open (resp. ijSO, i*j-sclopen) set U containing x and for each \( \alpha \in D \), \( \exists \beta \in D \) with \( \beta \geq \alpha \) such that \( A_\beta \cap U \neq \emptyset \).

**CHAPTER-THREE**

**ijSO Separation Axioms**

In this chapter we define weak type as well as strong type separation axioms in bitopological spaces, viz: \( ijSPT_k \) separation axioms, where \( k=0, 1, 2, 3, 4, 5; \) a bitopological space satisfying \( ijSPT_k \) separation axioms is said to be a \( ijSPT_k \) space. In this chapter we study some important properties of \( ijSPT_k \) bitopological spaces. With the notions of \( ijS^* \)-upper semi continuous and \( ijS^* \)-lower semi continuous real valued functions on a bitopological space we define some more strong separation axioms in a bitopological spaces. We also see that \( ijSPT_{k+1} \) space always implies \( ijSPT_k \) space, but converse may not be true in general. Some nice counterexamples are to be constructed in this chapter.

Lastly we define some weak separation axioms in case of fuzzy bitopological spaces and the relationship between crisp setting and fuzzy setting are to be study.

**3.1 Separation Axioms in Bitopological Spaces**

In this chapter I will always means the unit closed interval \([0, 1]\) with usual topology.

**Definition3.1.1.** Let \((X, \tau_1, \tau_2)\) be a bitopological space then the bitopological space \((X, \tau_1, \tau_2)\) is said to be

(i) \( ijSPT_0 \) space or satisfy the \( ijSPT_0 \)-separation axiom if for \( x, y \in X, x \neq y, \exists \) ijSO set \( U \) such that either \( x \in U \) and \( y \notin U \) or \( x \notin U \) and \( y \in U \).

(ii) \( ijSPT_1 \) space or satisfy the \( ijSPT_1 \)-separation axiom if for \( x, y \in X, x \neq y, \exists \) ijSO sets \( U \) and \( V \) such that \( x \in U \), \( x \notin V \) and \( y \in V \) and \( y \notin U \).
(iii) \(ij\text{SPT}_2\) or \(ij\text{SP}\) Hausdorff space or satisfy the \(ij\text{SPT}_2\)-separation axiom if for \(x, y \in X, x \neq y, \exists \text{ijSO sets } U \text{ and } V \text{ such that } x \in U, y \in V \text{ and } U \cap V = \emptyset\).

(iv) \(ij\text{SPT}_{2.5}\) space or satisfy the \(ij\text{SPT}_2\)-separation axiom if for any two distinct points \(x \text{ and } y\) there exist two \(ij\text{SO sets } U \text{ and } V \text{ containing } x \text{ and } y \text{ respectively such that } \text{ijsc}(U) \cap \text{ijsc}(V) = \emptyset\).

(v) \(ij\text{SP}\) regular space if for any \(x \in X \text{ and any } ij\text{SC set } F \text{ such that } x \in F, \exists \text{ijSO sets } U \text{ and } V \text{ such that } x \in U, F \subseteq V \text{ and } U \cap V = \emptyset\).

(vi) \(ij\text{SP}\) normal space if for any two disjoint \(ij\text{SC sets } F \text{ and } K, \exists \text{ijSO sets } U \text{ and } V \text{ such that } F \subseteq U, K \subseteq V \text{ and } U \cap V = \emptyset\).

Remark 3.1.2. We see that if in the bitopological space \((X, \tau_1, \tau_2)\) if \(\tau_j\) is the discrete topology then every \(ij\text{SO set}\) becomes a \(\tau_i\)-open set, in that case i.e if \(\tau_j\) is the discrete topology on \(X\) then if \((X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_k\)-space \((k=0, 1, 2, 2.5)\) or \(ij\text{SP}\) regular or \(ij\text{SP}\) normal then the topological space \((X, \tau_i)\) becomes a \(T_k\)-space, \((k=0, 1, 2, 2.5)\) or regular or normal accordingly.

Theorem 3.1.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space then if \((X, \tau)\) is a \(T_k\) space then the bitopological space \((X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_k\)-space for \(k=0, 1, 2, 2.5\). The converse is not true.

Proof: The proof follows from directly since every \(\tau_i\)-open set is also an \(ij\text{SO set}\) and every \(\tau_i\)-closed set is also an \(ij\text{SC set}\).

For the converse part let us see the following examples:

Example 3.1.4. (i) Let us consider the bitopological space \((X, \tau_1, \tau_2)\), where \(X=\{x, y, z\}\), \(\tau_1=\{\emptyset, \{x\}, \{y, z\}, X\}\), \(\tau_2=\{\emptyset, X\}\), all 12 \(SO\) sets are \(\emptyset, \{x\}, \{x, y\}, \{x, z\}, \{y, z\}\) and \(X\), so \((X, \tau_1, \tau_2)\) is \(12\text{SPT}_0\) space and also a \(12\text{SPT}_1\) space but the topological space \((X, \tau_1)\) is not a \(T_0\)-space nor a \(T_1\)-space.

(ii) Let us consider the bitopological space \((X, \tau_1, \tau_2)\), \(X=\{x, y, z\}\), \(\tau_1=\{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}\), \(\tau_2=\{\emptyset, X\}\), all 12 \(SO\) sets are \(\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}\) and \(X\), so \((X, \tau_1, \tau_2)\) is \(12\text{SPT}_2\) space but the topological space \((X, \tau_1)\) is not a \(T_2\)-space.
Theorem 3.1.5. Let \((X, \tau_1, \tau_2)\) be a bitopological space then the following statements are equivalent:

(i) \((X, \tau_1, \tau_2)\) is \(ij\)-SPT\(_0\) space.

(ii) For every pair of distinct points \(x\) and \(y\) in \(X\) either \(x\not\in \text{ijcl}(y)\) or \(y\not\in \text{ijcl}(x)\)

(iii) \(ij\)-semi closure of distinct points are distinct.

Proof: (i)⇒(ii) : Let \(x, y\in X, x\neq y\), so \(\exists\) ijSO set \(U\) such that either \(x\in U\) and \(y\not\in U\) or \(y\in U\) and \(x\not\in U\), so accordingly either \(x\not\in X-U, \{y\}\in X-U\) and so \(\text{ijcl}(y)\subseteq X-U\) and so \(x\not\in \text{ijcl}(y)\) or \(y\not\in X-U, \{x\}\in X-U\) and so \(\text{ijcl}(x)\subseteq X-U\) and so \(y\not\in \text{ijcl}(x)\).

(ii)⇒(iii) : Let \(x, y\in X, x\neq y\) so either \(x\not\in \text{ijcl}(y)\) or \(y\not\in \text{ijcl}(x)\) hence (iii) follows.

