through pipes and channels. He showed that the dispersion coefficient was dependent on the specific rheology of the fluid viz the yield stress for Bingham plastic and Casson fluids, and power law index for power law fluids which was valid for relatively large time. The axial dispersion was found to reduce due to the non-Newtonian rheology of the fluid. Dash et al. (2000) studied the dispersion of a solute in a Casson fluid in a conduit employing the generalized dispersion model of Gill and Sankarasubramanian (1970). It was shown that the effect of yield stress reduced the rate of dispersion of passive species in the flow. Their results agreed with those of Sharp (1993) for large times. Nagarani et al.
(2008) studied the exact analysis of unsteady convective diffusion in Casson fluid flow in an annulus with application to a catheterized artery. In this study they made an attempt to analyse the catheter insertion related errors in flow measurements in the cardiovascular system. Recently Ramana (2011) studied the dispersion in Herschel-Bulkley fluid in a conduit (pipe/channel) and in annular flow with a motivation to understand the dispersion of dyes / drugs / nutrients in the blood stream of the cardiovascular system modelling blood as a Herschel-Bulkley fluid. It is observed that non-Newtonian rheology has a significant influence on the rate of dispersion of the solute and the mean concentration.

The applications of magnetohydrodynamic principles in biology and medicine are abundant. It is known that the Lorentz’s force opposes the motion of a conducting fluid. As blood is an electrically conducting fluid (Katz and Kolin 1938), the principles of MHD may be used to decelerate the flow of blood in a human arterial system and thus it is useful in the treatment of certain cardiovascular disorders. Kolin (1936) and later Korchevskii et al. (1965) studied the influence of magnetic field in human system with a motivation to regulate the movement of blood. Vardanyan (1973) studied the effect of magnetic field on blood flow and this work was later corroborated by Sud et al. (1974, 1978) by considering different models. In all these studies the effect of magnetic field is found to slow down the speed of blood.
The dispersion of a solute in a laminar flow of an electrically conducting fluid in a two dimensional channel in the presence of a transverse magnetic field has been studied by Gupta and Chatterjee (1968) using both Taylor’s theory and Aris analysis. They showed that the solute is dispersed relative to a plane moving with the mean velocity of the flow with an effective diffusion coefficient which decreases with an increase in magnetic field. Annapurna and Gupta (1979) studied the dispersion of a solute in an electrically conducting fluid flow between two parallel plates in the presence of a uniform transverse magnetic field. The importance of dispersion in hydromagnetic flows has been discussed by Branover, et al. (1979). Deshikachar and Rao (1987) studied the axial molecular diffusion of a solute in the laminar flow of an electrically conducting fluid oscillating with zero mean velocity, between two parallel plates in the presence a transverse magnetic field using perturbation analysis. Rao and Deshikachar (1987) analysed the unsteady convective diffusion of a solute in a fully developed laminar flow of an incompressible, homogeneous, viscous fluid in an annular pipe by extending the analysis of Gill and Sankarasubramanian (1970).

In this chapter the dispersion of a solute in a Newtonian fluid flowing through a conduct (pipe / channel) is studied under the influence of a transverse magnetic field with a motivation to understand the influence of magnetic field on the rate of dispersion. The mathematical formulation of the problem in pipe flow and the corresponding solutions are presented in section
2.2. The channel flow analysis is presented in section 2.3. The effect of magnetic field (Hartmann number) on the dispersion coefficient and the overall dispersion process is discussed in section 2.4. The conclusions are presented in section 2.5.

\section{2.2 PIPE FLOW ANALYSIS}

\textbf{Mathematical Formulation}

Consider the dispersion of a solute that is initially of \( z_s \) units in length distributed in a straight circular tube of radius ‘a’ (The schematic diagram is shown in figure 2.1). The unsteady convective diffusion equation which describes the local concentration \( \bar{C} \) of the solute as a function of longitudinal (axial) coordinate \( z \), transverse (radial) coordinate \( r \) and time \( \tilde{t} \) can be written in the form

\[
\frac{\partial \bar{C}}{\partial \tilde{t}} + \bar{w} \frac{\partial \bar{C}}{\partial z} = D_m \left( \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \bar{C}}{\partial \tilde{r}} \right) + \frac{\partial^3 \bar{C}}{\partial z^3} \right)
\]  

(2.1)

where \( \bar{w} \) is the axial velocity of the fluid in pipe and \( D_m \) is coefficient of molecular diffusion (molecular diffusivity) which is assumed to be constant.

