Chapter 6

Dust-hole Equilibria in Vlasov-Poisson System

6.1 Background of the theory

The fluid approximation is sufficiently accurate to describe the majority of macroscopic (i.e. large scale) plasma phenomena that are typically encountered. The fluid model is sufficient for providing a good description of important types of wave-like behavior that are possible in a plasma. However, some phenomena, such as Landau damping, could no longer be understood macroscopically and fluid treatment was found to be inadequate. A velocity distribution function $f(x, \vec{v}, t)$ is needed to describe each of the species of particles in the plasma and the foundation of kinetic theory was laid down. Both Vlasov and Landau gave separate treatments for the wave theory, which are valid under different limits. Vlasov's treatment was apparently successful in reproducing the dispersion relation describing the effect of thermal motions on electron plasma waves and also other types of plasma waves. However, in this theory, the approximate description of thermal corrections to the dispersion relation for plasma waves was obtained under the condition
\[ \omega >> k v_{the} \ (v_{the} \rightarrow \text{thermal velocity of electrons}) \] taking a Maxwellian distribution for \( f_0 \). The integral equation in this theory becomes singular at \( v = \omega/k \). Landau used the method of Laplace transformation to obtain the full correct solution for the effects of a distribution of particle velocities on electron plasma waves, thereby correcting the treatment of Vlasov. His result extends the linear kinetic theory of small-amplitude perturbations to include the effect of particles traveling close to the wave's phase velocity, i.e. resonating with the wave. According to this theory, plasma waves are always slightly damped. Damping is exponentially small for long wavelengths (\( k\lambda_D << 1 \)) but is large (\( \gamma \sim \omega_p \)) for wavelengths of the order of Debye length (\( \lambda_D = v_{th}/\omega_p \)).

The small Landau damping for \( k\lambda_D << 1 \) can be interpreted as due to the fact that there are very few particles at \( v = w/k \approx w_p/k \approx v_{th}/k\lambda_D >> v_{th} \). Physically, it is clear that Landau damping is associated with those particles in the distribution that have a velocity nearly equal to the phase velocity of the wave, \( w/k \). Resonant particles travel along at almost the same speed as the wave and tend to see a relatively static electric field, rather than a rapidly fluctuating one. They can therefore exchange energy very effectively with the wave. The electrons with \( v \approx \omega/k \), (which are nearly resonant with the plasma wave in the Landau problem), see an essentially steady electric field which can be positive or negative depending on their phase relative to the wave. Thus, some nearly resonant particles are accelerated by the wave, while the others are decelerated. However, a Maxwellian distribution has more slower electrons than faster ones. Consequently, there are more particles being accelerated on average by this mixing process than being decelerated. This results in a net transfer of energy from the wave to the particles and hence, the wave is damped. Mathematically, linear wave theory of Vlasov-Poisson plasmas was completed by van Kampen [1995] and developed further by Case [1959; 1978] showing essentially that besides a discrete spectrum of normal modes belonging to a time asymptotic dispersion relation a continuum of modes and eigen
values exist. van Kampen approach consists in expressing the solution of the initial value problem of the linearized Vlasov-Poisson system in terms of the eigen functions of the equations. In a stable plasma, these eigen functions, known as van Kampen modes, turn out to be undamped, but with a singularity appearing. Thus, any well-behaved initial condition can be satisfied only by a continuous superposition of such modes. The relative phase between any two modes increases in time without limit, so that the constructive interference imposed by the initial condition dies away exponentially.

The van-Kampen approach, which gives undamped harmonic waves, does not have any physical solution because of presence of $\delta$-function at resonance velocity. Among these waves, there is one particular mode that corresponds to the Vlasov's dispersion relation which can be obtained from Landau's theory by setting $w = w_r$ and using only $\epsilon_r (k, w_r) = 0$. Vlasov mode is the only physical mode in the van Kampen continuum. The trapped dust-acoustic mode discussed here is resonant around dust-thermal velocity with the condition $\left| \frac{w_r}{k v_{td}} - x_0 \right| << 1$. This can be called as a consequence of Vlasov mode, which is obtained by considering the real part of the generalized dispersion modes.