(iii)⇒(i) : Let if possible \((X, \tau_1, \tau_2)\) is not a \(ij\)-SPT\(_0\) space, so \(\exists\) \(x, y\in X, x\neq y\) such that any ijSO set \(U\) contains \(x\) as well as \(y\), hence no \(ij\)SC set contains exactly one of \(x\) and \(y\) contradicting to (iii), hence (i) follows.

Theorem 3.1.6. A bitopological space \((X, \tau_1, \tau_2)\) is a \(ij\)-SPT\(_1\) space iff every singleton set in \(X\) is a \(ij\)SC set.

Proof: Let the space be \(ij\)-SPT\(_1\) space and \(x\in X\) then for every \(y\in X-\{x\}\) \(\exists\) a ijSO set \(U_y\) containing \(y\) such that \(x\not\in U_y\), i.e \(\{y\}\subseteq U_y\subseteq X-\{x\}\), so \(X-\{x\}=\cup\{\{y\}: y\in X-\{x\}\}\subseteq X-\{x\}\), hence \(\cup\{\{y\}: y\in X-\{x\}\}=X-\{x\}\), so \(X-\{x\}\) is a ijSO set and hence \(\{x\}\) is a \(ij\)SC set.

Conversely every singleton set in \(X\) is a \(ij\)SC set. Let \(x, y\in X, x\neq y\) \(\therefore,\) \(\{x\}\) and \(\{y\}\) are \(ij\)SC sets and \(U= X-\{y\}\) and \(V= X-\{x\}\) are two ijSO sets containing each of them but not the other, so \((X, \tau_1, \tau_2)\) is a \(ij\)-SPT\(_1\) space.

Theorem 3.1.7. Let \((X, \tau_1, \tau_2)\) be a \(ij\)-SPT\(_0\) space in which \(x\in \text{ijcl}(y)\) \(\Rightarrow\) \(y\in \text{ijcl}(x)\), then \((X, \tau_1, \tau_2)\) is a \(ij\)-SPT\(_1\) space.
Proof: Let $x, y \in X$, $x \neq y$, so either either $x \notin \text{ijscl}(y)$ or $y \notin \text{ijscl}(x)$. Let $x \notin \text{ijscl}(y)$, so if $y \in \text{ijscl}(x)$ then $x \in \text{ijscl}(y)$ which is impossible so $y \notin \text{ijscl}(x)$ and $U = X - \text{ijscl}(y)$ and $V = X - \text{ijscl}(x)$ are iSO sets such that $x \in U$, $x \notin V$ and $y \in V$ and $y \notin U$.

\[\therefore, (X, \tau_1, \tau_2) \text{ is a iSPT}_1 \text{ space.}\]

**Theorem 3.1.8.** Let $(X, \tau_1, \tau_2)$ be a bitopological space then the following statements are equivalent:

(i) $(X, \tau_1, \tau_2)$ is iSPT$_2$ space.

(ii) Let $x \in X$ then for each $y \neq x$ \exists a iSO set $U$ containing $x$ such that $y \notin \text{ijscl}(U)$.

(iii) For each $x \in X$, $\cap \{\text{ijscl}(U)\} : U$ is iSO sets containing $x\} = \{x\}$.

Proof: (i)$\Rightarrow$(ii) Since $x \neq y$ \exists iSO sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. \therefore, $U \subseteq V^c = X - V$, so $\text{ijscl}(U) \subseteq V^c$ and hence $y \notin \text{ijscl}(U)$.

(ii)$\Rightarrow$(iii) For each $y \neq x$ \exists a iSO sets $U_y$ containing $x$ such that $y \notin \text{ijscl}(U_y)$.

\[\therefore, \{x\} \subseteq \cap \{\text{ijscl}(U) : U \text{ is iSO sets containing } x\} = \{x\}.\]

The right inclusion is due to the fact that for any $y \neq x$, \exists a iSO sets $U_y$ containing $x$ such that $y \notin \text{ijscl}(U_y)$.

Hence $\cap \{\text{ijscl}(U) : U$ is iSO sets containing $x\} = \{x\}$.

(iii)$\Rightarrow$(i) Let $x, y \in X$, $x \neq y$, since $\cap \{\text{ijscl}(U) : U$ is iSO set containing $x\} = \{x\}$, \exists a iSO sets $U_y$ containing $x$ such that $y \notin \text{ijscl}(U_y)$.

\[\therefore, (\text{ijscl}(U_y))^c \text{ is a iSO set containing } y \text{ and } U_y \cap (\text{ijscl}(U_y))^c = \emptyset.\]

\[\therefore, X \text{ is a iSPT}_2 \text{ space.}\]

**Theorem 3.1.9.** A bitopological space $(X, \tau_1, \tau_2)$ is iSPT$_0$ space iff for any two fuzzy points $x_r$ and $y_s$ in $X$ with distinct supports, there exists a fuzzy open set $\alpha$ in $(X, \omega_\eta(X))$ such that $x_r \alpha \leq (y_s)^c$ or $y_s \alpha \leq (x_r)^c$.

Proof: Proof is similar to that of Theorem 3.1.11.
**Theorem 3.1.10.** A bitopological space \((X, \tau_1, \tau_2)\) is \(ij\text{-}SPT_1\) space iff for any two fuzzy points \(x_r\) and \(y_s\) in \(X\) with distinct supports, there exist two fuzzy open sets \(\alpha\) and \(\beta\) in \((X, \omega_b(X))\) such that \(x_r \alpha \leq (y_s)^c\) and \(y_s \beta \leq (x_r)^c\).

**Proof:** Proof is similar to that of Theorem 3.1.11.

**Theorem 3.1.11.** A bitopological space \((X, \tau_1, \tau_2)\) is \(ij\text{-}FSPT_2\) space iff for any two fuzzy points \(x_r\) and \(y_s\) in \(X\) with distinct supports, there exist two fuzzy open sets \(\alpha\) and \(\beta\) in \((X, \omega_b(X))\) such that \(x_r \alpha \leq \beta\) and \(y_s \beta \leq \alpha\). \(\alpha \wedge \beta = 0\).

**Proof:** Let \((X, \tau_1, \tau_2)\) is \(ij\text{-}SPT_2\) space. Let \(x_r\) and \(y_s\) be two fuzzy points in \(X\) such that \(x \neq y\), so \(\exists\) a \(ij\text{-}SO\) set \(U\) and \(V\) in \((X, \tau_1, \tau_2)\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\), now \(\chi_u\), \(\chi_v \in \omega_b(X)\) and \(x_r \chi_u\), \(y_s \chi_v\) and \(\chi_u \wedge \chi_v = 0\).