\textbf{Initial and Boundary Conditions}

The initial and boundary conditions for the slug input of solute length \( z_s \) under consideration are given by
\[
\bar{C}(0, \bar{r}, \bar{z}) = C_0 \quad \text{if} \quad |\bar{z}| \leq \bar{z}_s / 2 \quad (2.2a)
\]
\[
\bar{C}(0, \bar{r}, \bar{z}) = 0 \quad \text{if} \quad |\bar{z}| > \bar{z}_s / 2 \quad (2.2b)
\]
\[
\bar{C}(\bar{t}, \bar{r}, \infty) = 0 \quad (2.2c)
\]
\[
\frac{\partial \bar{C}}{\partial \bar{t}} (\bar{t}, 0, \bar{z}) = 0 = \frac{\partial \bar{C}}{\partial \bar{t}} (\bar{t}, a, \bar{z}) \quad (2.2d, e)
\]

Conditions (2.2a) and (2.2b) constitute the initial conditions, (2.2c) constitutes the far downstream condition, and (2.2d) and (2.2e) constitute respectively, the symmetry condition at the centre and the no-flux condition at impermeable wall of the pipe.

**Non – Dimensionalisation**

The following non – dimensional variables are considered

\[
C = \frac{\bar{C}}{C_0}, \quad w = \frac{\bar{w}}{w_0}, \quad r = \frac{\bar{r}}{a}, \quad z = \frac{D_m \bar{z}}{a^2 w_0}, \quad t = \frac{D_m \bar{t}}{a^2} \quad (2.3)
\]

where \( w_0 = -\frac{a^2}{4\mu_n} \frac{d \bar{p}}{d \bar{z}} \) \quad (2.4)

is the characteristic velocity (centerline velocity in a Poiseuille flow), \( \mu_n \) is the Newtonian viscosity of the fluid and \( \frac{d \bar{p}}{d \bar{z}} \) is the applied pressure gradient along the axis of the pipe.

The unsteady convective diffusion equation (2.1), in non – dimensional form, takes the form

\[
\frac{\partial C}{\partial t} + w \frac{\partial C}{\partial z} = \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{Pe^2} \frac{\partial^2}{\partial z^2} \right] C \quad (2.5)
\]
where \( Pe = \frac{aw_0}{D_m} \), is the Peclet number \( (2.6) \)

The initial and boundary conditions \( (2.2) \), in dimensionless form, are given by

\[
C(0, r, z) = 1 \quad \text{if} \quad |z| \leq z_s / 2 \quad (2.7a)
\]

\[
C(0, r, z) = 0 \quad \text{if} \quad |z| > z_s / 2 \quad (2.7b)
\]

\[
C(t, r, \infty) = 0 \quad (2.7c)
\]

\[
\frac{\partial C}{\partial r}(t, 0, z) = 0 = \frac{\partial C}{\partial r}(t, 1, z) \quad (2.7d, e)
\]

**Velocity distribution**

Consider the flow of Newtonian fluid in a circular pipe. Assume that the flow is axi-symmetric, fully developed, steady and laminar. A uniform magnetic field \( B_o \) is applied in the transverse direction. The equation of motion for a steady flow under the influence of the transverse magnetic field following Sanyal et al. (2007) and Sachin Shaw et al. (2010) is given by

\[
0 = -\frac{\partial \bar{p}}{\partial z} + \mu \bar{\nabla}^2 \bar{w} - \sigma B_o^2 \bar{w} \quad (2.8)
\]

where \( \frac{d \bar{p}}{d \bar{z}} \) is the applied pressure gradient along the axis of the pipe. \( \mu \) is the kinematic viscosity, \( \sigma \) is the electrical conductivity of the medium.