Very recently, Schamel[2000] has given a unified description of weak hole equilibria in collisionless plasma and predicted a trapped particle modes in dusty plasma for phase velocities in the thermal range. He has shown analytically that Landau and Vlasov theories of plasma waves are not the complete theories. The excitation of coherent phase-space vortices (holes) or related structures associated with particles trapping plays a crucial role in controlling the plasma dynamics and gives rise to new modes in plasma, which grow in amplitude until saturation and control the plasma response. Korn and Schamel [1996] have shown that linear theory is not adequate to represent the new kind of solutions, even in the limit of extremely small amplitude. Presence of trapping demands that possible modes arising are nonlinear from the
very beginning, even if their amplitudes are small. Hence, previous analysis of non-linear effects based on linear theory cannot predict correct modes [Korn & Schamel 1996, Schamel 2000].

The imaginary part of the dispersion relation obtained by linear Landau theory disappears when the effect due to trapped particle is taken into account. In the linear Landau case, the term $\phi \partial_v f_1$ in the Vlasov equation is assumed to be small compared with the term $\phi \partial_v f_0$. When one considers trapping of particles, this assumption is no longer valid. The derivative of the complete distribution function becomes zero or changes sign in the resonant region i.e. $\partial_v f_1$ is of the same order as $\partial_v f_0$ and hence, it is not justified to neglect $\partial_v f_1$ compared to $\partial_v f_0$. The new class of solutions cannot be described by a linear theory even in the limit of extremely small amplitudes. As a result of trapping, these modes are nonlinear from the outset, no matter how small their amplitudes [Korn et al. 1996].

The trapping effect has recently drawn attention because of the omnipresence of holes in collisionless laboratory and space plasma [Schamel 1986; Saeki et al. 1979; Temerin et al. 1982]. Observations as well as numerical simulations of a coasting ion (hadron) beam in circular accelerators and storage rings have shown the excitation of long-lived coherent structures superimposed on the beam. When such a beam interacts with the electromagnetic field induced by the ring environment, may become unstable at high intensities and can develop holes (or notches) in the longitudinal distribution function and in the associated line density. Such holes can also be excited externally by means of an applied voltage impressed on the beam [Schamel et al. 2000; Colestock et al. 1996].

A theory on dust hole equilibria in Vlasov-Poisson system is investigated in the following section. A new kind of dust-acoustic wave arising due to the trapping effect is found to exist in the dust thermal range. The effect of dust charge fluctuation on this new mode is also considered.
6.2 Trapped Dust Acoustic Wave: A new mode in dusty plasma

A continuum of nonlinear, steady-state, ultra low frequency acoustic wave is shown to exist, propagating near the dust-thermal velocity and being characterized by deficit of dust particles trapped in the trough of the wave potential. It is based on the theory of hole equilibria in Vlasov-Poisson systems proposed by Schamel [2000] where it is shown that linear theory is not adequate to represent the new kind of solutions even in the limit of extremely small amplitudes. Landau or Van Kampen modes and their general superpositions cannot describe the trapped particle modes due to an incorrect treatment of resonant particles for phase velocities in the thermal range.

6.2.1 Introduction:

In the last decade, many new kinds of wave modes that exist in low frequency mode are discovered in plasma with highly charged massive dust grains. The dust dynamics plays the key role in supporting these modes. Rao et al [1990] predicted for the first time the existence of Dust Acoustic Wave (DAW) which arises due to the restoring force provided by the plasma (electron and ion) thermal pressure while the inertia is due to the dust mass. The DAW propagates as the normal mode when the phase speed is much larger than the dust thermal speed. Another low frequency wave mode prevailing in dusty plasma is the Dust Lattice Wave (DLW) in the strong coupling regime. It was predicted by Melandso [1996]. Both of low-frequency modes are experimentally verified by recent laboratory experiments on dusty plasma [Barkan et al.1995; Prabhakara et al.1966; Thomson et al. 1997; Homann et al.1997]. Very recently, Rao has predicted the existence of another wave mode called Dust Coulomb Wave (DCW) [Rao 1999] in dense dusty plasma in presence of
the fluctuation in dust charge and number density perturbation. The Coulomb interaction between the grains plays a dominant role in providing the restoring force for these modes, while the inertia arises from the dust grain mass. All these three different modes DAW, DLW and DCW are found to exist in the regime \( v_{td} \ll \omega/k \ll v_{ti}, v_{te} \).