Conversely suppose the given condition holds. Let \(x\) and \(y\) are two distinct points on \(X\), so \(x_1\) and \(y_1\) be two fuzzy points in \(X\) with \(x \neq y\), so \(\exists f, g \in \omega_b(X)\) such that \(x_1 f, y_1 g f\) and \(f \wedge g = 0\), so \(1+f(x) > 1\) and \(1+g(y) > 1\), i.e. \(f(x) > 0\) and \(g(y) > 0\). Now \(U = f^{-1}(0, 1]\) and \(V = g^{-1}(0, 1]\) are \(ij\text{-}SO\) sets such that \(x \in U\), \(y \in V\) and also \(U \cap V = \emptyset\) (else if \(x \in U \cap V\), \(f(x) > 0\) and \(g(x) > 0\), so \((f \wedge g)(x) > 0\), i.e. \(f \wedge g \neq 0\) which is imposible), i.e. the bitopological space \((X, \tau_1, \tau_2)\) is \(ij\text{-}SPT_2\) space.

**Theorem 3.1.12.** Let \((X, \tau_1, \tau_2)\) be a bitopological space then the following statements are equivalent:

(i) \((X, \tau_1, \tau_2)\) is \(ij\text{-}SP\) regular space.

(ii) For each \(x \in X\) and \(ij\text{-}SO\) set \(U\) containing \(x\) \(\exists\) a \(ij\text{-}SO\) set \(V\) containing \(x\) such that \(V \subseteq \text{ijscl}(V) \subseteq U\).

(iii) For each \(x \in X\) and a \(ij\text{-}SC\) set \(F\) not containing \(x\) \(\exists\) a \(ij\text{-}SO\) set \(V\) containing \(x\) such that \(\text{ijscl}(V) \cap F = \emptyset\).

**Proof:** 

(i)\(\Rightarrow\)(ii) Let \(x \in X\) and \(U\) be a \(ij\text{-}SO\) set containing \(x\). \(\ldots, U^c = F\) is a \(ij\text{-}SC\) set not containing \(x\), so \(\exists\) disjoint \(ij\text{-}SO\) sets \(V\) and \(W\) such that \(x \in V\), \(F \subseteq W\) and \(V \cap W = \emptyset\). Now \(V \subseteq W^c \subseteq F^c = U\) so, \(V \subseteq \text{ijscl}(V) \subseteq W^c \subseteq F^c = U\).
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(ii)⇒(iii) Let $x \in X$ and $F$ be a ijSC set not containing $x$, $F^c$ is a ijSO set containing $x$, so $\exists$ a ijSO set $V$ containing $x$ such that $V \subseteq ijsc(V) \subseteq F^c$, $ijsc(V) \cap F = \emptyset$.

(iii)⇒(i) Let $x \in X$ and $F$ be a ijSC set not containing $x$, so $\exists$ a ijSO set $V$ containing $x$ such that $ijsc(V) \cap F = \emptyset$, $W = (ijsc(V))^c$ is a ijSO set containing $F$ and $V \cap W = \emptyset$, i.e. $(X, \tau_1, \tau_2)$ is ijSP regular space.

**Theorem 3.1.13.** Let $(X, \tau_1, \tau_2)$ be a bitopological space then the following statements are equivalent:

(i) $(X, \tau_1, \tau_2)$ is ijSP normal space.

(ii) For each ijSC set $E$ and ijSO set $U$ containing $E$, $\exists$ a ijSO set $V$ containing $E$ such that $V \subseteq ijsc(V) \subseteq U$.

(iii) For each pair of disjoint ijSC sets $E$ and $F$, $\exists$ a ijSO set $U$ containing $F$ such that $ijsc(U) \cap F = \emptyset$.

(iv) For each pair of disjoint ijSC sets $E$ and $F$, $\exists$ a ijSO sets $U$ and $V$ containing $E$ and $F$ respectively such that $ijsc(U) \cap ijsc(V) = \emptyset$.

**Proof:**

(i)⇒(ii) Let $E$ be a ijSC and $U$ be a ijSO set containing $E$, $E$ and $F^c = U^c$ are two disjoint ijSC sets, so $\exists$ two ijSO sets $V$ and $W$ containing $E$ and $F$ respectively such that $V \cap W = \emptyset$, so $V \subseteq W^c \subseteq F^c = U$, $V \subseteq ijsc(V) \subseteq W^c \subseteq U$.

(ii)⇒(iii) Let $E$ and $F$ are two disjoint ijSC sets, so $F^c$ is a ijSO set containing $E$, so $\exists$ a ijSO set $U$ containing $E$ such that $U \subseteq ijsc(U) \subseteq F^c$. $\therefore, ijsc(U) \cap F = \emptyset$.

(iii)⇒(iv) Let $E$ and $F$ are two disjoint ijSC sets, so $\exists$ a ijSO set $U$ containing $E$ such that $ijsc(U) \cap F = \emptyset$, again $F$ and $ijsc(U)$ are two disjoint ijSC sets, so $\exists$ another ijSO set $V$ containing $F$ and $ijsc(U) \cap ijsc(V) = \emptyset$.

(iv)⇒(i) Let $E$ and $F$ are two disjoint ijSC sets then, $\exists$ ijSO sets $U$ and $V$ containing $E$ and $F$ respectively such that $ijsc(U) \cap ijsc(V) = \emptyset$ and so $U \cap V = \emptyset$, i.e. $(X, \tau_1, \tau_2)$ is ijSP normal space.
Theorem 3.1.14. A $ijSP_{T_1}$ bitopological space is $ijSP_{T_0}$ space. A $ijSP_{T_2}$ bitopological space is $ijSP_{T_1}$ space.

Proof: Directly follows from the definitions.

Remark 3.1.15. The converse of above two statements is not true in general. We consider the following examples:

(i) We consider the bitopological space $(X, \tau_1, \tau_2)$, where $X=\{x, y\}$, $\tau_1=\{\emptyset, \{x\}, X\}$, $\tau_2=\{\emptyset, \{y\}, X\}$, all 12SO sets are $\emptyset, \{x\}$ and $X$, so $(X, \tau_1, \tau_2)$ is 12SPT$_0$ space but not a 12SPT$_1$ space.

(ii) Again we consider the bitopological space $(X, \tau_1, \tau_2)$, where $X=\{x, y, z\}$, $\tau_1=\{\emptyset, \{x\}, \{y, z\}, X\}$, $\tau_2=\{\emptyset, \{x\}, X\}$, all 12SO sets are $\emptyset, \{x\}, \{x, y\}, \{x, z\}, \{y, z\}$ and $X$, so $(X, \tau_1, \tau_2)$ is 12SPT$_1$ space but not 12SPT$_2$ space.

Theorem 3.1.16. Every $ijSPT_{2.5}$ space is $ijSPT_2$ space, but converse is not true.

Proof: Since $ijScl(U) \cap ijScl(V)=\emptyset$ implies $U \cap V=\emptyset$, so from the above definition it follows that every $ijSPT_{2.5}$ space is $ijSPT_2$ space.

Example 3.1.17. We consider the bitopological space $(X, \tau_1, \tau_2)$ where $X=\{(x, y) : y \geq 0, x, y \in \mathbb{Q}\}$ and $\tau_1=\text{Irrational slope topology}$ on $X$ (See page-93 of Steen and Seebach$^4$), $\tau_2=\text{discrete topology}$ on $X$. Then the bitopological space $(X, \tau_1, \tau_2)$ is 12SPT$_2$ space, but not 12SPT$_{2.5}$ space.