Solving the equation \( 2.8 \) along with the no slip condition and the velocity distribution of a fluid in a pipe, in non-dimensional form can be obtained as

\[
w = \frac{p}{M^2} \left[ 1 - \frac{I_o(Mr)}{I_o(M)} \right] \quad (2.9)
\]
Where \( M = B_0 a \sqrt{\frac{\sigma}{\mu}} \) is the Hartmann number, \( B_0 \) is strength of the magnetic field, \( \sigma \) is the electrical conductivity of the medium, \( \mu \) is the co-efficient of viscosity of blood. \( I_0 \) is the modified Bessel function of order zero of the first kind. The mean velocity of the fluid in dimensionless form is given by

\[
w_m = \frac{P}{M^2} - \frac{2 P}{M^3} \frac{I_1(M)}{I_0(M)}
\]

(2.10)

**Method of Solution**

Consider the convection across a plane moving with an average velocity \( w_m \) of the fluid. For this, define a new coordinate system moving with new axial coordinate \( z_1 \), given by

\[
z_1 = z - w_m t
\]

(2.11)

The solution of equation (2.5) along with the conditions (2.7) is formulated as a series expansion in \( \frac{\partial^j C_m}{\partial z_1^j} \), following Gill and Sankarasubramanian (1970), is given by

\[
C = C_m + \sum_{j=1}^{\infty} f_j(t, r) \frac{\partial^j C_m}{\partial z_1^j}
\]

(2.12)

where \( C_m = 2 \int_0^1 C r \, dr \)

(2.13)

is the mean concentration over a cross section.

On transforming the unsteady convective diffusion equation (2.5) into the moving co-ordinate system (r, \( z_1 \), t) where \( z_1 \) is given in equation (2.11) and substituting equation (2.12) into the transformed unsteady convective
equation, we get

\[
\frac{\partial C_m}{\partial t} + (w-w_m') \frac{\partial C_m}{\partial z_1} - \frac{1}{Pe^2} \frac{\partial^2 C_m}{\partial z_1^2} + \sum_{j=1}^{\infty} \left[ \left( \frac{\partial f_j}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_j}{\partial r} \right) \right) \frac{\partial^j C_m}{\partial z_1^j} + (w-w_m') f_j \frac{\partial^{j+1} C_m}{\partial z_1^{j+1}} \right] = 0
\]  

(2.14)

It is assumed that the process of distributing $C_m$ is diffusive in nature from the time ‘zero’, then following Gill and Sankarasubramanian’s (1970) the generalized dispersion model for $C_m$ can be written as

\[
\frac{\partial C_m}{\partial t} = \sum_{i=1}^{\infty} K_i(t) \frac{\partial^i C_m}{\partial z_1^i}
\]  

(2.15)

with dispersion coefficient $K_i$ as suitable functions of time $t$. The first two terms in the right hand side of equation (2.15) describe the transport of $C_m$ in axial direction $z_1$ through convection and diffusion respectively, and therefore, the coefficients $K_1$ and $K_2$ are termed as the longitudinal convection and diffusion coefficients for $C_m$. In order to obtain an exact solution for the unsteady convective diffusion problem (2.5) and (2.7) a sufficient number of terms in equation (2.12) and (2.15) must be considered. Substituting equation (2.15) in equation (2.14) and rearranging the terms, we obtain
\[
\left[ \frac{\partial f_1}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_1}{\partial r} \right) + (w - w_m) + K_1(t) \right] \frac{\partial C_m}{\partial z_1} + \left[ \frac{\partial f_2}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_2}{\partial r} \right) + (w - w_m) f_1 \right.
\]
\[
+ K_1(t) f_1 + K_2(t) - \frac{1}{Pe^2} \right] \frac{\partial^2 C_m}{\partial z_1^2} + \sum_{j=1}^n \left\{ \frac{\partial f_{j+2}}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_{j+2}}{\partial r} \right) + (w - w_m) f_{j+1}(t,r) \right.
\]
\[
- \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_{j+2}}{\partial r} \right) + (w - w_m) f_{j+1}(t,r) - \frac{1}{Pe^2} f_j(t,r) + \sum_{i=1}^{j-1} f_{j+2-i}(t,r) K_i(t) + K_{j+2}(t) \right\}
\]
\[
\frac{\partial^{j+2} C_m}{\partial z_1^{j+2}} = 0 \quad (2.16)
\]