The presence of dust particles can significantly modify the wave propagation characteristics in collisionless plasma. The mode propagation in presence of resonant particles for phase speeds near the thermal range is intrinsically nonlinear, and the usual concept of linear wave theory [Stix 1992] based on the Landau or the van Kampen prescription using the Vlasov equation is inadequate. The trapping of particles in the potential trough of the electrostatic structure has tremendous consequences. It not only affects the phase space topology in the vicinity of the structure, but also changes more or less the wave characteristics. When the particle distribution deviates from Maxwellian due to a wave perturbation of amplitude \( \psi \), however small it may be, some particles are trapped under this potential. Linearization is not allowed in such a situation. It is necessary to take nonlinear approach by taking trapping into account from the very beginning.

In this problem, we consider the effect of trapped dust particles on the propagation of electrostatic dust modes in the dust thermal speed range, and show that there exist a new class of stationary, nonlinear modes over a wide range of dust fugacity [Rao 1999]. These modes exist due to a deficit of dust particles trapped in the wave field, and are thus a consequence of nonlinearity which prevails even in the infinitesimal amplitude limit.

6.2.2 Formulation

The present problem is devoted to marginal modes in the dust thermal range. We consider an unmagnetized dusty plasma having electrons, ions and dust
particles. For simplicity, first it is considered the case when the grain charge is a constant. The effect of charge fluctuation are self-consistently included in a later section. The system is described by the coupled set of Vlasov-Poisson equations given as:

\[
\begin{align*}
\partial_t f_e + v \partial_x f_e + \phi' \partial_v f_e &= 0 \\
\mu_d \partial_t f_d + v_d \partial_x f_d + \phi' \theta_d \partial_u f_d &= 0 \\
\mu_i \partial_t f_i + u \partial_x f_i - \theta_i \phi' \partial_u f_i &= 0
\end{align*}
\] (6.1)\n
\[
\begin{align*}
\nfd &= Vd + Kd \n^2 \exp\left\{-\left(\frac{sgn \n v \sqrt{\epsilon_d + x_d}}{lj \n} \right)^2\right\} \exp\left\{-x_d^2\right\} \exp\left\{-\beta_d \epsilon_d\right\} \quad \epsilon_d > 0 \\
\phi'' &= n_e + Z_d n_d - n_i
\end{align*}
\] (6.4)

where space \(x\), time \(t\) and electric potential \(\phi\) are normalized by electron Debye length \(\lambda_{De}\), inverse of plasma frequency \(\omega_{pe}^{-1}\) and \(T_{ef}/c\), \(T_{ef}\) being the temperature in energy units of the electrons in unperturbed state. The velocities \(v, v_d\) and \(u\) of electrons, dusts and ions are normalized by \(v_{th_e} = \sqrt{2T_e/m_e}\), \(v_{th_d} = \sqrt{2T_d/m_d}\) and \(v_{th_i} = \sqrt{2T_i/m_i}\) respectively. \(\mu_d\) and \(\mu_i\) are defined as \(\mu_d := (m_d T_e/m_e T_d)^{1/2}\), \(\mu_i := (m_i T_e/m_e T_i)\). The charge of the dust grain is given by \(-Z_d e\) and \(\theta_d = T_e/T_d\).