Theorem 3.1.18. Let $(X, \tau_1, \tau_2)$ be a $ijSP$ normal bitopological space, $A$ and $B$ are two non-empty disjoint $ijSC$ sets, then there exists a real valued function

$$f : (X, \tau_1, \tau_2) \to \mathbb{R}, \mathbb{R} \text{ with the usual topology, such that}$$

(i) $0 \leq f(x) \leq 1, \forall x \in X,$

(ii) $f(A)=\{0\}$ and $f(B)=\{1\}$

(iii) $f$ is $ijS^*$ upper semi continuous and $ijS^*$ lower semi continuous function.

Proof: We know that the set $D = \left\{ \frac{m}{2^n} : n \in \mathbb{N}, 0 \leq m \leq 2^n \right\}$ is dense in $[0, 1]$ with usual topology on $[0, 1]$. 

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Let $U_0 = A$ and $U_1 = X - B$, so $U_0 \subseteq U_1$, and by ijSP normality there exists a ijSO set $U_{1/2}$ such that $U_0 \subseteq U_{1/2} \subseteq \text{ijsc}l(U_{1/2}) \subseteq U_1$. Again by ijSP normality we have two ijSO sets $U_{1/4}$ and $U_{3/4}$ such that $U_0 \subseteq U_{1/4} \subseteq \text{ijsc}l(U_{1/4}) \subseteq U_{1/2} \subseteq \text{ijsc}l(U_{1/2}) \subseteq U_{3/4} \subseteq \text{ijsc}l(U_{3/4}) \subseteq U_1$.

Continuing in this way we have for $r, s \in D$ and $r < s$, we have ijSO sets $U_r$ and $U_s$ such that,

$$U_0 \subseteq U_r \subseteq \text{ijsc}l(U_r) \subseteq U_s \subseteq \text{ijsc}l(U_s) \subseteq U_1.$$ 

Now we define a function $f : X \to \mathbb{R}$ as,

$$f(x) = \inf\{t \in D : x \in U_t\}$$

$$= 1, \text{ else.}$$

So, $0 \leq f(x) \leq 1$, $\forall x \in X$, hence (i) follows.

If $x \in A$, i.e $x \in U_0$, then $x \in U_r$, $\forall t \in D$, so $f(x) = 0$, i.e. $f(A) = \{0\}$.

Again, if $x \in B$, then $x \notin U_1$, i.e. $x \notin U_r$, $\forall t \in D$, so $f(x) = 1$, i.e. $f(B) = \{1\}$. Hence (ii) follows.

Let $r \in \mathbb{R}$, with out loss of generality we suppose $0 < r < 1$.

So $f^{-1}(r, \infty) = f^{-1}(r, 1]$ and $f^{-1}(-\infty, r) = f^{-1}[0, r)$.

Let $x \in f^{-1}(r, 1]$, so $r < f(x) \leq 1$, since $D$ is dense in $[0, 1]$, so $\exists s, t \in D$, such that $r < s < t < f(x)$, i.e. $x \notin U_s$, since $\text{ijsc}l(U_s) \subseteq U_s$, so $x \notin \text{ijsc}l(U_s)$, i.e.

$$x \in \bigcup_{s \in D \atop r < s} (X - \text{ijsc}l(U_s)).$$

Hence $f^{-1}(r, 1] \subseteq \bigcup_{s \in D \atop r < s} (X - \text{ijsc}l(U_s))$.

Again let $x \in \bigcup_{s \in D \atop r < s} (X - \text{ijsc}l(U_s))$, so $x \notin \text{ijsc}l(U_s)$ for some $s \in D$ and $r < s$,

i.e. $x \notin U_s$, so $f(x) > r$, i.e. $x \in f^{-1}(r, 1]$. Hence $\bigcup_{s \in D \atop r < s} (X - \text{ijsc}l(U_s)) \subseteq f^{-1}(r, 1]$.

So, $f^{-1}(r, \infty) = f^{-1}(r, 1] = \bigcup_{s \in D \atop r < s} (X - \text{ijsc}l(U_s))$, which is a ijSO set, so $f$ is ijS* lower semi continuous function.

Now, let $x \in f^{-1}[0, r)$, so $0 \leq f(x) < r$. Since $D$ is dense in $[0, 1]$, $\exists s \in D$, such that $f(x) < s < r$, i.e. $x \in U_s$, so $x \in \bigcup_{s \in D \atop s < r} U_s$, i.e. $f^{-1}[0, r) \subseteq \bigcup_{s \in D \atop s < r} U_s$.  

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Again let, $x \in \bigcup_{s \in D, s < r} U_s$, so $x \in U_s$ for some $s \in D$ and $s < r$, so $0 \leq f(x) \leq s < r$, i.e. $x \in f^{-1}[0, r)$, hence $\bigcup_{s \in D, s < r} U_s \subseteq f^{-1}[0, r)$.

So, $f^{-1}(-\infty, r) = f^{-1}[0, r) = \bigcup_{s \in D, s < r} U_s$ which is a $ijSO$ set, i.e. $f$ is a $ijS^*$ upper semi continuous function.

Hence the theorem follows.

**Corollary 3.1.19.** A bitopological space $(X, \tau_1 \sqcup \tau_2)$, where $(X, \tau_2)$ is extremely disconnected space is $12SP$ normal, then for any two non-empty disjoint $12SC$ sets $A$ and $B$ in $X$ there exists a $12S^*$ continuous function $f : (X, \tau_1 \sqcup \tau_2) \to I$, $I = [0, 1]$ with the usual topology, such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

**Proof:** The proof follows from the fact that in such bitopological spaces $f$ defined in the above theorem for $i = 1, j = 2$ becomes $12S^*$ continuous function.

The Converse of the Theorem 3.1.18 is also true, which can be seen from the following theorem:

**Theorem 3.1.20.** Let $(X, \tau_1, \tau_2)$ be a bitopological space such that for any two disjoint non-empty $ijSC$ set $A$ and $B$, there is a $ijS^*$ lower and $ijS^*$ upper semi continuous function $f : (X, \tau_1, \tau_2) \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$, $\forall x \in X$ and $f(A) = \{0\}$ and $f(B) = \{1\}$, then $(X, \tau_1, \tau_2)$ $ijSP$ normal space.

**Proof:** If we put $U = f^{-1}(-\infty, .1)$ and $V = f^{-1}(1.9, \infty)$, then $U$ and $V$ both are $ijSO$ sets and $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$, i.e. $(X, \tau_1, \tau_2)$ is a $ijSP$ normal space.