Comparing the coefficient of \( \frac{\partial^j C_m}{\partial z_1^j} \), \( j = 1, 2 \ldots \), we get an infinite set of differential equations given by

\[
K_1(t) + \frac{\partial f_1}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_1}{\partial r} \right) + (w - w_m) = 0 \quad (2.17a)
\]

\[
K_2(t) - \frac{1}{Pe^2} + \frac{\partial f_2}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_2}{\partial r} \right) + [(w - w_m) + K_1(t)] f_1 = 0 \quad (2.17b)
\]

\[
K_{j+2}(t) + \frac{\partial f_{j+2}}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_{j+2}}{\partial r} \right) + (w - w_m) f_{j+1} - \frac{1}{Pe^2} f_j + \sum_{i=1}^{j+1} K_i(t) f_{j+2-i} = 0 \quad (2.17c)
\]

for \( j = 1, 2 \ldots \) with \( f_0 = 1 \)

we get the initial and boundary conditions on \( f_j \)'s from equation (2.7) and (2.12), as

\[
f_j(0, r) = 0 \quad j = 1, 2 \ldots \quad (2.18a)
\]

\[
\frac{\partial f_j}{\partial r}(t,0) = 0 = \frac{\partial f_j}{\partial r}(t,1) \quad j = 1, 2 \ldots \quad (2.18 b,c)
\]

and from equation (2.12) and (2.13), the solvability condition is obtained as

\[
\int_0^1 f_j \, r \, dr = 0 \quad j = 1, 2 \ldots \quad (2.19)
\]
Multiplying equations (2.17a) (2.17b) and (2.17c) by r and integrating from 0 to 1, and using the condition (2.20), we have

$$K_1(t) = -2 \int_0^1 (w - w_m) r \, dr = 0 \quad (2.20)$$

$$K_2(t) = \frac{1}{pe^2} - 2 \int_0^1 f_1(t, r) w(r) \, r \, dr \quad (2.21)$$

and $$K_{j+2}(t) = -2 \int_0^1 f_{j+1}(t, r) w(r) \, r \, dr , \, j = 1, 2, \ldots \ldots \quad (2.22)$$

**Solution for \( f_1 \)**

In the series expansion of equation (2.12), the function \( f_1 \) is the most important coefficient as it gives the measure of deviation of the local concentration \( C \) from the mean concentration \( C_m \). The solution to the non-homogeneous parabolic partial differential equation (2.17a) and the conditions (2.18) can be written in the form

$$f_1(t, r) = f_{1s}(r) + f_{1t}(t, r) \quad (2.23)$$

where \( f_{1s}(r) \) is the large time solution which corresponds to Taylor-Aris’s dispersion theory and \( f_{1t} \) is the transient part which describes the time-dependent nature of the dispersion phenomena corresponding to a Newtonian model. From equations (2.17a) and (2.18) and using (2.20), we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_{1t}}{\partial r} \right) = -(w - w_m) \quad (2.24)$$

$$\frac{\partial f_{1t}}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f_{1t}}{\partial r} \right) \quad (2.25)$$
with the boundary and initial conditions

\[
\frac{df_{1s}}{dr}(r = 0) = 0 = \frac{df_{1s}}{dr}(r = 1) \quad (2.26a, b)
\]

\[
\frac{df_{1s}}{dr}(t, 0) = 0 = \frac{df_{1s}}{dr}(t, 1) \quad (2.27a, b)
\]

\[
f_{1s}(0, r) = -f_{1s}(r) \quad (2.27c)
\]

From the solvability condition (2.19), we have

\[
\int_0^1 f_{1s} r dr = \int_0^1 f_{1s} r dr = 0 \quad (2.28)
\]

The solution for \( f_{1s} \) is, obtained from equation (2.24) subject to the conditions (2.26) and (2.28), is given by

\[
f_{1s}(r) = \frac{P_o}{M^2} \left[ \frac{1}{M} \frac{I_1(M)}{I_o(M)} \frac{r^2}{2} - \frac{1}{M^2} \frac{I_o(M r)}{I_o(M)} \right] + B \quad (2.29)
\]

where \( B = -\frac{2}{M^3} \frac{P_o}{I_o(M)} \left[ \frac{1}{8} - \frac{1}{M^2} \right] \quad (2.30) \)