In the perturbed state, the solutions of these Vlasov equations for electrons, dust and ions respectively, can be given in the frame comoving with the wave in which a possible electrostatic structure \(\phi\) may become stationary, by an approximate ansatz in terms of constants of motion as,

\[
\begin{align*}
f_e &= \frac{1 + K_e}{\sqrt{2\pi}} \begin{cases} 
\exp\left\{-\left(\frac{sgn \n v \sqrt{\epsilon_e + x_e}}{lj \n} \right)^2\right\} & \epsilon_e > 0 \\
\exp\left\{-x_e^2\right\} \exp\left\{-\beta_e \epsilon_e\right\} & \epsilon_e \leq 0
\end{cases} \\
f_d &= \nu_d \frac{1 + K_d}{\sqrt{2\pi}} \begin{cases} 
\exp\left\{-\left(\frac{sgn \n v \sqrt{\epsilon_d + x_d}}{lj \n} \right)^2\right\} & \epsilon_d > 0 \\
\exp\left\{-x_d^2\right\} \exp\left\{-\beta_d \epsilon_d\right\} & \epsilon_d \leq 0
\end{cases}
\end{align*}
\] (6.5)\n
(6.6)
\[ f_i = \nu_i \frac{1 + K_i}{\sqrt{2\pi}} \left\{ \begin{array}{ll}
\exp\left[-(\text{sgn} v \sqrt{\epsilon_i} + x_i)^2\right] & \epsilon_i > 0 \\
\exp(-x_i^2)\exp(-\beta_i \epsilon_i) & \epsilon_i \leq 0
\end{array} \right. \]  \hspace{1cm} (6.7)

where we have defined

\[
\begin{align*}
\epsilon_e & := v^2 - \phi, \quad \epsilon_d := v_d^2 - Z_d \theta_d \phi, \quad \epsilon_i := u^2 + \theta_i (\phi - \psi) \\
x_s & := v_{ps}/v_{ts}, \quad s = e, i, d \\
K_s & := (k_{0s} \lambda_{D_s})^2 \psi/2, \quad s = d, e
\end{align*}
\]  \hspace{1cm} (6.8)

The parameter \( K_i \) will be self-consistently determined later. \( k_{0e} \) and \( k_{0d} \) are related to the wave number, which coincide with the actual wave number \( k \) for the case of electrostatic dust thermal mode. The justification of this will be clear later on.

The first parts of each of the equations from (6.5) to (6.7) describe free particles (with sign \( v < 0 \) or \( \text{sgn} \ v > 0 \)) with energy \( \epsilon_s > 0 \). The second parts on the other hand represent trapped electrons, dusts or ions with \( \epsilon_s < 0 \). \( \epsilon \) and \( \text{sgn} \ v \) are constants of motion. The latter is introduced to allow a nontrivial free distribution, which is nonsymmetric in velocity space. The distributions are continuous at the separatrix with \( \epsilon = 0 \). \( \phi(x) \) is the electrostatic potential, which is assumed to be periodic in space, under which particles are trapped. \( \phi(x) \) satisfies without loss of generality,

\[ 0 \leq \phi(x) \leq \psi \]  \hspace{1cm} (6.9)

where \( \psi \) is the amplitude of the perturbation. At the location where trapped particles are absent, \( \phi = 0 \) and the distribution functions defined by (6.5) - (6.7) reduce to the Maxwellian distributions. \( \beta \) is the trapping parameter which controls the state of trapped particles.

By an integration over the velocities and assuming \( \psi \ll 1 \) we get the following densities
\[ n_e = (1 + K_e) \left\{ 1 - \frac{1}{2} \frac{Z_r'(x_e)}{Z_e} \phi - \frac{4}{3} b(\beta_e, \sqrt{2x_e}) \phi^{3/2} + \ldots \right\} \] (6.10)

\[ n_d = \nu_d(1 + K_d) \left\{ 1 - \frac{1}{2} \frac{Z_r'(x_d)}{Z_d} \theta_d \phi - \frac{4}{3} b(\beta_d, \sqrt{2x_d}) (Z_d \theta_d \phi)^{3/2} + \ldots \right\} \] (6.11)