But in the similar fashion we do not get Tietze extension theorem in any general bitopological space with the notion of $ijSO$ sets, which can be seen from the following example:

**Example 3.1.21.** We consider the bitopological space $(X, \tau_1, \tau_2)$ where $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\tau_2 = \{\emptyset, \{a, b\}, \{c, d\}, X\}$.

All $12SO$ sets are: $\emptyset$, $\{a\}$, $\{a, b\}$, $\{b\}$, $\{b, c\}$, $\{a, b, c\}$, $\{a, b, d\}$ and $X$.
All 12SC sets are: X, \{b, c, d\}, \{c, d\}, \{a, c, d\}, \{a, d\}, \{d\}, \{a\}, \{c\} and \emptyset.

Let us take the 12SC set A=\{c, d\}, now the induced topologies on A are 
\[\tau_1^A = \{\emptyset, \{c\}, \{d\}, A\}, \quad \tau_2^A = \{\emptyset, A\}.\]

All 12SO sets in (A, \tau_1^A, \tau_2^A) are: \emptyset, \{c\}, \{d\} and A.

We consider the mapping \(f : (F, \tau_1^A, \tau_2^A) \to \mathbb{R}\), defined as \(f(c)=1\) and \(f(d)=2\). This mapping is 12S* upper semi continuous and 12S* lower semi continuous.

If possible let \(f\) has an extension on (X, \tau_1, \tau_2) as \(F : (X, \tau_1, \tau_2) \to \mathbb{R}\) which is 12S* upper semi continuous and 12S* lower semi continuous. Accordingly \(F(c)=1, F(d)=2\) which is impossible. So \(f\) does not have its Tietze type extension.

**Definition 3.1.22.** (i) A ijSPT$_1$, ijSP regular bitopological space is said to be a ijSPT$_3$ space. (ii) A ijSPT$_1$, ijSP normal bitopological space is said to be a ijSPT$_4$ space.

**Theorem 3.1.23.** (i) A ijSPT$_3$ space is ijSPT$_{2.5}$ space, but converse is not true.

(ii) A ijSPT$_4$ space is ijSPT$_3$ space, but converse is not true.

**Proof of 3.1.23.(i):** Let (X, \tau_1, \tau_2) be a ijSPT$_3$ space, x and y be two distinct points in X. Since (X, \tau_1, \tau_2) is ijSPT$_1$ space, so \{y\} is a ijSC set, and so U=X – \{y\} is a ijSO set and x \in U, so there exists a ijSO set V such that x \in V \subseteq ijscI(V) \subseteq U. Again there exists ijSO set L such that x \in L \subseteq ijscI(L) \subseteq V and L is ijSO set containing x. Now \{y\}=X – U \subseteq X – ijscI(V)=W \subseteq X – V \subseteq X – ijscI(L) and W is a ijSO set containing y. So, ijscI(W) \subseteq X – V \subseteq X – ijscI(L) and hence ijscI(L) \cap ijscI(W)=\emptyset, i.e. (X, \tau_1, \tau_2) is a ijSPT$_{2.5}$ space.

**Example 3.1.24(i)** The converse is not true, for that we consider the bitopological space (X, \tau_1, \tau_2) where X=P \cup L, where P=\{(x, y) \in \mathbb{R}^2 : y>0\} and L=\{(x, y) \in \mathbb{R}^2 : y = 0\}, \tau_1 is the Niemytzki’s Tangent Disc Topology on X (See page -100 of Steen and Seebach$^{17}$), \tau_2=usual topology on X induced from \mathbb{R}^2. Then (X, \tau_1, \tau_2) is a 12SPT$_{2.5}$ space, but not a 12SPT$_3$ space.

**Proof of 3.1.23.(ii):** The proof is similar that of 3.1.23(i) with the fact that in such spaces every singleton set is a ijSC set.
Example 3.1.24(ii) Let T be the product space \([0, \Omega] \times [0, \omega]\) with interval topology on \([0, \Omega]\) and \([0, \omega]\), where \(\Omega\) is the first uncountable ordinal and \(\omega\) is the first infinite ordinal. Let \(X= T_{\omega}=[0, \Omega] \times [0, \omega] - \{(\Omega, \omega)\}\) with the subspace topology \(\tau_1\) i.e. the deleted pancake topology (See page -106 of Steen and Seebach47) and \(\tau_2=\) discrete topology on X. Then \((X, \tau_1, \tau_2)\) is a 12SPT3 space, but not a 12SPT4 space.

3.2 Some Strong Separation Axioms using Real Valued Function

In this section we will discuss some separation axioms with the notion of a type of real valued function from a bitopological space to \(I=[0, 1]\) with the usual topology and the relation between these types of separation axioms and the separation axioms defined in last section.

Definition 3.2.1. A bitopological space \((X, \tau_1, \tau_2)\) is said to be completely ijSP Hausdorff if for any two distinct points \(x\) and \(y\) in \(X\), there exists a function
\[
f : (X, \tau_1, \tau_2) \rightarrow I
\]
which is simultaneously ijS* upper semi continuous and ijS* lower semi continuous such that \(f(x)=0\) and \(f(y)=1\).

Definition 3.2.2. A bitopological space \((X, \tau_1, \tau_2)\) is said to be completely ijSP regular if for any ijSC set \(F\) and for any point \(x\) outside \(F\), there exists a function
\[
f : (X, \tau_1, \tau_2) \rightarrow I
\]
which is simultaneously ijS* upper semi continuous and ijS* lower semi continuous such that \(f(x)=0\) and \(f(F)=\{1\}\).

Definition 3.2.3. A ijSPT1 bitopological space \((X, \tau_1, \tau_2)\) is said to be ijSPT3.5 if \((X, \tau_1, \tau_2)\) is completely ijSP regular.

Theorem 3.2.4. (i) A completely ijSP Hausdorff space is ijSP Hausdorff space, but converse is not true.

(ii) A ijSPT3.5 space is ijSPT3 space, but converse is not true.

(iii) A ijSPT4 space is ijSPT3.5 space, but converse is not true.
Proof of 3.2.4.(i): Let \((X, \tau_1, \tau_2)\) be completely ijSP Hausdorff space and \(x\) and \(y\) be two distinct points in \(X\), then there exists an function
\[
f : (X, \tau_1, \tau_2) \to \mathbb{R}
\]
which is simultaneously ijS* upper semi continuous and ijS* lower semi continuous such that \(f(x)=0\) and \(f(y)=1\).

Now, if we put \(U=f^{-1}[0, .1)\) and \(V=f^{-1}(.9, 1]\), then \(U\) and \(V\) both are ijSO sets and \(x \in U\), \(y \in V\), \(U \cap V=\emptyset\), i.e. \((X, \tau_1, \tau_2)\) is a ijSPT\(_2\) or ijSP Hausdorff space.

Example 3.2.5.(i) We consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=\{(x, y) : y \geq 0, x, y \in \mathbb{Q}\}\) and \(\tau_1=\)Irrational slop topology on \(X\) (See page -93 of Steen and Seebach\(^{47}\)), \(\tau_2=\)discrete topology on \(X\). Then the bitopological space \((X, \tau_1, \tau_2)\) is 12SP Hausdorff but not completely 12SP Hausdorff space.