From equation (2.25) subject to the conditions (2.27) and (2.28) the solution for \( f_{1t} \) is obtained as

\[
f_{1t} = \sum_{m=1}^\infty A_m J_0(\lambda_m r) e^{-\lambda_m^2 t} \quad (2.31)
\]

where \( A_m = -\frac{1}{\int_0^1 f_{1s} J_0(\lambda_m r) r dr} \)

\[
= -\frac{2}{J_o(\lambda_m)} \frac{P}{M^3} \frac{I_1(M)}{I_o(M)} \left[ \frac{1}{\lambda_m^2} - \frac{1}{\lambda_m^2 + M^2} \right] \quad (2.32)
\]
$J_0$ and $J_1$ are the Bessel functions of first kind of order zero and one respectively and $\lambda_m$’s are the solutions of the equation $J_1(x) = 0$

**Solution for $K_2$**

The coefficient $K_2(t)$ has a very significant role in the generalized dispersion model given by equation (2.15). It is known from equation (2.21), that $K_2$ depends on the function $f_1$. Substituting the expression of $f_{1s}$ and $f_{1t}$ and simplifying the equation (2.21), we can obtain $K_2$. Once $K_2(t)$ is known, then $f_2(t, r)$ can be obtained from equation (2.17 b) in a similar manner to that $f_1(t, r)$. Following similar procedure we can find $K_3(t), f_3(t, r), K_4(t), f_4(t, r)$ … etc. Since the expression for $f_1(t, r)$ and $K_2(t)$ are complicated in nature, it is very difficult to evaluate $f_2(t, r), K_3(t), …$ and so on. It was shown that in the absence of magnetic field (Gill and Sankarasubramanian, 1970) $K_3(t \rightarrow \infty) = -1/23040$ and the magnitude of higher order coefficients decrease further. We have not evaluated these coefficients which are likely to decrease further in magnitude due to the presence of magnetic field.

**Solution for mean concentration**

Neglecting $K_3(t)$ and higher order coefficients, the generalized dispersion model leads to

$$\frac{\partial C_m}{\partial t} = K_2(t) \frac{\partial^2 C_m}{\partial z_1^2} \tag{2.34}$$

The initial and boundary conditions for $C_m$ are given by
\[ C_m(0, z_1) = \begin{cases} 1 & \text{if } |z_1| \leq \frac{z_s}{2} \\ 0 & \text{if } |z_1| > \frac{z_s}{2} \end{cases} \] (2.35a)

\[ C_m(0, z_1) = 0 \quad \text{if } |z_1| > \frac{z_s}{2} \] (2.35b)

\[ C_m(t, \infty) = 0 \] (2.35c)

From equation (2.34) along with the help of the initial and boundary conditions (2.35) the solution for mean concentration can be obtained as

\[
C_m = \frac{1}{2} \left[ \text{erf} \left( \frac{1}{2} \frac{z_s - z_1}{2\sqrt{\xi}} \right) + \text{erf} \left( \frac{1}{2} \frac{z_s + z_1}{2\sqrt{\xi}} \right) \right] \tag{2.36}
\]

where \( \xi = \int_0^t K_s(t) \, dt \) \tag{2.37}

By using the expression for \( C_m \) and \( f_1 \) in equation (2.12) and neglecting the higher order terms, we can obtain the local concentration \( C \) to a first approximation as

\[
C(r, z_1, t) = C_m(z_1, t) + f_1(r, t) \frac{\partial C_m}{\partial z_1}(z_1, t). \tag{2.38}
\]

### 2.3 CHANNEL FLOW ANALYSIS

We consider the Newtonian fluid flowing in a channel that is assumed to be axi-symmetric, fully developed, steady and laminar. We use the Cartesian co-ordinate system \((\bar{x}, \bar{z})\), where \( \bar{x} \) denotes the transverse co-ordinate and \( \bar{z} \) denotes the axial co-ordinate, to describe the dispersion of a
solute in a channel. Following a similar procedure as in the case of pipe flow, the unsteady convective diffusion equation for the dispersion of the solute in the channel flow in dimensionless form can be written as

\[ \frac{\partial C}{\partial t} + w \frac{\partial C}{\partial z} = \left[ \frac{\partial^2 C}{\partial x^2} + \frac{1}{Pe^2} \frac{\partial^2 C}{\partial z^2} \right] \] \hfill (2.39)