\[ n_i = \nu_i(1 + K_i) \left\{ 1 - \frac{1}{2} \frac{Z_r'(x_i)}{Z_i} \theta_i (\psi - \phi) - \frac{4}{3} b(\beta_i, \sqrt{2x_i}) [\theta_i (\psi - \phi)]^{3/2} + \ldots \right\} \] (6.12)

where \( \beta_s, s = e, d, i \) denotes the trapping parameter of the \( s \)-th species. Furthermore, the function \( b(\beta, \sqrt{2x}) \) is defined by

\[ b(\beta, \sqrt{2x}) = \frac{1}{\sqrt{\pi}} (1 - \beta - 2x^2) \exp(-x^2) \] (6.13)

and \( z_r' \) represents the derivative of the real part of the plasma dispersion function, the latter being defined by,

\[ z_r(x) = \frac{1}{\sqrt{\pi}} P \int dt \frac{\exp(-t^2)}{t-x} \]

For the marginally stable mode near the dust thermal velocity regime, it will hold \( x_d = \frac{\mu}{k v_{td}} \sim O(1) \) where \( x_e \) and \( x_i \) are related to \( x_d \) through

\[ x_d = x_e \sqrt{\frac{T_e m_d}{T_d m_e}} = x_i \sqrt{\frac{T_i m_d}{T_d m_i}} \]

clearly, the terms under the square root in the above equations are typically very large and therefore, we have the inequalities, \( x_e \ll 1 \) and \( x_i \ll 1 \) from which it follows that

\[ -\frac{1}{2} Z_r'(x_{e,i}) = 1 \]

according to the expansions
\[-\frac{1}{2}Z_r'(x) = \begin{cases} 
1 - 2x^2(1 - 2x^2/3 + \ldots), & |x| < 1, \\
-\frac{1}{x_0}(x - x_0) + (x - x_0)^2, & |x - x_0| < 1, \\
-\frac{1}{2x^2}[1 + 3/(2x^3) + \ldots], & |x| > 1.
\end{cases} \tag{6.14}\]

In order to simplify the ensuing algebra, we shall now assume that the electrons and ions are isothermal. For ultra low-frequency modes under consideration, this is a very good assumption. Thus, for $\beta_e = 1$ and $\beta_i = 1$, the third term in eqs. (6.10) and (6.12) disappears and we get the simplified expressions

\[n_e = (1 + K_e)\{1 + \phi + \ldots\} \tag{6.15}\]

\[n_i = \nu_i(1 + K_i)\{1 + \theta_i(\psi - \phi) + \ldots\} \tag{6.16}\]

The normalization constants $\nu_i$ and $\nu_d$ can be determined from the quasineutrality condition. In case of a completely unperturbed plasma (no wave potential), $\psi \to 0, K_e, K_d, K_i \to 0$ and quasineutrality yields

\[n_e + Z_d n_d = n_i \]

or $1 + Z_d\nu_d = \nu_i > 1 \tag{6.17}\]

On the other hand, we consider the case of a solitary potential pulse $\phi$ with a finite but small amplitude $\psi$ for determining the constants $K_s, s = e, d, i$. At infinity, $\phi$ and its derivatives vanish, which as we will later see, implies $K_e = 0, K_d = 0$. Moreover, quasineutrality at $\phi = 0$ implies $K_i = -\theta_i\psi$ on use of the equation (6.17), so that in the lowest order, that means neglecting terms of order $\psi^2$ and higher, we get

\[n_i = \nu_i(1 - \theta_i\phi) \tag{6.18}\]

112
It may be noted that eqs. (6.15) and (6.18) are simply the lowest order contributions of the corresponding Boltzmann distributions.

Assuming a nearly sinusoidal wave with wavenumber \( k \), the relation between \( k_{0e}, k_{0d} \) and \( k \) turns out to be

\[
k^2 = k_{0e}^2 + \frac{k_{0d}^2}{Z_d \theta_d}
\]

(6.19)

In case of a stronger distortion of wave, the wavenumber \( k \) in (6.19) has to be replaced by a parameter \( k_0 \) which is related to \( k \) in a more complicated way [Schamel 1972].