Proof of 3.2.4.(ii): Let \((X, \tau_1, \tau_2)\) be ijSPT\(_{3.5}\) space and \(F\) be a ijSC set and \(x \notin F\) in \(X\), so there exists an function
\[
f : (X, \tau_1, \tau_2) \to I
\]
which is ijS* upper semi continuous and ijS* lower semi continuous such that \(f(x)=0\) and \(f(F)=\{1\}\).

Now, if we put \(U=f^{-1}[0, .1)\) and \(V=f^{-1}(.9, 1]\), then \(U\) and \(V\) both are ijSO sets and \(x \in U\), \(F \subseteq V\), \(U \cap V=\emptyset\), i.e. \((X, \tau_1, \tau_2)\) is a ijSPT\(_3\) space.

Example 3.2.5.(ii) We consider the Tychonoff Corkscrew topological space \((X, \tau_1)\) (See page -109 of Steen and Seebach\(^{47}\)) and \(\tau_2\) be the discrete topology on \(X\). Then the bitopological space \((X, \tau_1, \tau_2)\) is 12SPT\(_3\) space but not 12SPT\(_{3.5}\) space.

Proof of 3.2.4.(iii): It follows from the fact that every singleton set is a ijSC set and then using the Theorem 3.1.18.

Example 3.2.5.(iii) The converse is not true, for that we consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=P \cup L\), where \(P=\{(x, y) \in \mathbb{R}^2 : y>0\}\) and \(L=\{(x, y) \in \mathbb{R}^2 : y = 0\}\), \(\tau_1\) is the Niemytzki’s Tangent Disc Topology on \(X\) (See page -100 of Steen
and Seebach\textsuperscript{47}, $\tau_2=$ discrete topology on $X$ induced from $\mathbb{R}^2$. Then the bitopological space $(X, \tau_1, \tau_2)$ is $12\text{SPT}_{3.5}$ space but not $12\text{SPT}_4$ space.

**Definition 3.2.6.** A bitopological space $(X, \tau_1, \tau_2)$ is said to be perfectly ijSP normal if for any two non-empty disjoint ijSC set $F$ and $K$, there exists a functions

$$f: (X, \tau_1, \tau_2) \rightarrow I$$

which is simultaneously ijS* upper semi continuous and ijS* lower semi continuous such that $f^{-1}(0)=F$ and $f^{-1}(1)=K$.

**Definition 3.2.7.** A ijSPT\textsubscript{1} bitopological space is called a ijSPT\textsubscript{5} space if it is perfectly ijSP normal.

**Theorem 3.2.8.** A perfectly ijSP normal is ijSP normal space.

**Proof:** Let $F$ and $K$ be two disjoint ijSC sets in $(X, \tau_1, \tau_2)$, so there exists an function

$$f: (X, \tau_1, \tau_2) \rightarrow I$$

which is simultaneously ijS* upper semi continuous and ijS* lower semi continuous such that $f^{-1}(0)=F$ and $f^{-1}(1)=K$.

Now, if we put $U=f^{-1}[0, .1)$ and $V=f^{-1}(.9, 1]$, then $U$ and $V$ both are ijSO sets and $F \subseteq U$, $K \subseteq V$, $U \cap V=\emptyset$, i.e. $(X, \tau_1, \tau_2)$ is a ijSP normal space.

For the converse we consider following example

**Example 3.2.9.** We consider the bitopological space $(X, \tau_1, \tau_2)$ where $X=\{a, b, c\}$, $\tau_1=\{\emptyset, \{a\}, \{b, c\}, X\}$ and $\tau_2=\{\emptyset, \{c\}, X\}$.

All 12SO sets are: $\emptyset, \{a\}, \{a, b\}, \{b, c\}$ and $X$

All 12SC sets are: $\emptyset, \{b, c\}, \{c\}, \{a\}$ and $X$

It is easy to see that $(X, \tau_1, \tau_2)$ is 12SP normal space. If possible let $(X, \tau_1, \tau_2)$ is completely 12SP normal space, so there exists a function

$$f: (X, \tau_1, \tau_2) \rightarrow I$$

which is simultaneously 12S* upper semi continuous and 12S* lower semi continuous such that and $f^{-1}\{0\} = \{a\}$ and $f^{-1}\{1\} = \{c\}$. In this case $0 < f(b)$
<1, which is impossible since \( f^{-1}(r, 1] = \{c\} \) for \( f(b) < r < 1 \), which is not a ijSO set. So, the bitopological space is not a perfectly12SP normal.

**Corrolary3.2.10.** A ijSPT\(_3\) space is ijSPT\(_4\) space. But converse is not true.

**Proof:** The proof follows from the definitions and above theorem.

For the converse we consider the following example:

**Example3.2.11.** We consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=\{a, b, c\}\), \(\tau_1=\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}\) and \(\tau_2=\{\emptyset, X\}\).

All 12SO sets are: \(\emptyset, \{a\}, \{a, b\}, \{b\}, \{b, c\}, \{a, c\}, \text{and} \ X\)

All 12SC sets are: \(\emptyset, \{b, c\}, \{c\}, \{a, c\}, \{a\}, \{b\}\) and \(X\)

It is easy to see that \((X, \tau_1, \tau_2)\) is 12SPT\(_1\) space and also 12SP normal space, so a 12SPT\(_4\) space. If possible let \((X, \tau_1, \tau_2)\) is 12SPT\(_5\) space so perfectly 12SP normal space, so there exists a function.

\[ f : (X, \tau_1, \tau_2) \rightarrow I \]

which is 12S* upper semi continuous and 12S* lower semi continuous such that and \(f^{-1}\{0\} = \{a\}\) and \(f^{-1}\{1\} = \{c\}\). In this case \(0 < f(b) < 1\), which is impossible since \(f^{-1}(r, 1] = \{c\}\) for \(f(b) < r < 1\) which is not a 12SO set. So, the bitopological space is not a 12SPT\(_5\) space.

From whole discussion we have that:

\[ \text{ijSPT}_5 \Rightarrow \text{ijSPT}_4 \Rightarrow \text{ijSPT}_{3.5} \Rightarrow \text{ijSPT}_3 \Rightarrow \text{ijSPT}_{2.5} \Rightarrow \text{ijSPT}_2 \Rightarrow \text{ijSPT}_1 \Rightarrow \text{ijSPT}_0 \]

Here the reverse implications may not hold.
3.3 Some Relationships with Fuzzy Bitopological Spaces

In this section we will define some separation axioms in fuzzy bitopological spaces and also find some relation between the separation axioms of a bitopological space with the separation axioms in fuzzy bitopological spaces.