and \( Pe = \frac{a w_0}{D_m} \), is the Peclet number

The initial and boundary conditions, in the non-dimensional form are given as

\[ C(0, x, z) = 1 \quad \text{if} \quad |z| \leq z_s / 2 \quad \text{(2.40a)} \]

\[ C(0, x, z) = 0 \quad \text{if} \quad |z| > z_s / 2 \quad \text{(2.40b)} \]

\[ C(t, x, \infty) = 0 \quad \text{(2.40c)} \]

\[ \frac{\partial C}{\partial x}(t, 0, z) = 0 = \frac{\partial C}{\partial x}(t, 1, z) \quad \text{(2.40 d,e)} \]

where ‘a’ is the half of the channel width, and \( w_0 \) is the characteristic velocity which is given by

\[ w_0 = -\frac{a^2}{4 \mu_w} \frac{d \bar{p}}{d \bar{z}} \quad \text{(2.41)} \]

The velocity distribution for an axi-symmetric, fully developed, steady, laminar flow of a Newtonian fluid in a channel, in non-dimensional form, is obtained as

\[ w = \frac{P}{M^2} \left[ 1 - \frac{\cosh M x}{\cosh M} \right] \quad \text{(2.42)} \]
Where \( M = B_0 a \sqrt{\frac{\sigma}{\mu}} \) is the Hartmann number, \( B_0 \) is strength of the magnetic field, \( \sigma \) is the electrical conductivity of the medium, \( \mu \) is the co-efficient of viscosity of blood. \( I_0 \) is modified Bessel function of order zero of the first kind.

The mean velocity, in dimensionless form is given by

\[
w_m = \frac{P}{M^2} \left[ 1 - \frac{1}{M} \frac{\sinh M}{\cosh M} \right]
\]

In order to analyse the dispersion of a solute in the present channel case, the unsteady convective diffusion equation (2.39) has to be solved for the local concentration \( C \) subject to the conditions (2.40) with the axial velocity \( w \) given in equation (2.42). In a similar manner to that of pipe flow analysis, we proceed to find the solution for the problem in a co-ordinate system moving with the average velocity \( w_m \) of the fluid. In the channel case the mean concentration \( C_m \) is defined as

\[
C_m = \frac{1}{L} \int_0^L C \, dx
\]

There will be minor modifications in the solvability condition (2.20) due to change in the definition of \( C_m \) and expressions for \( K_j \), \( j = 1, 2, \ldots \).

Following a similar process followed in the case of pipe flow analysis, the solution for \( f_{ls}, f_{lt} \) and \( K_2 \) can be obtained.

The steady state solution \( f_{ls} \) is obtained as
The solution for $f_{1t}$ is given by

$$f_{1t} = \sum_{m=1}^{\infty} A_m e^{-\lambda_m t} \cos (\lambda_m x)$$

(2.46)

where $A_m = \frac{\int_0^1 f_{1s} \cos (\lambda_m x) \, dx}{\int_0^1 \cos^2 (\lambda_m x) \, dx} = -2 \int_0^1 f_{1s} \cos (\lambda_m x) \, dx$

(2.47)

$\lambda_m = n \pi$, $n = 1, 2, \ldots$.

### 2.4 RESULTS AND DISCUSSION

The objective of the present study is to understand the effect of magnetic field on the dispersion of a solute in a Newtonian fluid flowing in a conduit (pipe/channel). This study facilitates to know the dispersion of drugs and nutrients in circulatory system and to measure the cardiac output employing the Indicator Dilution Technique. This analysis can also be utilized to artificial blood handling devices such as blood oxygenators and hemodialysers. As some devices involve parallel plates and membranes, the study of dispersion in parallel plates geometry is also investigated.

The generalized dispersion model of Gill and Sankarasubramanian (1970) is used to study the process of unsteady convective diffusion. As a result the entire process of dispersion is described in terms of a simple
diffusion process with apparent dispersion coefficient \( K_2 \) as a function of time.