Finally inserting the density expressions (6.11), (6.15) and (6.18) into the Poisson equation, \( \phi'' = n_c + Z_d n_d - n_i \), and using (6.17), we obtain

\[
\phi''(x) = \frac{(k \lambda_{De})^2}{2} \psi + \{1 + \nu_d Z_d^2 \theta_d [-\frac{1}{2} Z_d'(x_d)] + \nu_i \theta_i \} \phi \\
- \theta_d^{3/2} Z_d^{5/2} \nu_d \frac{1}{3} b(\beta_d, \sqrt{2x_d})
\]

(6.20)

Equation (6.20) can be integrated and recast in the form of the energy of a quasi-particle, namely, \( (\phi')^2/2 + V(\phi) = 0 \) where the effective potential \( V(\phi) \) is given by

\[
- V(\phi) = \frac{(k \lambda_{De})^2}{2} \psi \phi + ... \frac{\phi^2}{2} - \theta_d^{3/2} Z_d^{5/2} \nu_d \frac{8}{15} b(\beta_d, \sqrt{2x_d}) \phi^{5/2},
\]

(6.21)

where we have assumed \( V(0) = 0 \), and the expression in ... corresponds to that given in equation (6.20).

For the existence of a solution, it is necessary that \( V(\psi) = 0 \), which gives

\[
(k \lambda_{De})^2 + \{...\} = \frac{16}{15} \theta_d^{3/2} Z_d^{5/2} \nu_d b(\beta_d, \sqrt{2x_d}) \sqrt{\psi}.
\]

(6.22)

Equation (6.22) can be considered as the nonlinear dispersion relation for the trapped modes since it allows the determination of the wave phase speed rep-
resented by \(x_d\). Equation (6.22) can be written as
\[
-\frac{1}{2} Z'_r(x_d) = -(k\lambda_Dd)^2[1 + (k\lambda_D)^{-2}] + \frac{16}{15} b(\beta_d, \sqrt{2}x_d)\sqrt{\theta_dZ_d\psi} \equiv D, \quad (6.23)
\]
with
\[
(\lambda_D)^{-2} = (\lambda_{D*})^{-2} + (\lambda_{Dt})^{-2}. \quad (6.24)
\]
An analytic solution of equation (6.23) can be found [Korn et al. 1996; Schamel 2000] in the limit \(|D| \ll 1\). Using the expansion of \(-\frac{1}{2} Z'_r(x)\) around \(x = x_0 \equiv 0.924\), namely,
\[
-\frac{1}{2} Z'_r(x) = -\frac{1}{x_0}(x - x_0) + \ldots, \quad (6.25)
\]
we get
\[
x_d = x_0(1 - D).
\]
Thus, the phase speed of this wave is given by
\[
\frac{\omega}{k} = v_d x_d = 0.924v_d[1 + (k\lambda_Dd)^2(1 + \frac{1}{k^2\lambda_d^2})
- \frac{16}{15} b(\beta_d, \sqrt{2}x_0)\sqrt{\theta_dZ_d\psi}]. \quad (6.26)
\]
As seen from equation (6.26), the phase speed of the new wave lies in the ther­
al range of the dust particle distribution. We shall term this wave Thermal
Dust Acoustic Wave (TDAW).

To find the corresponding quasi-potential, we utilize equation (6.22) to
simplify (6.21). By inserting the expression \{\ldots\} from equation (6.22) into
equation (6.21), we get
\[
-V(\phi) = \frac{(k\lambda_Dd)^2}{2} \phi(\psi - \phi) + \frac{8}{15} \theta_d^{3/2} z_d^{5/2} v_d b(\beta_d, \sqrt{2}x_0) \phi^2(\sqrt{\psi} - \sqrt{\phi}),
\]
\[
\equiv \frac{(k\lambda_Dd)^2}{2} \phi^2[\phi(1 - \phi) + S\phi^2(1 - \sqrt{\phi})], \quad (6.27)
\]
where \(\phi = \phi/\psi\), and