**Definition 3.3.1.** A fuzzy bitopological space \((X, \delta_1, \delta_2)\) is said to be:

(a) \(ijFSPT_0\) space iff for any two fuzzy points \(x, y \in X, x \neq y\), \(\exists \ ijFSO\) set \(\alpha\) such that \(x, \alpha \subseteq (y)\) \(\alpha\) or \(y, \alpha \subseteq (x)\) \(\alpha\)

(b) \(ijFSPT_1\) space iff for any two fuzzy points \(x, y \in X, x \neq y\), \(\exists \ ijFSO\) sets \(\alpha\) and \(\beta\) such that \(x, \alpha \subseteq (y)\) \(\alpha\) and \(y, \beta \subseteq (x)\) \(\beta\)

(c) \(ijFSPT_2\) space iff for any two fuzzy points \(x, y \in X, x \neq y\), \(\exists \ ijFSO\) sets \(A\) and \(B\) such that \(x, \alpha \subseteq \alpha\) \(\alpha\), \(y, \beta \subseteq \beta\) and \(\alpha \cap \beta = 0\).

**Remark 3.3.2.** If a bitopological space \((X, \delta_1, \delta_2)\) is \(ijFSPT_0\) space then it may not be a \(jiFSPT_0\).

*e.g.*: \(X = \{x, y\}, \delta_1 = \{1, 0, \{x.5\}\}, \delta_2 = \{1, 0\}\) then \((X, \delta_1, \delta_2)\) is \(12FSPT_0\) space but not a \(21FSPT_{0\gamma}\)-space.

**Remark 3.3.3.** If a fuzzy bitopological space \((X, \delta_1, \delta_2)\) is \(ijFSPT_1\) space then it may not be a \(jiFSPT_1\).

*e.g.*: \(X = \{x, y\}, \delta_1 = \{1, 0, \{x.5\}\}, \delta_2 = \{1, 0\}\) then \((X, \delta_1, \delta_2)\) is \(12FSPT_1\) space but not a \(21FSPT_1\)-space.

**Remark 3.3.4.** If a fuzzy bitopological space \((X, \delta_1, \delta_2)\) is \(ijFSPT_2\) space then it may not be a \(jiFSPT_2\).

*e.g.*, The fbt-space \((X, \delta_1, \delta_2)\) defined in the Remark 3.3.3 is a \(12FSPT_2\) space but not a \(21FSPT_2\)-space.
Theorem 3.3.5. For any fuzzy bitopological space \((X, \delta_1, \delta_2)\), \(ij\text{FSPT}_2 \Rightarrow ij\text{FSPT}_1 \Rightarrow ij\text{FSPT}_0\).

Proof: The proof directly follows from the Definition 3.3.1.

Remark 3.3.6. A \(ij\text{FSPT}_0\) space may not be a \(ij\text{FSPT}_1\)-space or \(ij\text{FSPT}_2\)-space.

e.g.: The fbt-space \((X, \delta_1, \delta_2)\) defined in the Remark 3.3.2 is a \(12\text{FSPT}_0\)-space but not a \(12\text{FSPT}_1\)-space nor a \(12\text{FSPT}_2\)-space.

Remark 3.3.7. A \(ij\text{FSPT}_1\) space may not be a \(ij\text{FSPT}_2\).

e.g.: \(X=\{x, y, z\}, \delta_1=\{1, 0, \{x, d\}, \{y, z, d\}\}, \delta_2=\{1, 0\}\) then \((X, \delta_1, \delta_2)\) is a \(12\text{FSPT}_1\) space but not a \(12\text{FSPT}_2\)-space.

Theorem 3.3.8. Let \((X, \tau_1, \tau_2)\) bitopological space, then \((X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_0\) space iff \((X, \omega(\tau_1), \omega(\tau_2))\) is a \(ij\text{FSPT}_0\)-space.

Proof: Let \((X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_0\) space and \(x, y\) be two fuzzy points in \(X\), with \(x \neq y\), so \(\exists \ ij\text{SO} \ set \ A \ in \ (X, \tau_1, \tau_2) \ such \ that \ x \in A \ and \ y \not\in A \ or \ y \in A \ and \ x \not\in A\), let \(x \in A \ and \ y \not\in A \) (other case is similar) \(\chi_A\) is a \(ij\text{FSO}\) in \((X, \omega(\tau_1), \omega(\tau_2))\) and \(\chi_A(x)=1 \ and \ \chi_A(y)=0 \ i.e. \ x \ q \ \chi_A \leq (y, 1)^c\) Hence \((X, \omega(\tau_1), \omega(\tau_2))\) is a \(ij\text{FSPT}_0\)-space.

Conversely, let \((X, \omega(\tau_1), \omega(\tau_2))\) be a \(ij\text{FSPT}_0\)-space. Let \(x, y \in X \ and \ x \neq y\), then for any \(r \in [0, 1)\), \(x_{1-r}\) and \(y_{1-r}\) are two fuzzy points with distinct supports, so \(\exists \ ij\text{SO} \ set \ A \ in \ (X, \omega(\tau_1), \omega(\tau_2)) \ such \ that \ x_{1-r} q \ A \leq (y, 1)^c\) or \(y_{1-r} q \ A \leq (x, 1)^c\).

Let \(x_{1-r} q \ A \leq (y, 1)^c\), (other case is similar) \(\therefore 1-r+A(x) \geq 1 \ and \ A(y) \leq r, \ i.e. \ A(x)>r \ and \ A(y) \leq r \ \therefore \ x \in A^{-1}(r, 1] \ and \ y \not\in A^{-1}(r, 1]\) but \(A^{-1}(r, 1]\) is \(ij\text{SO}\) set in \((X, \tau_1, \tau_2)\).

\(\therefore (X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_0\) space.

Theorem 3.3.9. Let \((X, \tau_1, \tau_2)\) bitopological space, then \((X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_1\) space iff \((X, \omega(\tau_1), \omega(\tau_2))\) is a \(ij\text{FSPT}_1\)-space.

Proof: The proof is similar to that of Theorem 3.3.8

Theorem 3.3.10. Let \((X, \tau_1, \tau_2)\) bitopological space, then \((X, \tau_1, \tau_2)\) is a \(ij\text{SPT}_2\) space iff \((X, \omega(\tau_1), \omega(\tau_2))\) is a \(ij\text{FSPT}_2\)-space.
**Proof:** Let \((X, \tau_i, \tau_j)\) be a \(ij\)-SPT\(_2\) space and \(x_r\) and \(y_s\) be two fuzzy points in \(X\), with \(x \neq y\), so \(\exists\) \(ij\)SO sets \(A\) and \(B\) in \((X, \tau_i, \tau_j)\) such that \(x \in A\), \(y \in B\) and \(A \cap B = \emptyset\). \(\chi_A\) and \(\chi_B\) are \(ij\)FSO in \((X, \omega(\tau_1), \omega(\tau_2)), \chi_A(x) = 1, \chi_A(y) = 0\) and \(\chi_B(x) = 1, \chi_B(y) = 0\), i.e. \(x_q \chi_A\), \(y_q \chi_B\) and since \(A \cap B = \emptyset\), so \(\chi_A \cap \chi_B = 0\).