The dispersion coefficient \( K_2 \) is found to be influenced significantly by the magnetic field. The time dependent nature of the dispersion coefficient \( K_2 \) versus time for different values of magnetic field (Hartmann number) for dispersion in pipe flow is described in Fig 2.2(a). The corresponding plots in channel case are shown in Fig 2.2(b). The dispersion coefficient \((K_2 - 1/Pe^2)\) becomes essentially a constant for large values of time. Taylor’s theory is applicable to the dispersion of the passive tracer in flow after the time at which \( K_2(t) - \frac{1}{Pe^2} \) attains the asymptotic value of the dispersion coefficient, while for small values of time the approximation corresponding to Lighthill (1966) holds good. It is also observed that the time beyond which the Taylor’s theory is applicable is unaffected by the presence of magnetic field. The time taken to reach the steady state is observed to be dependent on the magnetic field. In the absence of magnetic field, the time to reach the steady state is 0.5 (Gill and Sankarasubramanian 1970). In the presence of magnetic field this steady state is reached faster and this critical value reduces as Hartmann number increases. When \( M = 3 \) the time to reach this critical value is almost half of the time corresponding to the case when \( M = 1 \). In the channel case a similar behavior is noticed. However, the time to attain this critical value is more compared to that of pipe flow analysis. The presence of magnetic field in a pipe (channel) reduces the dispersion coefficient. Increase in the
Hartmann number still decreases the dispersion coefficient. Gupta and Chatterjee (1968) and Annapurna and Gupta (1979) observed the same in the channel case. In pipe flow analysis (channel) when Hartmann number is 2 the dispersion is reduced by 2 (4) times of the corresponding value for $M = 1$. When $M = 3$ this reduction factor is observed to be 6 (17). From Fig 2.3(a, b) it is noticed that the dispersion coefficient in pipe (channel) flow analysis decreases with increase in Hartmann number and as $M$ approaches 5 (4), $K_2$ approaches the value zero. In this case flow becomes more plug like and the dispersion disappears.

The time evaluation of the function $f_1$ for dispersion in pipe and channel flows is described in Fig 2.4(a, b). $f_1$ provides a measure of deviation in the local concentration $C$ from the mean concentration $C_m$. At time $t = 0$ $f_1$ is uniformly zero over the entire cross-section of the pipe (channel). $f_1$ is noted to attain its steady state value $f_{1s}$ as $t$ increases which is also shown in Fig 2.5(a, b).

The effect of magnetic field on $f_1$ is shown in Fig 2.6(a, b). The presence of magnetic field is seen to reduce the magnitude of the peak values of $f_1$. In the pipe (channel) case when $M = 3$ there is a 3 fold (four fold) reduction in the magnitude of $f_1$ at $r = 0$ corresponding to the case when $M = 1$. It is noticed from Fig 2.5 and Fig 2.6 that the functions $f_1$ and $f_{1s}$ pass through a common point for all times and for all values of the magnetic field. At this point $f_1$ and $f_{1s}$ are zero and the local concentration $C$ of the solute
becomes equal to the mean concentration $C_m$. Therefore, this point shall be considered as the centre of mass of the solute over a cross section of the pipe/channel. This centre of mass of the solute is independent of time and Hartmann number. It is also observed that centre of mass of the solute in the pipe flow analysis occurs nearer to pipe wall while it is the midway of the channel width.

Fig 2.7 (a, b) describes the variation of mean concentration with time when Hartmann number $M = 1$, pressure gradient $P = 1$ and axial distance $z = 0.5$. It is observed that the peak values of the mean concentration $C_m$ in pipe (channel) occurs at $t = 4.64$ (2.08) for different lengths of slug inputs of solute. The peak value of $C_m$ increases with increase in slug input length. There is a fivefold enhancement in $C_m$ when $z_s$ is increased from 0.004 to 0.019 and a two fold increase is noticed when $z_s = 0.008$. Exactly a similar trend is observed in channel flow also. However, the magnitudes of the peak values of $C_m$ are less in channel.

Fig 2.8(a, b) depicts the variation of mean concentration for different values of Hartmann number. It is observed that as $M$ increases the value of $C_m$ is also increased and the time taken to attain this peak value of $C_m$ also increases both in pipe and channel cases. In pipe flow the peak value in the absence of the magnetic field occurs at $t = 1$ (Gill and Sankarasubramanian (1970)). The presence of magnetic field takes more time to attain the peak
value. When $M = 1$ the peak value of $C_m$ in pipe (channel) occurs at $t = 4.65 (2.08)$ while it is at $t = 9.75 (6.72)$ when $M = 3$.