114
Note that the above two assumptions, namely, (i) \( k \) represents the actual wave number, and (ii) \(|D|\) is small, require \( S \) to be small, that is,

\[
|S| \ll 1 \quad (6.29)
\]

One way to achieve this is to assume \( b(\beta_d, 1.307) \) to be small. From equation (6.13), we then get

\[
\beta_d \sim -0.71 \quad (6.30)
\]

which is negative. Thus, the dust distribution function must have a notch at the phase velocity. If \( S \) is exactly zero, equation (6.27) corresponds to a harmonic wave with a phase velocity given by (6.26) without the last term proportional to \( b\sqrt{\psi} \). This would be the corresponding Vlasov mode [Schamel 2000; Stix 1992] transferred to the dust case. On the other hand, if \( S \) is nonzero but small, equation (6.27) represents a periodic wave with wavenumber \( k \).

However, an explicit evaluation shows [Schamel 1972] that an infinite number of higher harmonics is already involved with this mode, being an intrinsic feature of the kinetic mode. This is in contrast to hydrodynamic modes where only the second harmonic would be involved for small distortions.

In general, one can show [Korn et al. 1996] that a solution exists as long as

\[
-2 \leq S \leq \infty. \quad (6.31)
\]

In the extreme case, we obtain solitary waves for \( \phi \), namely, a solitary hump for \( S = \infty \), and a solitary trough for \( S = -2 \). It is also seen [Schamel 2000] that \( S > 0 \)(\( S < 0 \)) represents rarefaction (compression) waves with a depression (compression) in the number density. The propagation velocity is subsonic (supersonic) with respect to the harmonic TDAW defined by
which follows from equation (6.26) in the limit \( b \to 0 (S \to 0) \). For this particular mode with \( S = 0 \), the left hand side of equation (6.22) agrees with the real part of Landau's dispersion relation which, assuming a real value of \( \omega \), is given by,

\[
\epsilon_r(k, \omega_r) = 1 + \sum_{j=e,i} \frac{1}{k^2 \lambda_{Dj}^2} \left[-\frac{1}{2} Z'_r(x_j) \right] = 0
\]  

(6.33)

for \( |x_e| \ll 1 \) and \( |x_i| \ll 1 \). In the case of a two-component plasma \( j = e, i \), equation (6.33) is just the dispersion relation obtained by Vlasov [1945; Vlasov 1992]. This also coincides with that particular van Kampen mode in which the factor in front of the \( \delta \)-function contribution to the perturbed distribution function disappears. A justification of this particular solution in terms of a nonlinear theory involving trapped particles from the outset has recently been given in reference [Schamel 2000].

We shall now briefly discuss on the modifications that arise when grain charge fluctuations are self-consistently included into the analysis of the trapped modes. In this case, equation (6.23) can be modified under the limit \( S \to 0 \) to the form [Rao 2000]

\[
\frac{1}{2} Z'_r(x_d) = -(k\lambda_D)^2 \left[1 + \frac{1}{(k\lambda_D)^2} \right] - \begin{cases} 
(k\lambda_D)_d^2 \sqrt{\pi} e^{\Lambda^2} \frac{1}{(k\lambda_R)^2} \frac{\omega_1}{k_{vd}}, & \text{for } \omega_r >> \omega_1, \\
\frac{\omega_1}{k_{vd}} \frac{\sqrt{\pi}}{(k\lambda_R)^2} e^{\Lambda^2} (1 - erf \Lambda) & \text{for } \omega_r << \omega_1,
\end{cases}
\]

(6.34)

where \( \Lambda = \omega_1/k_{vd} \) and \( \lambda_R = \lambda_D/\sqrt{\delta} \). Here, the dust fugacity \( f \) is defined by \( f = 4\pi n_0 d \lambda_D^2 R \), where \( R \) is the grain size(radius), \( \delta \equiv \omega_2/\omega_1 \) is the ratio of the charging frequencies \( \omega_1 \) and \( \omega_2 \) given in reference [Rao 2000], and is typically
of the order of unity for most dusty plasmas of practical interest. The last term on the right hand side of equation (6.34) is the new contribution arising due to the grain charge fluctuations.