Hence \((X, \omega(\tau_1), \omega(\tau_2))\) is a \(ij\FSPT\(_2\)-space.

Conversely, let \((X, \omega(\tau_1), \omega(\tau_2))\) be a \(ij\FSPT\(_r\)-space. Let \(x, y \in X\) and \(x \neq y\), then for any \(r \in [0,1)\), \(x_{1-r}\) and \(y_{1-r}\) are two fuzzy points with distinct supports, so \(\exists\) \(ij\)FSO sets \(A\) and \(B\) in \((X, \omega(\tau_1), \omega(\tau_2))\) such that \(x_{1-r} \in A\), \(y_{1-r} \in B\) and \(A \cap B = 0\).

\[x = A^t\langle r, 1 \rangle \text{ and } y = B^t\langle r, 1 \rangle.\]
But \(A^t\langle r, 1 \rangle\) and \(A^t\langle r, 1 \rangle\) are \(ij\)SO set in \((X, \tau_1, \tau_2)\). If \(z \in A^t\langle r, 1 \rangle \cap A^t\langle r, 1 \rangle\), then \(A(z) > r\) and \(B(z) > r\), i.e. \((A \cap B) > r\), which contradicts the fact that \(A \cap B = 0\), so \(A^t\langle r, 1 \rangle \cap A^t\langle r, 1 \rangle = \emptyset\).

\[\therefore (X, \tau_i, \tau_j)\] is a \(ij\SPT\(_2\) space.

**Definition 3.3.11.** A fuzzy bitopological space \((X, \delta_1, \delta_2)\) is said to be \(t-ij\SPT\(_k\)–space iff \((X, u_t(\delta_1), u_t(\delta_1))\) is a \(ij\SPT\(_k\)-space \((k = 0, 1, 2), t \in [0, 1)\).

**Definition 3.3.12.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(t-ij\FSPT\(_k\)-space iff \((X, \omega_t(\tau_1), \omega_t(\tau_2))\) is a \(ij\FSPT\(_k\)-space \((k = 0, 1, 2), t \in [0, 1)\).

**Theorem 3.3.13.** Let \((X, \tau_i, \tau_j)\) be a bitopological space, if \((X, \tau_i, \tau_j)\) is a \(t-ij\FSPT\(_0\) space then \((X, \tau_i, \tau_j)\) is a \(ij\SPT\(_0\) space.

**Proof:** Let \((X, \tau_i, \tau_j)\) be a bitopological space, then if \((X, \tau_i, \tau_j)\) is a \(t-ij\FSPT\(_0\) space, \((X, \omega_t(\tau_1), \omega_t(\tau_2))\) is \(ij\FSPT\(_r\)-space. Let \(x, y \in X\) and \(x \neq y\), \(x_{t-1}\) and \(y_{t-1}\) are two fuzzy points with distinct supports. So \(\exists\) \(ij\)FSO set \(A\) in \((X, \omega_t(\tau_1), \omega_t(\tau_2))\) such that \(x_{t-1}qA \leq t\langle 1 \rangle\) or \(y_{t-1}qA \leq t\langle 1 \rangle\). Let \(x_{t-1}qA \leq t\langle 1 \rangle\) (other case is similar), \(\therefore 1-t+A(x) > 1\) and \(A(y) \leq t i.e. A(x) > t\) and \(A(y) \leq t\), \(\therefore x \in A^{t-1}(t, 1]\) and \(y \notin A^{t-1}(t, 1]\) but \(A^{t-1}(t, 1]\) is \(ij\SO\) set in \((X, \tau_i, \tau_j)\)

\[\therefore (X, \tau_i, \tau_j)\] is \(ij\SPT\(_0\) space.
Theorem 3.3.14. Let $(X, \tau_1, \tau_2)$ be a bitopological space, if $(X, \tau_1, \tau_2)$ is a $t$-ijFSPT$_1$ space then $(X, \tau_1, \tau_2)$ is a ijSPT$_1$ space.

**Proof:** The proof is as similar that of the Theorem 3.3.13.

Theorem 3.3.15. Let $(X, \tau_1, \tau_2)$ be a bitopological space, if $(X, \tau_1, \tau_2)$ is a $t$-ijFSPT$_0$ space then $(X, \tau_1, \tau_2)$ is a ijSPT$_0$ space.

**Proof:** The proof is as similar that of the Theorem 3.3.13.

Theorem 3.3.16. Let $(X, \delta_1, \delta_2)$ be a fuzzy bitopological space, if $(X, \omega_t(\iota_t(\delta_1)), \omega_t(\iota_t(\delta_2)))$ is a ijFSPT$_2$-space then $(X, \delta_1, \delta_2)$ is a $t$-ijSPT$_2$ space.

**Proof:** Let $(X, \omega_t(\iota_t(\delta_1)), \omega_t(\iota_t(\delta_2)))$ is a ijFSPT$_0$-space.

Let $x, y \in X$ and $x \neq y$, say $x_{1-t}$ and $y_{1-t}$ are two fuzzy points with distinct supports. so $\exists$ ijFSO set $A$ in $(X, \omega_t(\iota_t(\delta_1)), \omega_t(\iota_t(\delta_1)))$ such that $x_{1-t}A \leq (y_{1-t})^c$ or $y_{1-t}A \leq (x_{1-t})^c$.

Let $x_{1-t}A \leq (y_{1-t})^c$, (other case is similar) $\therefore 1-t+A(x)\geq 1$ and $A(y) \leq t$ i.e. $A(x)\geq t$ and $A(y) \leq t$. $\therefore x \in A^{-1}(t,1]$ and $y \notin A^{-1}(t,1]$ but $A^{-1}(t,1]$ is ijSO set in $(X, \iota_t(\delta_1), \iota_t(\delta_1))$ $\therefore (X, \delta_1, \delta_2)$ is a $t$-ijSPT$_0$ space.

Theorem 3.3.17. Let $(X, \delta_1, \delta_2)$ be a fuzzy bitopological space, if $(X, \omega_t(\iota_t(\delta_1)), \omega_t(\iota_t(\delta_2)))$ is a ijFSPT$_1$-space then $(X, \delta_1, \delta_2)$ is a $t$-ijSPT$_1$ space.

**Proof:** Proof is similar to that of the theorem 3.3.16.

Theorem 3.3.18. Let $(X, \delta_1, \delta_2)$ be a fuzzy bitopological space, if $(X, \omega_t(\iota_t(\delta_1)), \omega_t(\iota_t(\delta_2)))$ is a ijFSPT$_2$-space then $(X, \delta_1, \delta_2)$ is a $t$-ijSPT$_2$ space.

**Proof:** Proof is similar to that of the Theorem 3.3.16.