The variation of mean concentration $C_m$ with axial distance $z$ for different slug input lengths of solute is presented in Fig 2.9 (a, b). The peak value of concentration in pipe (channel) occurs at $z = 0.003 (0.005)$. As the length of slug input of the solute increases the peak value also increase. In the pipe flow analysis when $M = 1$ the peak value increases from 0.761 to 1 when $z_s$ changes its value from 0.004 to 0.019. In the channel flow analysis there is a fivefold increase in the peak value of $C_m$.

The plot of the variation of $C_m$ versus $z$ for different values of $M$ is presented in Fig 2.10 (a, b). It is noticed that the peak value of $C_m$ increases with increase in $M$. However, the peak value is drifted to right of the origin as $M$ takes higher values in pipe case while in the case of channel the peak values occurs at $z = 0$.

### 2.5 CONCLUSIONS

The objective of the present investigation is to study the effect of a magnetic field on the process of dispersion. The convective dispersion process is analysed applying the generalized dispersion model suggested by Gill and Sankarasubramanian (1970). It is observed that the diffusion coefficient, which describes the dispersion process, is influenced by a magnetic field. It is observed that the results on $K_2(t) - \frac{1}{Pe^2}$ agree with that of Taylor’s theory for
large times and for small values of time the results agree with Lighthill. It is also observed that the time beyond which the Taylor theory is applicable is unaffected by the presence of magnetic field. The effect of magnetic field is to reduce the rate of dispersion of the solute in the fluid flow. It is observed that the presence of magnetic field in both pipe and channel cases time taken for the dispersion coefficient to reach a steady state is more and the time further increases with increase in the magnetic field. The time to attain these critical values is more than channel case. Enhancement in magnetic field reduces the dispersion. In channel case the rate of reduction in $K_2$ is found to be more. It is found that the centre of mass of the solute is independent of time and magnetic field. This center of mass occurred nearer to the pipe wall in the pipe flow analysis where as it is found in the way of the width of the channel. The time taken for the mean concentration to attain the peak value is found to increase in magnetic field in both pipe and channel cases. The values of $C_m$ in pipe flow analysis are drifted to the right of origin along the axial direction as $M$ increases, while in the corresponding channel case they are attained at the origin.
Fig 2.2 Variation of dispersion coefficient $K_2(1/Pe^2)$ verses time $t$ for different values of $M$ (a) Pipe (b) Channel
Fig 2.3 Variation of dispersion coefficient $K_2 - (1/Pe^2)$ verses $M$ for $t = 0.05$
(a) Pipe  (b) Channel
Fig 2.4 Distribution of dispersion function $f_1$ for different values of time $t$ when $M=1$
(a) Pipe  (b) Channel
Fig 2.5 Distribution of steady state dispersion function $f_{1s}$ for different values of $M$ when $t=0.5$  
(a) Pipe  (b) Channel
Fig 2.6 Distribution of dispersion function $f_1$ for different values of $M$ when $t=0.05$
(a) Pipe (b) Channel
Fig 2.7 Distribution of dispersion function $f_1$ for different values of M when $t=0.05$
(c) Pipe  (d) Channel
Fig 2.7 Variation of mean concentration $C_m$ with time for different values of $Z_s$ when $M=1$ and $z=0.5$  (a) Pipe  (b) Channel
Fig 2.8 Variation of mean concentration $C_m$ with time for different values of $M$ when $Z_u=0.019$ and $Z=0.5$ (a) Pipe (b) Channel
Fig 2.9 Variation of mean concentration $C_m$ with axial distance $z$ for different values of $Z_s$ when $t=0.03$ and $M=1$  (a) Pipe  (b) Channel
Fig 2.10 Variation of mean concentration $C_m$ with axial distance $z$ for different values of $M$ when $t=0.03$ and $Z_s=0.04$  (a) Pipe  (b) Channel
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Fig 2.1 Schematic diagram of velocity profile for a Newtonian fluid in a circular tube in presence of magnetic field and absence of magnetic field