It may be noted that the concept of fugacity is most useful in classifying dusty plasmas as tenuous (low fugacity, \( f \ll 1 \)), dilute (medium fugacity, \( f \sim 1 \)) or dense (high fugacity, \( f \gg 1 \)). On the other hand, \( \lambda_R \) is the effective screening length arising due to grain collective interactions, and plays a fundamental role in the dense regime like that of the Debye length (\( \lambda_D \)) in the tenuous regime. It may be noted that the two scale-lengths are related to the fugacity through the relation, \( f = \frac{\lambda^2_D}{\lambda^2_R} \). Following the analysis described above, the corresponding dispersion relation for the harmonic TDAW including the grain charge fluctuations is given by

\[
\omega = 0.924 k \nu_{td} \left[ 1 + \frac{\lambda^2_D}{\lambda^2_R} \left\{ (1 + k^2 \lambda^2_D) + \left( \frac{\lambda_D}{\lambda_R} \right)^2 \sqrt{\pi} e^{\lambda^2} \frac{\omega_1}{k \nu_{td}} \right\} \right]
\]

(6.35)

which is valid for \( \omega_r \gg \omega_1 \). On the other hand, in the opposite case when \( \omega_r \ll \omega_1 \), we obtain

\[
\omega = 0.924 k \nu_{td} \left[ 1 + \frac{\lambda^2_D}{\lambda^2_R} \left\{ (1 + k^2 \lambda^2_D) + \left( \frac{\lambda_D}{\lambda_R} \right)^2 \right\} \right]
\]

(6.36)

Clearly it follows that from equation (6.36) that the term appearing due to the grain charge fluctuations plays a role only when \( \lambda^2_D \approx \lambda^2_R \), that is, in the dilute regime (\( f \delta \sim 1 \)) or when \( \lambda^2_D \gg \lambda^2_R \), that is, in the dense regime (\( f \delta \gg 1 \)). In the case of tenuous dusty plasmas with low fugacity \( f \delta \ll 1 \), that is \( \lambda^2_D \ll \lambda^2_R \), equation (6.35) and (6.36) reduce to equation (6.32) and hence the charge fluctuations will have no effect. This is in accordance with the observation[Rao 1999; Rao 2000] that dusty plasmas support in the dense regime a new class of linear normal modes called “Dust-Coulomb Waves” (DCWs) when the grain charge fluctuations are self-consistently taken into account. These modes exist in a frequency regime much lower than the DAW regime, and are accompanied
by both number density as well as grain charge fluctuations. Physically, they are driven by an effective pressure [Rao 2000] called “Coulomb pressure”, and can be considered as the electrostatic analog of the hydromagnetic modes. The modes given by equation (6.36) for the high fugacity regime can thus be considered as the dispersion relation for harmonic “Thermal Dust-Coulomb Waves” (TDCWs). It should be noted that when the parameter $S$ is non-zero and finite, in which case an infinite number of higher harmonics is involved with the mode, the effect of charge fluctuations is still an open question.

6.3 Conclusion:

To summarize, we have considered the effect of trapped dust particles on the propagation of ultra low-frequency, electrostatic dust modes over the entire range of fugacity from the tenuous to the dense regime. In the tenuous regime when grain charge fluctuations can be neglected, we have shown the existence of stationary thermal dust acoustic wave (TDAW) which propagates without any damping in the dust thermal speed range. On the other hand, when the grain charge fluctuations are accounted for in the dense regime, we obtain the corresponding mode for the case of thermal dust Coulomb wave (TDCW) in the harmonic regime. Both the modes essentially arise due to a deficit of trapped dust particles corresponding to the existence of a notch in the dust distribution function.
Fig 119: the real part of the plasma dispersion function \(-\frac{1}{2}Z_r(x)\) as a function of \(x\).